Option Valuation Under A Regime-Switching Model Using the Fast Fourier Transform

by

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ABSTRACT

ELHAM SOHRABI. Option Valuation Under A Regime-Switching Model Using the Fast Fourier Transform. (Under the direction of Dr. Adriana Ocejo Monge)

The global financial crisis had severe implications on the real economy. For the US alone, Luttrell et al. [55] estimate output losses in the range of 6 to 14 trillion USD. It is hence not surprising that policy makers are keen to develop models which can issue warning signals ideally sufficiently early to implement policies that increase the resilience of financial institutions and ultimately mitigate at least some of the risks and costs associated with financial crises. Hence regime-switching models have been used extensively to identify business cycle turning points. Specifically, regime-switching models have the capability to incorporate the changes of the model dynamics brought by the changing macroeconomics conditions. Regime-switching models typically use the states of a modulating Markov chain to represent the states of an economy, depicted by some macroeconomics indicators. By adopting this methodology, regime-switching models can incorporate the impacts of structural changes in macroeconomics conditions on asset price dynamics and the stochastic evolution of investment opportunity sets, for example. Consequently, it is practical to consider the valuation of financial derivatives under regime-switching models.

In this thesis, we consider valuation of different types of options where the underlying asset price or commodity spot price is governed by a regime-switching model. We adopt an observable, continuous-time, finite-state Markov chain. We mostly focus on obtaining analytical formula of the so-called characteristic function for logarithm of commodity spot price, futures price and stock price.

Chapter 1 is organized as follows. Section 1 describe options and its pricing model. Section 2 provides a literature review for regime-switching model, stochastic interest rate models and fast Fourier transform. Section 3 presents a brief introduction of the

basic definitions and mathematical tools to be used in this thesis such as Markov chain setup model and fast Fourier transform (FFT). We describe how to obtain analytical pricing formula using inverse Fourier transform and discretize the pricing formula via FFT.

The outline of Chapter 2 is as follows. In Section 1, we first briefly introduce the motivation behind developing a fast Fourier transform approach for option pricing when the underlying asset process is governed by a regime-switching model. Section 2 describes the risk-neutral world and the asset price dynamics where under risk-neutral probability measure follows a regime-switching geometric Brownian motion. Section 3 presents the derivation for obtaining an analytical pricing formula for the two-state case and general case via the inverse Fourier transform. Then Section 4 calculates the inverse Fourier transform via the fast Fourier transform, providing an easier and faster way to calculate options prices. Section 5 introduces other numerical methods to compare with FFT results. As usual, we try to implement Monte Carlo simulation, as frequently serves as a benchmark for testing other numerical methods. A novel semi-Monte Carlo simulation algorithm is presented by Liu el. at [1] that can be also used as benchmark values in numerical experiments. To price our path dependent European call options, we require the stock price trajectory $\{S_t\}_{t\in[0,T]}$. Furthermore, we reported numerical results in Section 6 & 7 and provide further remarks and conclutions about the chapter in section 8. All of our proofs and Python programming are placed in the Appendix. It is our hope that this information when combined with some familiarity with the language, or at least an error checking IDE, that any reader will be able to replicate our results with little trouble.

In Chapter 3, we first state our motivation in section 1. Section 2 presents the Markovian regime-switching Ornstein-Uhlenbeck model. In this section, we discuss a Markovian regime-switching extension to the Ornstein-Uhlenbeck model for evaluating European-style commodity options and futures options. The main feature of our model is that model parameters, the mean-reverting level and the volatility of the commodity spot price, are governed by an observable continuous-time, finite-state, Markov chain. In Section 3, we first consider the valuation of commodity options and then the valuation of commodity futures options using inverse Fourier transform. The final section provides concluding remarks. All proofs in this chapter are standard and involve the use of standard mathematical techniques.

We also extend our work in chapter 4 to investigate the pricing of European-style commodity options and futures options with a Markovian regime-switching Hull–White stochastic interest rate model. The parameters of this model, including the mean-reversion level, the volatility of the stochastic interest rate, and the volatility of the commodity spot price are modulated by an observable, continuous-time, finite-state Markov chain. We start with introducing a risk-neutral probability measure. To take the zero-coupon bond value as the numéraire, a measure change technique is applied to change the risk-neutral probability measure into a forward measure. We then obtained a closed-form expression for the characteristic function of the logarithmic commodity price and futures price. Eventually, chapter 5 shows future directions and some potential future works.

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CHAPTER 1: INTRODUCTION

1.1 Overview

A "derivative" is a contract between two or more parties whose value is based on an agreed-upon underlying financial asset (like a security) or set of assets (like an index). Common underlying instruments include bonds, commodities, currencies, interest rates, market indexes and stocks. Derivatives have been about to be as the virtually important monetary instruments for centuries. The valuation of derivatives has been a long-lasting issue. There are many different types of derivatives. The most common derivative types are futures contracts, forward contracts, options and swaps. Amongst the different kinds of derivatives, options play a carrying a lot of weight role in the financial market. Options contracts have been known for decades. The Chicago Board Options Exchange was established in 1973, which set up a regime using standardized forms and terms and trade through a guaranteed clearing house. Trading activity and academic interest has increased since then (Brealey and Myers [53]).

Today, many options are created in a standardized form and traded through clearing houses on regulated options exchanges, while other over-the-counter options are written as bilateral, customized contracts between a single buyer and seller, one or both of which may be a dealer or market-maker. Hull [52] defines an option as follows: "An option is a financial derivative that represents a contract sold by one party (the option writer) to another party (the option holder). The contract offers the buyer the right, but not the obligation, to buy (call) or sell (put) a security or other financial asset at an agreed-upon price (the strike price) during a certain period of time or on a specific date (exercise date)".

There are many different types of options that can be traded and these can be categorized in a number of ways. In a very broad sense, there are two main types: calls and puts. Calls give the buyer the right to buy the underlying asset, while puts give the buyer the right to sell the underlying asset. Along with this clear distinction, options are also usually classified based on whether they are American style or European style. This has nothing to do with geographical location, but rather when the contracts can be exercised.

Options can be further categorized based on the method in which they are traded, their expiration cycle, and the underlying security they relate to. There are also other specific types and a number of exotic options that exist. Options and financial products with embedded-option features have become so important that we can hardly find an investment portfolio without these products. Consequently, the valuation of such well known financial derivatives deserves in a superior way attention.

After the introduction of well-known Black and Scholes [22]'s work, option valuation has played a vital role in the development of modern finance. The valuation of options has been a theoretically and practically important topic in the area of finance. In mathematical finance, the Black–Scholes equation is a partial differential equation (PDE) governing the price evolution of a European call or European put under the Black–Scholes model. The well-known Black-Scholes-Merton have attracted a lot of attention for quite a while due to the easy implementation of the closed-form option pricing formula. However, numerous empirical studies have revealed that the Black-Scholes-Merton model doesn't satisfy the ability to describe some vital features of the underlying assets, like no dividends are paid out during the life of the option, there are no transaction costs in buying the option, the risk-free rate and volatility of the underlying are known and constant, the returns on the underlying are normally distributed. In order to overcome the all mentioned shortcomings and improve the efficiency of the Black-Scholes-Merton model, both academic researchers and industry practitioners have dedicated efforts to extend the Black-Scholes-Merton model in various possible directions, including jump-diffusion models (Merton [30]), stochastic volatility models (Hull and White [31]; Wiggins [32]; Heston [33]; etc.), regime-switching models, etc.

1.2 Literature review

In this dissertation, we investigate options valuation under Regime-Switching (RS) model with stochastic interest rate using the fast Fourier transform (FFT). Therefore, it's beneficial to know about the background of Regime-Switching model, Stochastic interest rate and fast Fourier transform in the following subsections. This section presents a brief literature review of RS, Stochastic interest rate and FFT to be used in the subsequent chapters.

