Part I (MULTIPLE CHOICE, CALCULATORS NOT ALLOWED).

1. The limit

$$\lim_{x\to-\infty}\left(2x^2+x-1\right)$$

(a) is equal to 2.
(b) is equal to ∞.
(c) is equal to -2.
(d) is equal to $-\infty$.
(e) is equal to -1.

2. The limit

$$\lim_{x\to\frac{1}{5}^+}\frac{x}{10x-2}$$

(a) is equal to $-\frac{1}{2}$.
(b) is equal to $\frac{1}{10}$.
(c) is equal to ∞.
(d) is equal to 0.
(e) is equal to $-\infty$.

3. The limit

$$\lim_{x\to-\infty}x^2e^x$$

(a) is equal to 1.
(b) is equal to 0.
(c) is equal to -1.
(d) is equal to $-\infty$.
(e) is equal to ∞.

4. The limit

$$\lim_{x\to-3}\frac{x^2+6x+9}{x^2+2x-3}$$

(a) is equal to -1.
(b) is equal to $\frac{3}{2}$.
(c) is equal to 0.
(d) is equal to -3.
(e) does not exist.

5. The limit

$$\lim_{x\to 0}\frac{2x}{\sqrt{x+1}-1}$$

(a) is equal to -2.
(b) is equal to 0.
(c) is equal to 2.
(d) is equal to 4.
(e) is equal to ∞.

6. The limit

$$\lim_{x\to 5^-}\frac{|x-5|}{x-5}$$

(a) is equal to -1.
(b) is equal to 0.
(c) is equal to 1.
(d) is equal to ∞.
(e) is equal to $-\infty$.

7. The limit

$$\lim_{x\to 0}\frac{\sin(4x)}{2x}$$

(a) is equal to 0.
(b) is equal to 1.
(c) is equal to 2.
(d) is equal to ∞.
(e) is equal to $-\infty$.

8. $\frac{d}{dx}(2x\sin(x)) =$
(a) $2x$.
(b) $2\sin(x)$.
(c) $2x\cos(x)$.
(d) $2\sin(x)+2x\cos(x)$.
(e) $2\sin(x)+2\cos(x)$.

9. The derivative of function

$$f(x) = (5x^2-2x)^3$$

(a) is $f'(x) = (5x^2-2x)^3$.
(b) is $f'(x) = 3(10x-2)^2$.
(c) is $f'(x) = 10(5x^2-2x)^2$.
(d) is $f'(x) = (30x-6)(5x^2-2x)^2$.
(e) is $f'(x) = (10x-2)(5x^2-2x)^2$.

10. The derivative of function

$$f(t) = \frac{1}{\sqrt{2t}}$$

(a) is $f'(t) = -\frac{1}{2t\sqrt{2t}}$.
(b) is $f'(t) = -\frac{1}{\sqrt{2t}}$.
(c) is $f'(t) = -\frac{2}{\sqrt{2t}}$.
(d) is $f'(t) = \frac{1}{2t\sqrt{2t}}$.
(e) is $f'(t) = \frac{1}{\sqrt{2t}}$.

11. If $g(\theta) = \tan^2(5\theta)$, then $g'(\theta) =$

(a) $2\tan(5\theta)$.
(b) $2\tan(5\theta)\sec^2(5\theta)$.
(c) $10\tan(5\theta)\sec^2(5\theta)$.
(d) $10\tan^2(5\theta)\sec(5\theta)$.
(e) $10\tan^2(5\theta)\sec^2(5\theta)$.

12. If $y = \frac{5x}{x^3-4}$, then $\frac{dy}{dx} =$

(a) $\frac{5}{3x^2}$.
(b) $\frac{5}{3x^2-4}$.
(c) $\frac{x^4-15x^3-4x}{(x^3-4)^2}$.
(d) $\frac{20x^3-20}{(x^3-4)^2}$.
(e) $\frac{-10x^3-20}{(x^3-4)^2}$.

13. The derivative of function

$$f(x) = \frac{x^2+4x+3}{\sqrt{x}}$$

(a) is $f'(x) = \frac{3}{2}x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}$.
(b) is $f'(x) = -\frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$.
(c) is $f'(x) = -\frac{3}{2}x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}$.
(d) is $f'(x) = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$.
(e) is none of the above.

14. If $f(x) = \sin(x)e^{2x}$, then the second derivative $f''(x)$ is equal to

(a) $-\sin(x)e^{2x}$.
(b) $-4\sin(x)e^{2x}$.
(c) $-\sin(x)e^{2x} + 2\cos(x)e^{2x}$.
(d) $-\sin(x)e^{2x} + 2\cos(x)e^{2x} + 4\sin(x)e^{2x}$.
(e) $-\sin(x)e^{2x} + 4\cos(x)e^{2x} + 4\sin(x)e^{2x}$.