1.2.1 Regime-Switching model

The global financial crisis had severe implications on the real economy. For the US alone, Luttrell et al. [55] estimate output losses in the range of 6 to 14 trillion USD. It is hence not surprising that policy makers are keen to develop models which can issue warning signals ideally sufficiently early to implement policies that increase the resilience of financial institutions and ultimately mitigate at least some of the risks and costs associated with financial crises. Hence Regime-Switching models have been used extensively to identify business cycle turning points. Specifically, regime-switching models have the capability to incorporate the changes of the model dynamics brought by the changing macroeconomics conditions. Consequently, regime-switching models have attracted considerable interests and have been applied to various financial areas, such as option pricing, bond pricing, stock returns, etc. The history of regime-switching models can be traced back to the works of Quandt [34] and Goldfeld and Quandt [35]. The regime-switching model by Hamilton [26] is one of the most popular nonlinear time series models in the literature.

1.2.2 Stochastic interest rate models

An interest rate is the rate at which interest is paid by a borrower for the use of money that they borrow from a lender. Interest's rates are fundamental to a capitalist society. Interest rates are normally expressed as a percentage rate over the period of one year. Interest rates are also a tool of monetary policy and are taken into account when dealing with variables like investment, inflation, and unemployment. In traditional actuarial investigations, the interest rate is assumed to be deterministic and hence there is only one source of uncertainty, the mortality uncertainty, to be considered. Concerns about the effects of including a stochastic interest rate in the model have been growing during the last decade. The literature has tended to focus on annuities and the model adopted to describe the interest rate uncertainty, in a continuous framework, has usually involved the use of a Brownian motion. When the market rates are high, volatility is expected to be high or when interest rates are low, volatility will be low. Therefore, different stochastic interest rate models have been proposed and helped to overcome the disadvantage of the constant interest rate assumption under the Black-Scholes-Merton model. Some popular stochastic interest rate models include those proposed by Vasicek [23], Cox et al. [24], Hull and White [25], among others. One common feature of these models is the mean-reverting property of the interest rate. The short-term effectiveness of these models were justified by many empirical studies. Due to the advantages of regime-switching models, it is reasonable to expect that regime-switching stochastic interest rate models may improve the long-term effectiveness of the existing stochastic interest rate models. Examples of regime-switching stochastic interest rate diffusion models can be found in

Elliott and Mamon [27], Elliott and Wilson [28] and Elliott and Siu [18]. By adopting the method of stochastic flows, Siu [19] considered the valuation of a bond under a jump-augmented Vasicek model. A partial differential equation approach was applied in Shen and Siu [21] to obtain an exponential affine formula for a zero-coupon bond.

1.2.3 The fast Fourier transform

Various techniques have been devised to determine the valuation of financial derivatives. Among all, the Fourier transform has been widely applied to the valuation of financial derivatives. The faster calculation speed of the discrete Fourier transform against for example monte carlo simulation may be one of the main reasons why the fast Fourier transform (FFT) method attracts so much attention from both academics and industry. The first of these Fourier methods is actually the application of the Gil-Palaez inversion formula to finance. This idea originates from Heston [33]. However, singularities in the integrand prevent it to be an accurate method. The second attempt, more recent technique, was first proposed by Carr & Madan [2] by applying the FFT method to price European-style options under the variance gamma (VG) model. Since then, the FFT method has been applied to the valuation of options under different models. For example, Benhamou [39] discussed the valuation of discrete Asian options in non-lognormal density cases. Dempster and Hong [40] presented a two-dimensional FFT and considered the valuation of spread options under a three-factor stochastic volatility model. Cérny [6] discussed applications of the FFT in finance. By adopting the FFT technique, Liu et al. [1] investigated the valuation of options under a regime-switching model and Wong and Guan [41] considered the valuation of American options under a Lévy process.

FFT relys on the availability of the so-called characteristic function of the logarithm of the stock price. Given any such characteristic function, one can develop a simple analytic expression for the Fourier transform of the option value. Indeed, for a wide class of stock models characteristic functions have been obtained in closed form even if the risk-neutral densities (or probability mass functions) themselves are not available explicitly such as all the mentioned studies in the above paragraph. The tremendous speed of the FFT allows option prices for a huge number of strikes to be evaluated very rapidly. Although the FFT approach is significantly faster than other numerical methods, such as finite difference method and Monte Carlo simulation, it still has approximation errors when we adopt a discrete sum to approximate the integral. To control the approximation errors, Carr and Madan [2] discussed the selection of the upper limit of the integral and gave a sufficient condition to guarantee the square integrability property of the dampened pricing formula. Numerical errors in discretizing the pricing formula were discussed in Lee [42]. Liu et al. [1] also showed that the errors are small.

1.3 Preliminaries

This section presents a brief introduction of the basic definitions and tools to be used in the subsequent chapters. The contents in this section are mainly based on Elliott et al. [15] and Carr and Madan [2].

1.3.1 Markov chain

Let $\mathcal{T} = [0, T]$ be the time horizon ($T < \infty$). Define $(\Omega, \mathcal{F}, (\mathcal{F}(t)_{t\in\mathcal{T}}), \mathcal{P})$ to be a filtered complete probability space, where $\mathcal{F}(t)_{t\in\mathcal{T}}$ is a right continuous $\mathcal{P}$-complete filtration. Let $\mathbf{X} = \{\mathbf{X}(t)|t \in \mathcal{T}\}$ denote a continuous-time, finite-state Markov chain defined on $(\Omega, \mathcal{F}, \mathcal{P})$ with a finite state space S . Following Elliott et al. [15], without loss of generality, the state space of the Markov chain can be identified as a finite set of unit vectors $\varepsilon = \{e_1, e_2, ..., e_N\}$, where $e_i = (0, ..., 1, ..., 0) \in \mathbb{R}^N$ with 1 at its i-th position and 0 elsewhere.

Let $\mathcal{F}^x(t)$ denote the filtration generated by the Markov chain $\mathbf{X}$. By adopting the canonical state space ε, Elliott et al. [15] in Appendix B Lemma 1.1 obtained the following semi-martingale representation of the Markov chain:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}\,\mathbf{X}(s)\mathrm{d}s + \mathbf{M}(t)$$

where $\mathbf{A} = [a_{ij}]_{N\times N}$ is the generator of the Markov chain and $\{\mathbf{M}(t)|t \in \mathcal{T}\}$ is a right continuous martingale with respect to $\mathcal{F}^x(t)$.

1.3.2 The fast Fourier transform

The fast Fourier transform (FFT) was introduced in Carr and Madan [2]. For the sake of completeness, we present a brief introduction of the application of FFT. Define by $S(T)$ the value of the underlying asset at the maturity time T and K the strike price. Suppose that the price of a T-maturity European-style call option at time 0 is given by

$$C(0,T,K) = \mathbb{E}\Big\{\mathrm{e}^{-rT}\big(S(T) - K\big)^+\Big\} \tag{1.1}$$

where $\mathbb{E}[.]$ denote the expectation under the risk-neutral probability measure.

Let $s(T) = \ln S(T)$ and $\kappa = \ln K$ denote the logarithmic of the asset price at time T and the strike value, respectively. Then

$$C(0,T,\kappa) = \mathbb{E}\Big\{\mathrm{e}^{-rT}\big(\mathrm{e}^{s(T)} - \mathrm{e}^{\kappa}\big)^+\Big\}. \tag{1.2}$$

Now let's derive the Fourier transform of $C(0,T,\kappa)$. Assume $f(s)$ is the probability density function of $s(T)$:

$$\begin{aligned}
\hat{C}(0,T,u) &= \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa} C(0,T,\kappa)\mathrm{d}\kappa \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa} \mathbb{E}\big\{\mathrm{e}^{-rT}\big(\mathrm{e}^{s(T)} - \mathrm{e}^{\kappa}\big)^{+}\big\}\mathrm{d}\kappa \\
&= \mathrm{e}^{-rT} \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa} \int_{-\infty}^{\infty} \big(\mathrm{e}^{s} - \mathrm{e}^{\kappa}\big)^{+} f(s)\,\mathrm{d}s\,\mathrm{d}\kappa \\
&= \mathrm{e}^{-rT} \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa} \int_{\kappa}^{\infty} (\mathrm{e}^{s} - \mathrm{e}^{\kappa}) f(s)\,\mathrm{d}s\,\mathrm{d}\kappa \\
&= \mathrm{e}^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{s} \mathrm{e}^{-iu\kappa} (\mathrm{e}^{s} - \mathrm{e}^{\kappa})\,\mathrm{d}\kappa\, f(s)\,\mathrm{d}s \qquad (1.3)
\end{aligned}$$

where we changed the order of integration by Fubini's theorem and used the result $\int_s^{\infty} \mathrm{e}^{-iu\kappa}(\mathrm{e}^{s} - \mathrm{e}^{\kappa})^{+}\,\mathrm{d}\kappa = 0$. If we evaluate the inner integral of (1.3), we have

$$\begin{aligned}
\int_{-\infty}^{s} \mathrm{e}^{-iu\kappa}(\mathrm{e}^{s} - \mathrm{e}^{\kappa})\,\mathrm{d}\kappa &= \int_{-\infty}^{s} \mathrm{e}^{-iu\kappa}\,\mathrm{e}^{s}\,\mathrm{d}\kappa - \int_{-\infty}^{s} \mathrm{e}^{-iu\kappa}\,\mathrm{e}^{\kappa}\,\mathrm{d}\kappa \\
&= \mathrm{e}^{s} \frac{\mathrm{e}^{-iu\kappa}}{-iu}\Big|_{-\infty}^{s} - \frac{\mathrm{e}^{(1-iu)\kappa}}{(1-iu)}\Big|_{-\infty}^{s}. \qquad (1.4)
\end{aligned}$$

As we can see, the first term of (1.4) is undetermined due to $\lim_{\kappa\to-\infty} \mathrm{e}^{-iu\kappa} \neq 0$, while in second term of (1.4) $\lim_{\kappa\to-\infty} \mathrm{e}^{(1-iu)\kappa}$ converges to zero.

To get around the "undetermined" problem, Carr and Madan [2] introduced a dampning parameter α to modify the call option price. The so-called dampened call price is defined as

$$c(0,T,\kappa) = \exp(\alpha\kappa) C(0,T,\kappa).$$

The Fourier transform of the dampened call option price is given by

$$\hat{c}(0,T,u) = \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa}\, c(0,T,\kappa)\,\mathrm{d}\kappa.$$

Next we are showing that the dampening prameter α can force convergence, thereby permitting a computable Fourier tranform.

$$\begin{aligned}
\hat{c}(0,T,u) &= \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa} c(0,T,\kappa)\mathrm{d}\kappa \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa}\mathrm{e}^{\alpha\kappa}\mathbb{E}\big\{\mathrm{e}^{-rT}\big(\mathrm{e}^{s(T)} - \mathrm{e}^{\kappa}\big)^{+}\big\}\mathrm{d}\kappa \\
&= \mathrm{e}^{-rT}\int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa}\int_{-\infty}^{\infty} \mathrm{e}^{\alpha\kappa}\big(\mathrm{e}^{s(T)} - \mathrm{e}^{\kappa}\big)^{+} f(s(T))\,\mathrm{d}s\,\mathrm{d}\kappa \\
&= \mathrm{e}^{-rT}\int_{-\infty}^{\infty} \mathrm{e}^{\alpha\kappa}\mathrm{e}^{-iu\kappa}\int_{\kappa}^{\infty} (\mathrm{e}^{s} - \mathrm{e}^{\kappa}) f(s)\,\mathrm{d}s\,\mathrm{d}\kappa \\
&= \mathrm{e}^{-rT}\int_{-\infty}^{\infty}\int_{-\infty}^{s} \mathrm{e}^{(\alpha-iu)\kappa}(\mathrm{e}^{s} - \mathrm{e}^{\kappa})\,\mathrm{d}\kappa\, f(s)\,\mathrm{d}s \qquad (1.5)
\end{aligned}$$

Let's compare the inner integral in (1.3) to (1.5). It can be seen that the term $\int_{-\infty}^{s} \mathrm{e}^{-iu\kappa}(\mathrm{e}^{s} - \mathrm{e}^{\kappa})\,\mathrm{d}\kappa$ has been changed to $\int_{-\infty}^{s} \mathrm{e}^{(\alpha-iu)\kappa}(\mathrm{e}^{s} - \mathrm{e}^{\kappa})\,\mathrm{d}\kappa$.

Now if we evaluate the inner integral of (1.5), we have

$$\begin{aligned}
\int_{-\infty}^{s} \mathrm{e}^{(\alpha-iu)\kappa}(\mathrm{e}^{s} - \mathrm{e}^{\kappa})\,\mathrm{d}\kappa &= \int_{-\infty}^{s} \mathrm{e}^{(\alpha-iu)\kappa}\,\mathrm{e}^{s}\,\mathrm{d}\kappa - \int_{-\infty}^{s} \mathrm{e}^{(\alpha-iu)\kappa}\,\mathrm{e}^{\kappa}\,\mathrm{d}\kappa \\
&= \mathrm{e}^{s}\frac{\mathrm{e}^{(\alpha-iu)\kappa}}{(\alpha-iu)}\Big|_{-\infty}^{s} - \frac{\mathrm{e}^{(\alpha+1-iu)\kappa}}{(\alpha+1-iu)}\Big|_{-\infty}^{s} \\
&= \mathrm{e}^{s}\frac{\mathrm{e}^{(\alpha-iu)s}}{(\alpha-iu)} - \frac{\mathrm{e}^{(\alpha+1-iu)s}}{(\alpha+1-iu)} \\
&= \frac{\mathrm{e}^{(\alpha+1-iu)s}}{(\alpha-iu)(\alpha+1-iu)}. \qquad (1.6)
\end{aligned}$$

Given $\alpha > 0$, the exponential terms vanish for $\kappa = -\infty$:

$$\lim_{\kappa\to-\infty} \mathrm{e}^{(\alpha-iu)s} = \lim_{\kappa\to-\infty} \mathrm{e}^{(\alpha+1-iu)s} = 0. \qquad (1.7)$$

One can easily prove that $\alpha > 0$ is to ensure the square integrability for call options and $\alpha < 0$ is to ensure the square integrability for put options.

The pricing formula for the European-style call option can be obtained via the inverse Fourier transform as follows:

$$C(0,T,\kappa) = \frac{\mathrm{e}^{-\alpha\kappa}}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{iu\kappa}\, \hat{c}(0,T,u)\,\mathrm{d}u = \frac{\mathrm{e}^{-\alpha\kappa}}{\pi} \int_{0}^{\infty} \mathrm{e}^{iu\kappa}\, \hat{c}(0,T,u)\,\mathrm{d}u. \tag{1.8}$$

where the second term comes from the fact that call option is a real number. This implies the Fourier transform $\hat{c}(0,T,u)$ is odd in its imaginary part so that

$$\mathrm{Im}\{\hat{c}(0,T,u)\} = -\mathrm{Im}\{\hat{c}(0,T,-u)\} \tag{1.9}$$

and even in its real part so that

$$\mathrm{Re}\{\hat{c}(0,T,u)\} = \mathrm{Re}\{\hat{c}(0,T,-u)\}. \tag{1.10}$$

Thus this allows to rewrite the pricing integral as the second term in (1.8).