15. The derivative of function

$$f(x) = \ln \frac{x-1}{\sqrt{x+2}}$$

(a) is $f'(x) = \frac{\sqrt{x+2}}{x-1}$.
(b) is $f'(x) = \frac{1}{(x-1)\sqrt{x+2}}$.
(c) is $f'(x) = \frac{1}{2(x-1)\sqrt{x+2}}$.
(d) is $f'(x) = \frac{1}{x-1} - \frac{1}{x+2}$.
(e) is $f'(x) = \frac{1}{x-1} - \frac{1}{2(x+2)}$.

Part II (MULTIPLE CHOICE, CALCULATORS ALLOWED).

1. For the function $f(x)$ whose graph is given in Figure 1, choose the correct statement.

(a) $f(3) = \lim_{x\to 3} f(x)$.
(b) $\lim_{x\to 3^-} f(x) = 4$.
(c) $f(x)$ is right continuous at $x = 3$.
(d) $f(x)$ has an infinite discontinuity at $x = 3$.
(e) $2 = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$.

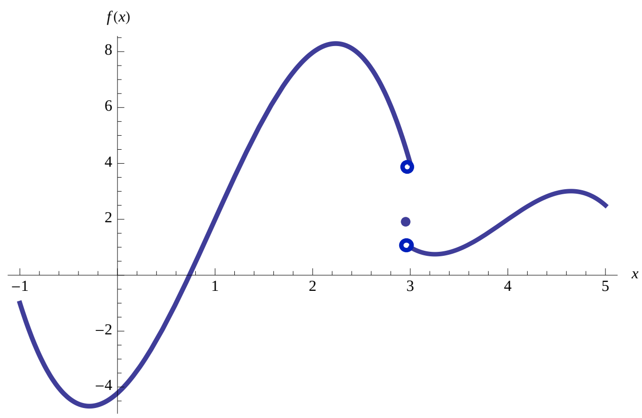

Figure 1: The graph of $f(x)$.

2. By the Squeeze Theorem, the limit

$$\lim_{x\to 0} 2x^2 \sin\left(\frac{2}{x^3}\right)$$

(a) is equal to $\lim_{x\to 0} 4x\cos(\frac{2}{3x^2})$.
(b) is equal to 0.
(c) is equal to ∞.
(d) is equal to $-\infty$.
(e) is equal to 2.

3. Given a piecewise-defined function

$$f(x) = \begin{cases} 2\,|x-3| & x \geq 1, \\ x^2+1 & 0 \leq x < 1, \\ \frac{1}{x+1} & x < 0. \end{cases}$$

(a) The domain of $f(x)$ is all the real numbers except at $x = -1, 0, 1, 3$;
the graph of $f(x)$ is continuous except at $x = -1$.
(b) The domain of $f(x)$ is all the real numbers except at $x = -1, 0, 1,$;
the graph of $f(x)$ is continuous except at $x = -1$.
(c) The domain of $f(x)$ is all the real numbers except at $x = -1, 1,$;
the graph of $f(x)$ is continuous except at $x = -1, 1$.
(d) The domain of $f(x)$ is all the real numbers except at $x = -1$;
the graph of $f(x)$ is continuous except at $x = -1, 1$.
(e) The domain of $f(x)$ is all the real numbers except at $x = -1$;
the graph of $f(x)$ is continuous except at $x = -1, 1, 3$.

4. Suppose $f(x)$ is a continuous function on $x \in [-1, 3]$, and $f(-1) = 4, f(3) = 7$. By the Intermediate Value Theorem, we can conclude that

(a) there exist a number $c \in (-1, 3)$ such that $f(c) = 0$.
(b) there exist a number $c \in (-1, 3)$ such that $f(c) = 5$.
(c) there exist a number $c \in (4, 7)$ such that $f(c) = 0$.
(d) there exist a number $c \in (4, 7)$ such that $f(c) = 5$.
(e) there exist a number $c \in (-1, 7)$ such that $f(c) = 0$.

5. Suppose a and b are finite real numbers. The limit

$$\lim_{x \to \infty} \frac{-a}{e^{2x} + b}$$

(a) is equal to $-\frac{a}{b}$.
(b) is equal to 0.
(c) is equal to $\frac{a}{b}$.
(d) is equal to $-\frac{a}{2b}$.
(e) does not exist.