To derive the Fourier transform of the dampened call option price, one standard way is to utilize the relationship between the Fourier transform of the dampened call option price and the characteristic function of the logarithmic asset price. In this thesis, under regime-switching models, the conditional characteristic function of the logarithmic asset price given $\mathcal{F}^X(t)$ has to be derived first. To illustrate the method, we present the details in the present context.

Let $\mathbf{F}_{s(T)|\mathcal{F}^X(T)}(s)$ denote the conditional density function of $s(T)$ given $\mathcal{F}^X(T)$. Then for each $t \in [0,T]$ and $u \in \mathbb{R}$

$$\begin{aligned}
\hat{c}(0,T,u) &= \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa} c(0,T,\kappa)\mathrm{d}\kappa \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa}\mathrm{e}^{\alpha\kappa}\mathbb{E}\big\{\mathrm{e}^{-rT}(\mathrm{e}^{s(T)} - \mathrm{e}^{\kappa})^{+}\big\}\mathrm{d}\kappa \\
&= \mathbb{E}\Big\{ \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa}\mathrm{e}^{\alpha\kappa}\mathbb{E}\big\{\mathrm{e}^{-rT}(\mathrm{e}^{s(T)} - \mathrm{e}^{\kappa})^{+}|\mathcal{F}^X(T)\big\}\mathrm{d}\kappa \Big\} \\
&= \mathbb{E}\Big\{ \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa}\mathrm{e}^{\alpha\kappa}\mathrm{e}^{-rT} \int_{\kappa}^{\infty} \big(\mathrm{e}^{s} - \mathrm{e}^{\kappa}\big)\mathbf{F}_{s(T)|\mathcal{F}^X(T)}(s)\mathrm{d}s\mathrm{d}\kappa \Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\bigg\{ \int_{-\infty}^{\infty} \mathrm{e}^{-rT} \mathbf{F}_{s(T)|\mathcal{F}^X(T)}(s) \int_{-\infty}^{s} \big(\mathrm{e}^{s}\mathrm{e}^{(\alpha-iu)\kappa} - \mathrm{e}^{(1+\alpha-iu)\kappa}\big)\mathrm{d}\kappa \mathrm{d}s \bigg\} \\
&= \mathbb{E}\bigg\{ \int_{-\infty}^{\infty} \mathrm{e}^{-rT} \mathbf{F}_{s(T)|\mathcal{F}^X(T)}(s) \bigg(\frac{\mathrm{e}^{(1+\alpha-iu)s}}{(\alpha-iu)} - \frac{\mathrm{e}^{(1+\alpha-iu)s}}{(1+\alpha-iu)} \bigg) \mathrm{d}s \bigg\} \\
&= \mathbb{E}\bigg\{ \mathrm{e}^{-rT} \bigg(\frac{\phi_{\mathcal{F}_T}(-i(1+\alpha)-u)}{(\alpha-iu)} - \frac{\phi_{\mathcal{F}_T}(-i(1+\alpha)-u)}{(1+\alpha-iu)} \bigg) \bigg\} \\
&= \frac{\mathrm{e}^{-rT}\mathbb{E}\Big\{ \phi_{\mathcal{F}_T}(-i(1+\alpha)-u) \Big\}}{(\alpha-iu)(1+\alpha-iu)}
\end{aligned}
\tag{1.11}
$$

where

$$
\phi_{\mathcal{F}_T}(\nu) = \mathbb{E}\{\mathrm{e}^{ius(T)}|\mathcal{F}^X(T)\} = \int_{-\infty}^{\infty} \mathrm{e}^{i\nu\, s}\, \mathbf{F}_{s(T)|\mathcal{F}^X(T)}(s)\, \mathrm{d}s \tag{1.12}
$$

is the conditional characteristic function of $s(T)$ given $\mathcal{F}^X(T)$.

The third equality in (1.11) holds by the well-known property of conditional expectations($\mathbb{E}\{\mathbb{E}\{X|Y\}\} = \mathbb{E}\{X\}$), and fifth equality holds by Fubini's theorem since the modified call price is bounded.

There are many ways to define the discrete Fourier transform (DFT), varying in the sign of the exponent, normalization, etc. Since we are going to use Python implementation, therfore we are following the same definition for DFT given in Python package numpy. The DFT is defined in Python is

$$
A_k = \sum_{m=0}^{n-1} a_m \exp\Big\{ -2\pi i \frac{mk}{n} \Big\}, \quad k = 0, ..., n-1. \tag{1.13}
$$

The inverse DFT is defined as

$$
a_m = \frac{1}{n} \sum_{k=0}^{n-1} A_k \exp\Big\{ 2\pi i \frac{mk}{n} \Big\} \quad m = 0, ..., n-1.
$$

It differs from the Fourier transform by the sign of the exponential argument and the default normalization by $\dfrac{1}{n}$.

Given the Fourier transform function $\hat{c}(0, T, u)$, the modified call option price

$c(0,T,\kappa)$ can be obtained by the inverse Fourier transform as described in (1.8)

$$c(0,T,\kappa) = \frac{\exp(-\alpha\kappa)}{\pi} \int_0^\infty \mathrm{e}^{iu\kappa} \hat{c}(0,T,u)\mathrm{d}u, \quad \forall \quad -\infty < \kappa < \infty. \tag{1.14}$$

Set $u_j = j\triangle_u$, $j = 0,1,...,N-1$, where $\triangle u$ is the grid size in the variable u. Then (1.14) can be approximated by the following summation:

$$c(0,T,\kappa) \approx \frac{1}{\pi} \sum_{j=0}^{N-1} \mathrm{e}^{iu_j\kappa} \hat{c}(0,T,u_j)\triangle_u. \tag{1.15}$$

Next, let $\triangle_\kappa$ be the grid size in κ and choose a grid along the log strike κ as below:

$$\kappa_l = (l - \frac{N}{2})\triangle_\kappa, \quad l = 0,1,...,N-1. \tag{1.16}$$

Then

$$\begin{aligned} c(0,T,\kappa_l) &\approx \frac{1}{\pi} \sum_{j=0}^{N-1} \mathrm{e}^{iu_j\kappa_l} \hat{c}(0,T,u_j)\triangle_u, \quad l = 0,1,...,N-1 \\ &= \frac{1}{\pi} \sum_{j=0}^{N-1} \mathrm{e}^{ij\triangle_u(l-\frac{N}{2})\triangle_\kappa} \hat{c}(0,T,u_j)\triangle_u, \quad l = 0,1,...,N-1 \\ &= \frac{1}{\pi} \sum_{j=0}^{N-1} \mathrm{e}^{ijl\triangle_u\triangle_\kappa} \mathrm{e}^{-ij\frac{N}{2}\triangle_u\triangle_\kappa} \hat{c}(0,T,u_j)\triangle_u, \quad l = 0,1,...,N-1. \end{aligned} \tag{1.17}$$

If we set

$$\triangle_u\triangle_\kappa = \frac{2\pi}{N}, \tag{1.18}$$

then we have

$$c(0,T,\kappa_l) \approx \frac{1}{\pi} \sum_{j=0}^{N-1} \mathrm{e}^{ijl\frac{2\pi}{N}} \mathrm{e}^{-ij\pi} \hat{c}(0,T,u_j)\triangle_u, \quad l = 0,1,...,N-1. \tag{1.19}$$

Using Simpson's rule for numerical integration, define a sequence of weighting factors by

$$w(j) = \begin{cases} \frac{1}{3}, & \text{if } j = 0, \\ \frac{4}{3}, & \text{if } j \text{ is odd}, \\ \frac{2}{3}, & \text{if } j \text{ is even}. \end{cases}$$

Then

$$c(k_l) \approx \frac{\triangle_u}{\pi N} \sum_{j=0}^{N-1} \mathrm{e}^{ijl\frac{2\pi}{N}} \mathrm{e}^{-ij\pi} \hat{c}(0, T, u_j) w(j) N, \quad l = 0, 1, ..., N-1. \tag{1.20}$$

Comparing (1.20) with (1.13), it is easily seen that $c(0, T, \kappa)$ can be obtained by taking the Fourier transform of the sequence $\left\{\mathrm{e}^{-ij\pi} \hat{c}(0, T, u_j) w(j) N\right\}$, for $j = 0, 1, ..., N-1$.

The fast Fourier transform algorithm developed by Cooley and Tukey [54] and later extended by many others provide a more efficient algorithm for calculating DFT or inverse DFT with sample points that are powers of two. That is, $N = 2^p$, $p \in \{1, 2, ...\}$. The Cooley-Tukey FFT algorithm can reduce the number of multiplications from N^2 to $N\log N$.

CHAPTER 2: FFT APPROACH FOR PRICING A EUROPEAN CALL OPTION UNDER A REGIME-SWITCHING MODEL

In this chapter, we introduce the fast Fourier transform (FFT) approach to option valuation, where the underlying asset price is governed by a regime-switching geometric Brownian motion.

2.1 Motivation

The fast Fourier transform (FFT) is a numerical approach for pricing options which utilizes the characteristic function of the underlying instrument's price process. The fast Fourier transform is a significant computational method in scientific computing and it has been widely applied to financial engineering, specifically in options pricing. FFT approach makes use of the characteristic function of the underlying asset price. The use of the fast Fourier transform method is motivated by the following reasons: the algorithm has speed advantage (especially over Monte Carlo Simulation and PDE). This enables the Fourier transform algorithm to calculate prices accurately for a whole range of strikes. The characteristic function of the log-price is known and has a simple form for many models considered in literature while the density is often not known in the closed form. The models meet this requirement include the stochastic volatility models, the affine jump diffusions, and the exponential Lévy models, among others; see Carr and Madan [2], Carr and Wu [3], and Duffie et al. [4] for detailed discussions of these models. However, FFT approach is only applicable to problems for which the characteristic functions of the underlying price process can be obtained analytically. Because of its prevalence, increasing research efforts have been devoted to the FFT approach in option pricing. For example, Carr and Madan [2] illustrated the fundamental idea of using FFT for valuing European options based on the Black-Scholes setting and applied it to the variance gamma (VG) model (see Madan et al. [5]). Ĉerný [6] presented a detailed discussion on the implementation of FFT to option pricing.

Along another line, considerable attention has been focused on the regime-switching diffusion models for asset prices recently. In this setting, model parameters (rate of return, volatility, and risk-free interest rate) are assumed to depend on a finite-state, observable Markov chain, whose states represent different ”states of the world” or regimes, which can describe various randomly changing economical factors. By incorporating an observable Markov chain into the formulation, the regime-switching framework can capture the effect of those less frequent but significant events that have impact on the individual asset price behavior (especially for long-term dynamics). This is a major advantage compared with other models, see Yao et al. [7], and Zhang [8], among others for discussions on considerations leading to this modelling approach.

In this chapter, the fast Fourier transform (FFT) approach is applied for pricing European-style call options, where the underlying asset price is governed by a regime-switching geometric Brownian motion (RSGBM). An FFT method for the regime-switching model is developed. For the two states case, numerical result is provided, however, for the general case where the number of states is more than two ($m > 2$), the fundamental matrix sulotion is not known explicitly. We use monte-carlo simulation as well as a novel method called semi-MC simulation to compare with our FFT results for the two case. We also use analytical sulotion for the case where the drift and interest rate don't depend on time and compare the results with what obtained by FFT, MCS and Semi-MCS.

2.2 Regime-switching model and risk-neutral option pricing

We consider a continuous-time economy with a finite time horizon $[0, T]$ where $T < \infty$. Suppose that $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space, where $\mathcal{P}$ is a risk-neutral probability measure. Let $\alpha(t)$ be a finite-state, continuous time, observable Markov chain with a finite state space $\mathcal{M} = \{1, ..., m\}$, which may represent general market trends. For example, when $m = 2$, $\alpha(t) = 1$ may denote a bull market and $\alpha(t) = 2$ a bear market.

Let's assume that under the risk-neutral measure $\mathcal{P}$, the dynamics of the underlying asset value, S_t, is given by

$$\mathrm{d}S_t = \mu(\alpha(t))\, S_t\, \mathrm{d}t + \sigma(\alpha(t))\, S_t\, \mathrm{d}W_t, \quad t \geq 0 \tag{2.1}$$

where $S(0) = S_0 > 0$ is the initial underlying asset price, W_t is a standard Brownian motion independent of $\alpha(t)$, and $\mu(\alpha(t))$ and $\sigma(\alpha(t))$ are the risk-free drift rate and volatility of the underlying asset, respectively. We assume that $\mu(j)$ and $\sigma(j)$ are positive constant, for each $j \in \mathcal{M}$.

Under the risk-neutral probability measure $\mathcal{P}$, the price of T-maturity European-style call options at time 0 with strike $K > 0$ is given as follows:

$$C(K) = \mathbb{E}\left\{\exp\left(-\int_0^T r(\alpha(t))\mathrm{d}t\right)(S_T - K)^+\right\}. \tag{2.2}$$

where the instantaneous risk-free interest rate, $r(\alpha(t))$ also depends on $\alpha(t)$ with $r(j) > 0$, for each $j \in \mathcal{M}$.

Following the notation in Lui, Zhang and Yin [1], let $k = \ln\left(\frac{K}{S_0}\right)$ and $S_T = S_0 \mathrm{e}^{X_T}$. Note that $k = 0$ (when $K = S_0$) will be always corresponding to the at-the-money case.

Then (2.2) can be written as

$$C(k) = S_0 \mathbb{E}\left\{ \exp\left(-\int_0^T r(\alpha(t))\mathrm{d}t \right) (\mathrm{e}^{X_T} - \mathrm{e}^k)^+ \right\}. \tag{2.3}$$

2.3 Fourier transform of the option price

In this section, we would like to apply Fourier transform to European-style call option price given by equation (2.3). Let's first recall the definition of Fourier transform (FT) and inverse Fourier transform (IFT) for continuous functions. The Fourier transform of the function f is traditionally denoted by $\hat{f}$. There are several common conventions for defining the Fourier transform of an integrable function $f : \mathbb{R} \mapsto \mathbb{C}$. Here we will use the following definition:

$$\hat{f}(u) = \int_{-\infty}^{\infty} \mathrm{e}^{-iux} f(x)\mathrm{d}x,$$

for any real number u.

When the independent variable x represents time, the transform variable u represents frequency. Under suitable conditions, f is determined by $\hat{f}$ via inverse Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} \mathrm{e}^{iux} \hat{f}(u)\mathrm{d}u,$$

for any real number x.