6. By the definition, the derivative of $f(x) = \sqrt{2x-5}$ is

(a)

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{2x+2h-5}-\sqrt{2x-5}}{h}.$$

(b)

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{2x+2h-5}+\sqrt{2x-5}}{h}.$$

(c)

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{2x+h-5}-\sqrt{2x-5}}{h}.$$

(d)

$$f'(x) = \lim_{h\to a} \frac{\sqrt{2x+h-5}+\sqrt{2x-5}}{h}.$$

(e)

$$f'(x) = \lim_{h\to a} \frac{\sqrt{2x+h-5}-\sqrt{2x-5}}{h}.$$

7. The following table gives values of functions $f(x)$ and $g(x)$ as well as their first derivatives for various x values. Choose the correct statement.

x	1	2	3	4	5	6	7	8	9
$f(x)$	3	-5	2	10	4	-6	5	-4	6
$g(x)$	1	0	-1	1	-2	3	8	9	-3
$f'(x)$	5	2	4	-3	7	1	-2	6	-5
$g'(x)$	3	7	9	2	5	-3	1	4	6

(a) $(f \circ g)'(4) = 6$ and $(f^{-1} \circ g)(6) = 2$.
(b) $(f \circ g)'(4) = 5$ and $(f^{-1} \circ g)(6) = 2$.
(c) $(f \circ g)'(4) = 6$ and $(f^{-1} \circ g)(6) = 1$.
(d) $(f \circ g)'(4) = 10$ and $(f^{-1} \circ g)(6) = 1$.
(e) $(f \circ g)'(4) = 3$ and $(f^{-1} \circ g)(6) = 2$.

8. Suppose $s(t) = 25t^2 + 30t + 14$ is the position of an object at time t which is moving along a straight line.

(a) The average velocity over the time interval $[a, a+h]$ is $50a + 25h + 30$; the instantaneous velocity at $t = a$ is $50a + 30$.
(b) The average velocity over the time interval $[a, a+h]$ is $5a + h + 5$; the instantaneous velocity at $t = a$ is $50a + 30$.
(c) The average velocity over the time interval $[a, a+h]$ is $50a + 25h + 30$; the instantaneous velocity at $t = a$ is $\frac{25}{2}a^3 + 15a^2 + 14a$.
(d) The average velocity over the time interval $[a, a+h]$ is $5a + h + 5$; the instantaneous velocity at $t = a$ is $\frac{25}{2}a^3 + 15a^2 + 14a$.
(e) The average velocity over the time interval $[a, a+h]$ is $2a + h + 5$; the instantaneous velocity at $t = a$ is $25a^2 + 30a + 14$.

9. The linearization of the function $f(x) = \sqrt{x}$ at $x = 9$ is

(a) $y = \frac{x}{6} - \frac{3}{2}$, and the approximation of $\sqrt{8.5}$ by the linearization is 2.9167.
(b) $y = \frac{x}{6} - \frac{3}{2}$, and the approximation of $\sqrt{8.5}$ by the linearization is 2.9155.
(c) $y = \frac{x}{6} + \frac{3}{2}$, and the approximation of $\sqrt{8.5}$ by the linearization is 2.9167.
(d) $y = \frac{x}{6} + \frac{3}{2}$, and the approximation of $\sqrt{8.5}$ by the linearization is 2.9155.
(e) It is not possible to use the linearization method to approximate $\sqrt{8.5}$.

10. Let $\tan^{-1}(x)$ be the inverse function of $\tan(x)$. Then the derivative

$$\frac{d}{dx}\tan^{-1}(ax)$$

(a) is equal to $\frac{a}{1+(ax)^2}$.
(b) is equal to $\frac{a}{\sqrt{1-(ax)^2}}$.
(c) is equal to $-a\tan^{-2}(ax)$.
(d) is equal to $a\sec^2(ax)$.
(e) is equal to $-a\tan^{-2}(ax)\sec^2(ax)$.

11. Let $x^3 + 3x^4y = 4y^2 - 9$. Use implicit differentiation to find the derivative y'.

(a)

$$y' = \frac{3x^2 + 12x^3y}{8y - 3x^4}.$$

(b)

$$y' = \frac{3x^2 + 12x^3y + 3x^4}{8y}.$$

(c)

$$y' = \frac{x^3 + 9}{4y - 3x^4}.$$

(d)

$$y' = \frac{3x^2}{4y - 3x^4}.$$

(e)

$$y' = \frac{4y^2 - 9 - x^3}{3x^4}.$$

12. Suppose the derivative of function $f(x)$ satisfy $2 \le f'(x) \le 4$ for $-1 \le x \le 1$. By the Mean Value Theorem, we can conclude that the difference $f(1) - f(-1)$

(a) is between -4 and -2.
(a) is between -2 and 2.
(b) is between 0 and 4.
(c) is between 4 and 8.
(d) is between 8 and 16.

13. Suppose $f(x) = 5(x+3)(x-7)$. Then

(a) $x = -3$ is a critical point of $f(x)$ and $f(x)$ is concave up for $x \in \mathbb{R}$.
(b) $x = 7$ is a critical point of $f(x)$ and $f(x)$ is concave up for $x \in \mathbb{R}$.
(c) $x = 2$ is a critical point of $f(x)$ and $f(x)$ is concave up for $x \in \mathbb{R}$.
(d) $x = -3$ is a critical point of $f(x)$ and $f(x)$ is concave down for $x \in \mathbb{R}$.
(e) $x = 7$ is a critical point of $f(x)$ and $f(x)$ is concave down for $x \in \mathbb{R}$.