For the rest of this dissertation, we use above definitions for FT and IFT. Assume $f(x)$ is the probability density function of X_T. Now let's derive the Fourier transform of $C(k)$.

$$
\begin{aligned}
\hat{C}(u) &= \int_{-\infty}^{\infty} \mathrm{e}^{-iuk} C(k)\mathrm{d}k \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-iuk} S_0 \mathbb{E}\Big\{\exp\Big(-\int_0^T r(\alpha(t))\mathrm{d}t\Big)(\mathrm{e}^{X_T} - \mathrm{e}^k)^+\Big\}\mathrm{d}k \\
&= S_0 \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa} \int_{-\infty}^{\infty} \exp\Big(-\int_0^T r(\alpha(t))\mathrm{d}t\Big)\big(\mathrm{e}^x - \mathrm{e}^k\big)^+ f(x)\,\mathrm{d}x\,\mathrm{d}k \\
&= S_0 \int_{-\infty}^{\infty} \exp\Big(-\int_0^T r(\alpha(t))\mathrm{d}t\Big)\mathrm{e}^{-iuk} \int_k^{\infty} (\mathrm{e}^x - \mathrm{e}^k) f(x)\,\mathrm{d}x\,\mathrm{d}k \\
&= S_0 \int_{-\infty}^{\infty} \exp\Big(-\int_0^T r(\alpha(t))\mathrm{d}t\Big) f(x) \int_{-\infty}^{x} \mathrm{e}^{-iuk}(\mathrm{e}^x - \mathrm{e}^k)\,\mathrm{d}k\,\mathrm{d}x
\end{aligned}
\tag{2.4}
$$

where we changed the order of integration by Fubini's theorem and used the result $\int_x^\infty \mathrm{e}^{-iuk}(\mathrm{e}^x - \mathrm{e}^k)^+\,\mathrm{d}k = 0$. If we evaluate the inner integral of (2.4), we have

$$
\begin{aligned}
\int_{-\infty}^{x} \mathrm{e}^{-iuk}(\mathrm{e}^x - \mathrm{e}^k)\,\mathrm{d}k &= \int_{-\infty}^{x} \mathrm{e}^{-iuk}\,\mathrm{e}^x\,\mathrm{d}k - \int_{-\infty}^{x} \mathrm{e}^{-iuk}\,\mathrm{e}^k\,\mathrm{d}k \\
&= \mathrm{e}^x \frac{\mathrm{e}^{-iuk}}{-iu}\Big|_{-\infty}^{x} - \frac{\mathrm{e}^{(1-iu)k}}{(1-iu)}\Big|_{-\infty}^{x}.
\end{aligned}
\tag{2.5}
$$

As we can see, the first term of (2.5) is not integrable since $\lim_{k\to-\infty} \mathrm{e}^{-iuk} \neq 0$, while in second term of (2.5) $\lim_{k\to-\infty} \mathrm{e}^{(1-iu)k}$ converges to zero.

To obtain a squared integrable function with respect to k, Carr and Madan [2] introduced a dampning parameter ρ to modify the call option price. The so-called dampened call price is defined as

$$
c(k) = \mathrm{e}^{\rho k}\frac{C(k)}{S_0}, \quad -\infty < k < \infty \tag{2.6}
$$

where $\rho > 0$ is a prespecified positive number (dampening factor). We explained in details in Chapter 1, Section 1.3.2 why we need positive dampening factor for call and negative dampening factor for put options.

The Fourier transform of the dampened call option price is given by:

$$\hat{c}(u) = \int_{-\infty}^{\infty} \mathrm{e}^{-iuk} c(k) \mathrm{d}k, \quad u \in (-\infty, \infty). \tag{2.7}$$

Once we find a closed form for Fourier transform of the dampened call option price, the pricing formula for the European-style call option can be obtained via the inverse Fourier transform as follows:

$$C(k) = \frac{\mathrm{e}^{-\rho k} S_0}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{iuk} \, \hat{c}(u) \, \mathrm{d}u = \frac{\mathrm{e}^{-\alpha k} S_0}{\pi} \int_{0}^{\infty} \mathrm{e}^{iuk} \, \hat{c}(u) \, \mathrm{d}u. \tag{2.8}$$

where the second term comes from the fact that call option is a real number. This implies the Fourier transform $\hat{c}(u)$ is odd in its imaginary part so that

$$\mathrm{Im}\{\hat{c}(u)\} = -\mathrm{Im}\{\hat{c}(-u)\} \tag{2.9}$$

and even in its real part so that

$$\mathrm{Re}\{\hat{c}(u)\} = \mathrm{Re}\{\hat{c}(-u)\}. \tag{2.10}$$

Thus this allows to rewrite the pricing integral as the second term in (2.8).

Therefore, all we need to do now is to find a closed form for Fourier transform of the dampened call option price. Let $\mathcal{F}_T$ be the σ-algebra generated by the Markov chain $\alpha(t)$, $0 \leq t \leq T$, that is, $\mathcal{F}_T = \sigma\{\alpha(t), 0 \leq t \leq T\}$. Note that W_t is still a Brownian motion (BM) with respect to the filteration of Markov chain since we consider BM to be independent of Markov chain. Let $f_{\mathcal{F}_T}(x)$ be the conditional density function of X_T given $\mathcal{F}_T$.

Then the Fourier transform of the dampened call option price, $c(k)$, is calculated as follows:

$$
\begin{aligned}
\hat{c}(u) &= \int_{-\infty}^{\infty} \mathrm{e}^{-iuk}\, c(k)\, \mathrm{d}k = \int_{-\infty}^{\infty} \mathrm{e}^{-iuk}\, \mathrm{e}^{\rho k}\, \mathbb{E}\big\{\mathrm{e}^{-\int_0^T r(\alpha(t))\mathrm{d}t}\, (\mathrm{e}^{X_T} - \mathrm{e}^{k})^{+}\big\}\, \mathrm{d}k \\
&= \mathbb{E}\Big\{ \int_{-\infty}^{\infty} \mathrm{e}^{-iuk}\mathrm{e}^{\rho k}\mathbb{E}\big\{\mathrm{e}^{-\int_0^T r(\alpha(t))\mathrm{d}t}(\mathrm{e}^{X_T} - \mathrm{e}^{k})^{+}|\mathcal{F}_T\big\}\mathrm{d}k\Big\} \\
&= \mathbb{E}\Big\{ \int_{-\infty}^{\infty} \mathrm{e}^{-iuk}\mathrm{e}^{\rho k}\mathrm{e}^{-\int_0^T r(\alpha(t))\mathrm{d}t} \int_{k}^{\infty} \big(\mathrm{e}^{x} - \mathrm{e}^{k}\big) f_{\mathcal{F}_T}(x)\mathrm{d}x\mathrm{d}k\Big\} \\
&= \mathbb{E}\Big\{ \int_{-\infty}^{\infty} \mathrm{e}^{-\int_0^T r(\alpha(t))\mathrm{d}t} f_{\mathcal{F}_T}(x) \int_{-\infty}^{x} \big(\mathrm{e}^{x}\mathrm{e}^{(\rho-iu)k} - \mathrm{e}^{(1+\rho-iu)k}\big)\mathrm{d}k\mathrm{d}x\Big\} \\
&= \mathbb{E}\Big\{ \int_{-\infty}^{\infty} \mathrm{e}^{-\int_0^T r(\alpha(t))\mathrm{d}t} f_{\mathcal{F}_T}(x)\Big(\frac{\mathrm{e}^{(1+\rho-iu)x}}{(\rho-iu)} - \frac{\mathrm{e}^{(1+\rho-iu)x}}{(1+\rho-iu)}\Big)\mathrm{d}x\Big\} \\
&= \mathbb{E}\Big\{\mathrm{e}^{-\int_0^T r(\alpha(t))\mathrm{d}t}\Big(\frac{\phi_{\mathcal{F}_T}(-i(1+\rho)-u)}{(\rho-iu)} - \frac{\phi_{\mathcal{F}_T}(-i(1+\rho)-u)}{(1+\rho-iu)}\Big)\Big\} \\
&= \frac{\mathbb{E}\Big\{\mathrm{e}^{-\int_0^T r(\alpha(t))\mathrm{d}t}\phi_{\mathcal{F}_T}(-i(1+\rho)-u)\Big\}}{(\rho-iu)(1+\rho-iu)}
\end{aligned}
\tag{2.11}
$$

where

$$
\phi_{\mathcal{F}_T}(u) = \mathbb{E}\{\mathrm{e}^{iuX_T}|\mathcal{F}_T\} = \int_{-\infty}^{\infty} \mathrm{e}^{iux} f_{\mathcal{F}_T}(x)\mathrm{d}x \tag{2.12}
$$

is the conditional characteristic function of X_T given $\mathcal{F}_T$.

Note that the third equality in (2.10) holds by the well-known property of conditional expectations($\mathbb{E}\{\mathbb{E}\{X|Y\}\} = \mathbb{E}\{X\}$), and we changed the order of integration by Fubini's theorem in fifth equality since the modified call price is bounded.

Call values are determined by substituting (2.11) into (2.8) and performing the required integration. To find the explicit form for the required integration, we still need to find the expectation in (2.11). Thus we first try to find $\phi_{\mathcal{F}_T}(u)$. To do this end, we need to know the distribution of X_T. Recall $S_T = S_0\,\mathrm{e}^{X_T}$, therefore

$$
S_T = S_0 \mathrm{exp}\Big(\int_0^T \big[\mu(\alpha(s)) - \frac{1}{2}\sigma^2(\alpha(s))\big]\mathrm{d}s + \int_0^T \sigma(\alpha(s))\mathrm{d}W_s\Big), \quad t \geq 0.
$$

To simplify our notation, let's define

$$L_T = \int_0^T \mu(\alpha(t))\mathrm{d}t, \quad V_T = \int_0^T \sigma^2(\alpha(t))\mathrm{d}t, \quad R_T = \int_0^T r(\alpha(t))\mathrm{d}t. \tag{2.13}$$

Then given $\mathcal{F}_T$, X_T has Gaussian distribution with mean $(L_T - \frac{1}{2}V_T)$ and variance V_T. It follows that

$$\phi_{\mathcal{F}_T}(u) = \exp\Big(iu(L_T - \frac{1}{2}V_T) - \frac{1}{2}u^2 V_T\Big). \tag{2.14}$$

Plugging (2.14) in (2.11), we have

$$\begin{aligned}\hat{c}(u) = \frac{1}{(\rho - iu)(1+\rho - iu)}\mathbb{E}\Big\{\exp\Big((1+\rho)\Big(L_T + \frac{1}{2}\rho V_T\Big) - R_T \\ - \frac{1}{2}u^2 V_T - iu\Big(L_T + (\frac{1}{2} + \rho)V_T\Big)\Big)\Big\}.\end{aligned} \tag{2.15}$$

As we can see, still we haven't been able to find a closed form for $\hat{c}(u)$ as it is necessary to calculate the expectation with respect to L_T , V_T and R_T in (2.15). Note that L_T , V_T and R_T are random variables as their value depends on how much time Markov chain spent in state j for example. Therefore, it's useful to define the sojourn time of the Markov chain $\alpha(t)$ in state j during the interval $[0, T]$.

$$T_j = \int_0^T \mathbf{1}_{\{\alpha(t)=j\}}\mathrm{d}t, \quad j \in \mathcal{M} \tag{2.16}$$

Then $\sum_{j=1}^m T_j = T$. Then the three random variables L_T , V_T and R_T defined in (2.13) can be rewritten as

$$\begin{aligned}L_T &= \sum_{j=1}^{m-1}\Big(\mu(j) - \mu(m)\Big)T_j + \mu(m)T, \\ V_T &= \sum_{j=1}^{m-1}\Big(\sigma^2(j) - \sigma^2(m)\Big)T_j + \sigma^2(m)T,\end{aligned} \tag{2.17}$$

$$R_T = \sum_{j=1}^{m-1} \Big(r(j) - r(m) \Big) T_j + r(m) T.$$

Using (2.17) in (2.15), we obtain that

$$\hat{c}(u) = \frac{1}{(\rho - iu)(1 + \rho - iu)} \exp\Big(B(u) T \Big) \mathbb{E} \Big\{ \exp\Big(i \sum_{j=1}^{m-1} A(u, j) T_j \Big) \Big\} \tag{2.18}$$

where for $j = 1, ..., m-1$,

$$\begin{aligned} A(u, j) = &-u \Big[(\mu(j) - \mu(m)) + \Big(\frac{1}{2} + \rho \Big) (\sigma^2(j) - \sigma^2(m)) \Big] \\ &+ \frac{1}{2} u^2 (\sigma^2(j) - \sigma^2(m)) i + \Big[\Big(r(j) - r(m)) - \\ &(1 + \rho)(\mu(j) - \mu(m) \Big) - \frac{1}{2} \rho (1 + \rho)(\sigma^2(j) - \sigma^2(m)) \Big] i, \end{aligned}$$

$$\begin{aligned} B(u) = &-iu \Big[\mu(m) + \Big(\frac{1}{2} + \rho \Big) \sigma^2(m) \Big] - \frac{1}{2} u^2 \sigma^2(m) + \\ &(1 + \rho)\mu(m) - r(m) + \frac{1}{2} \rho (1 + \rho) \sigma^2(m). \end{aligned} \tag{2.19}$$

Therefore, the determination of $\hat{c}(u)$ reduces to calculating the characteristic function of the random vector $(T_1, ..., T_{m-1})'$. For the two states case, we only need to find the characterisitic function of random sojourn time T_1, as by finding T_1 implies that $T_2 = T - T_1$. We adopt the same methodology in Liu el at [1] for the two states case. However, for general case, when $m > 2$, we need to find the characteristic function of the random vector $(T_1, ..., T_{m-1})'$. We are going to use a modification of proof of lemma 1 in Buffington and Elliott [11].

Let the generator of the Markov chain $\alpha(\cdot)$ be given by an $m \times m$ matrix $Q =$

$(q_{ij})_{m\times m}$ such that $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{m} q_{ij} = 0$ for each $j \in \mathcal{M}$. Let

$$\mathbf{I}(t) = \Big(\mathbf{1}_{\{\alpha(t)=1\}}, \mathbf{1}_{\{\alpha(t)=2\}}, ..., \mathbf{1}_{\{\alpha(t)=m\}}\Big)' \in \mathcal{R}^{m\times 1}, \tag{2.20}$$

denote the vector of indicator functions. Then it is shown by Yin and Zhang [10, Lemma 2.4, Chapter 2] that

$$M(t) = \mathbf{I}(t) - \int_0^t Q'\mathbf{I}(s)\mathrm{d}s \tag{2.21}$$

is a martingale, where Q' denotes the transpose of Q. This implies

$$\mathrm{d}\mathbf{I}(t) = Q'\mathbf{I}(t)\mathrm{d}t + \mathrm{d}M(t). \tag{2.22}$$

Let's simplify our notation in (2.18) in order to determine the characteristic function of the random vector $(T_1, ..., T_{m-1})'$.

$$\mathbb{E}\bigg\{\exp\bigg(i\sum_{j=1}^{m-1}\theta_j T_j\bigg)\bigg\} = \mathbb{E}\bigg\{\exp\bigg(i\sum_{j=1}^{m-1}\theta_j \int_0^T \mathbf{1}_{\{\alpha(t)=j\}}\mathrm{d}t\bigg)\bigg\}. \tag{2.23}$$

We would like to generalize our method by calculating the following characteristic function

$$\mathbb{E}\bigg\{\exp\bigg(i\sum_{j=1}^{m-1}\theta_j \int_t^u \mathbf{1}_{\{\alpha(t)=j\}}\mathrm{d}t\bigg)\bigg\}, \quad \forall \quad u \in [t, T]. \tag{2.24}$$

Define a random vector

$$\mathbf{Z}(t,u) = \Big(z_1(t,u), z_2(t,u), ..., z_m(t,u)\Big)' \in \mathcal{R}^{m\times 1} \tag{2.25}$$

for any $u \in [t, T]$, where

$$z_j(t,u) = \exp\left(i \sum_{j=1}^{m-1} \theta_j \int_t^u \mathbf{1}_{\{\alpha(s)=j\}} \mathrm{d}s\right) \mathbf{1}_{\{\alpha(u)=j\}}, \tag{2.26}$$

in other words, by using the given definition (2.21) for $\mathbf{I}(u)$

$$\mathbf{Z}(t,u) = \exp\left(i \sum_{j=1}^{m-1} \theta_j \int_t^u \mathbf{1}_{\{\alpha(s)=j\}} \mathrm{d}s\right) \mathbf{I}(u).$$

Consequently,

$$\mathrm{d}\mathbf{Z}(t,u) = \left(i \sum_{j=1}^{m-1} \theta_j \int_t^u \mathbf{1}_{\{\alpha(s)=j\}} \mathrm{d}s\right) \mathbf{Z}(t,u)\mathrm{d}u + \exp\left(i \sum_{j=1}^{m-1} \theta_j \int_t^u \mathbf{1}_{\{\alpha(s)=j\}} \mathrm{d}s\right) \mathrm{d}\mathbf{I}(u). \tag{2.27}$$

We would like to simplify (2.27) by denoting $\Theta = \mathbf{diag}(\theta_1, \theta_2, ..., \theta_{m-1}, 0)$ where $\mathbf{diag}(\theta_1, \theta_2, ..., \theta_{m-1}, 0)$ is the diagonal matrix with diagonal entries $\theta_1, \theta_2, ..., \theta_{m-1}, 0$.