14. Suppose we know the derivative of function $f(x)$ is $f'(x) = 5x + \frac{1}{2x}$. Then we can use the antiderivative formulae to find the original function as

(a)
$$f(x) = 5 - \frac{2}{x^2} + C.$$

(b)
$$f(x) = \frac{5}{2}x^2 - \frac{2}{x^2} + C.$$

(c)
$$f(x) = \frac{5}{2}x^2 + \frac{1}{2} + C.$$

(d)
$$f(x) = 5 + \frac{1}{2}\ln|x| + C.$$

(e)
$$f(x) = \frac{5}{2}x^2 + \frac{1}{2}\ln|x| + C.$$

15. To find the root of the equation

$$\cos x = x,$$

Newton's method provides the iteration formula

(a)

$$x_{n+1} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}.$$

(b)

$$x_n = x_{n+1} + \frac{\cos x_{n-1} - x_{n-1}}{\sin x_{n-1} + 1}.$$

(c)

$$x_{n+1} = x_n + \frac{\sin x_n - 1}{\cos x_n - x_n}.$$

(d)

$$x_{n+1} = x_n - \frac{\sin x_n - 1}{\cos x_n - x_n}.$$

(e)

$$x_n = x_{n+1} - \frac{\sin x_{n-1} - 1}{\cos x_{n-1} - x_n}.$$

Part III (FREE RESPONSE, CALCULATORS ALLOWED). You must show your work in order to receive credit.

1. [**Application of Differential and Related Rates**] A snowball is melting as its radius, surface area and volume are all decreasing, but at different rates. Suppose it is a perfect sphere and the surface area is a quadratic function of the radius: $A = 4\pi r^2$.

(a) What is the exact change in the surface area (ΔA) when the diameter changes from 8cm to 7.8cm?

(b) Calculate the differential dA and use it to approximate the change in part (a).

(c) Use the Chain Rule to differentiate the equation of the surface area with respect to time t and establish a function relating the rate of change in the surface area and the rate of change in radius.

(d) Suppose the surface area is decreasing at a rate of 0.2 cm^2/min when the diameter of the snowball is 8cm, how fast is the radius decreasing?

(e) Use (d) to approximate how long it takes for the diameter to change from 8cm to 7.8cm.

2. [Exponential Model for Population Growth] The world population was 2,560 million in 1950 and 6,080 million in 2000. Assume the growth rate of the population $P'(t)$ is proportional to the size of the population $P(t)$: $P'(t) = kP(t)$.

(a) Write down the population model as an exponential function of time ($P(t)$) explicitly: calculate all the parameters involved in the model using the data provided above.

(b) What is the predicted world population in 2050?

(c) When is the world population predicted to reach 10 billion?

(d) How fast was the population growing in 1950 and 2000 respectively?

3. [Sketch a Function Using Calculus] Let $f(x) = \frac{x^2}{x-1}$.

(a) What is the domain of the function $f(x)$? Are there any horizontal or vertical asymptotes?

(b) Calculate the derivative $f'(x)$ and find all critical points.

(c) Where is the function $f(x)$ increasing and where is it decreasing? Find the local maximum and local minimum points.

(d) Calculate the second derivative $f''(x)$.

(e) Where is the function $f(x)$ concave up and where is it concave down?

(f) Are there any inflection points? If the answer is 'yes', find the inflection points. If the answer is 'no', justify your conclusion.

(g) Plot a few points. Using the information from part (a)-(f) to give a detailed sketch of the graph of $f(x)$.

4. [Optimization Problem] A cylindrical can is to be made to hold 355 ml (=355 cm^3) of soda drink. If we were going to be concerned solely with reducing the manufacturing cost, then we would like to find the dimension of the can that produces the smallest surface area while keeping the volume constant at 355 ml.

(a) Suppose the soda can has perfect cylindrical shape comprised of the top and bottom circular disks with radius ‘r’, and the side with height ‘h’. Write down the volume V and surface area A of the soda can as functions of two variables: height ‘h’ and radius ‘r’.

(b) Use the constraint that the volume is fixed at 355 ml to rewrite the surface area A as a function of only one variable: radius ‘r’.

(c) We choose a reasonable range for the radius r to be between 2cm and 5cm. Use the Closed Interval Method to find the smallest surface area A.

(d) What is the dimension of the soda can ('h' and 'r') when the minimal surface area is achieved?

Solution to the Multiple Choice Questions:

Part I: bcbcd acdda cedee

Part II: bbdbb adaca accea