Note

$$\left(i \sum_{j=1}^{m-1} \theta_j \int_t^u \mathbf{1}_{\{\alpha(s)=j\}} \mathrm{d}s\right) \mathbf{Z}(t,u) = i\Theta\mathbf{Z}(t,u). \tag{2.28}$$

Using (2.28) and the martingale property (2.22), we have

$$\mathrm{d}\mathbf{Z}(t,u) = i\Theta\mathbf{Z}(t,u)\mathrm{d}u + Q^{'}\mathbf{Z}(t,u)\mathrm{d}u + \exp\left(i \sum_{j=1}^{m-1} \theta_j \int_t^u \mathbf{1}_{\{\alpha(s)=j\}} \mathrm{d}s\right) \mathrm{d}M(u), \tag{2.29}$$

then

$$\mathrm{d}\mathbf{Z}(t,u) = (i\Theta + Q^{'})\mathbf{Z}(t,u)\mathrm{d}u + \exp\left(i \sum_{j=1}^{m-1} \theta_j \int_t^u \mathbf{1}_{\{\alpha(s)=j\}} \mathrm{d}s\right) \mathrm{d}M(u). \tag{2.30}$$

Consequently,

$$\mathbf{Z}(t,u) = \mathbf{Z}(t,t) + \int_t^u (i\Theta + Q^{'})\mathbf{Z}(t,s)\mathrm{d}s + \int_t^u \exp\left(i\sum_{j=1}^{m-1}\theta_j \int_t^u \mathbf{1}_{\{\alpha(s)=j\}}\mathrm{d}s\right)\mathrm{d}M(s). \tag{2.31}$$

Taking expectations from both sides, we get

$$\mathbb{E}\{\mathbf{Z}(t,u)\} = \mathbf{Z}(t,t) + \int_t^u (i\Theta + Q^{'})\mathbb{E}\{\mathbf{Z}(t,s)\}\mathrm{d}s. \tag{2.32}$$

Note that $\mathbf{Z}(t,t) = \mathbf{I}(t)$, where $\mathbf{I}(t)$ is $m \times m$ identity matrix. By differentiation, we have

$$\mathrm{d}\mathbb{E}\{\mathbf{Z}(t,u)\} = \big(i\Theta + Q^{'}\big)\mathbb{E}\{\mathbf{Z}(t,u)\}\mathrm{d}u, \quad \mathbb{E}\{\mathbf{Z}(t,t)\} = \mathbf{I}(t). \tag{2.33}$$

Hence $\mathbb{E}\{\mathbf{Z}(t,u)\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension m:

$$\frac{\mathbb{E}\{\mathbf{Z}(t,u)\}}{\mathrm{d}u} = \left(i\Theta + Q^{'}\right)\mathbb{E}\{\mathbf{Z}(t,u)\}, \quad \mathbb{E}\{\mathbf{Z}(t,t)\} = \mathbf{I}(t). \tag{2.34}$$

Since $\big(i\Theta + Q^{'}\big)$ is not time dependent, the solution to (2.34) is given by

$$\mathbb{E}\{\mathbf{Z}(t,u)\} = \mathbf{I}(t)\exp\Big(\big(i\Theta + Q^{'}\big)(u-t)\Big). \tag{2.35}$$

Thus for $t = 0$ and $u = T$, we have

$$\mathbb{E}\{\mathbf{Z}(0,T)\} = \mathbf{I}(0)\exp\Big(\big(i\Theta + Q^{'}\big)T\Big). \tag{2.36}$$

Consequently, the characteristic function can be determined by

$$\begin{aligned}
\mathbb{E}\left\{\exp\left(i\sum_{j=1}^{m-1}\theta_j T_j\right)\right\} &= \mathbb{E}\left\{\exp\left(i\sum_{j=1}^{m-1}\theta_j \int_0^T \mathbf{1}_{\{\alpha(t)=j\}}\mathrm{d}t\right)\right\} \\
&= \sum_{j=1}^{m}\mathbb{E}\{z_j(0,T)\} = \mathtt{1}_m^{'}\mathbb{E}\{\mathbf{Z}(0,T)\} \\
&= \mathtt{1}_m^{'}\mathbf{I}(0)\exp\Big(\big(i\Theta + Q^{'}\big)T\Big), \qquad (2.37)
\end{aligned}$$

where $\mathtt{1}_m^{'} \in \mathbb{R}^{m\times 1}$. Setting $\theta_j = A(u,j)$ in (2.37) and then using the result in (2.18), we obtain the Fourier transform $\hat{c}(u)$, which can then be used in the inverse transform to determine the option price.

We closely follows the innovative way proposed by Liu et. al [1] for deriving a simple form for the two-state case. Let the generator of the Markov chain $\alpha(\cdot)$ be given by

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix} \qquad (2.38)$$

where λ_1 is the jump rate from state 1 to state 2 and λ_2 is the jump rate from state 2 to state 1. In this case, we need to find the characteristic function of T_1, the sojourn time in state 1.

Assume the initial state $\alpha(0) = j_0$. Define

$$\phi_{j_0}(\theta,T) = \mathbb{E}\{\mathrm{e}^{i\theta T_1}|\alpha(0) = j_0\}, \quad j_0 = 1,2. \qquad (2.39)$$

Then $\phi_1(\theta,T)$ and $\phi_2(\theta,T)$ satisfy the following system of integral equations (see the Appendix A.1):

$$\begin{aligned}
\phi_1(\theta,T) &= \mathrm{e}^{i\theta T}\mathrm{e}^{-\lambda_1 T} + \int_0^T \mathrm{e}^{i\theta t}\phi_2(\theta,T-t)\lambda_1\mathrm{e}^{-\lambda_1 t}\mathrm{d}t, \\
\phi_2(\theta,T) &= \mathrm{e}^{-\lambda_2 T} + \int_0^T \mathrm{e}^{i\theta t}\phi_1(\theta,T-t)\lambda_2\mathrm{e}^{-\lambda_2 t}\mathrm{d}t. \qquad (2.40)
\end{aligned}$$

In order to find $\phi_1(\theta,T)$ and $\phi_2(\theta,T)$ we are going to use Laplace transforms. Let's denote $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Recall

$$\begin{aligned}
&\mathcal{L}\{\mathrm{e}^{at}\} = \frac{1}{s-a}, \\
&\mathcal{L}\bigg\{\int_0^t f(t)g(t-u)du\bigg\} = F(s) \times G(s).
\end{aligned}$$

Taking Laplace transforms, we obtain the following system of algebraic equations:

$$\begin{aligned}
\mathcal{L}\{\phi_1(\theta,T)\} &= \frac{1}{s+\lambda_1-i\theta} + \frac{\lambda_1}{s+\lambda_1-i\theta}\mathcal{L}\{\phi_2(\theta,T)\}, \\
\mathcal{L}\{\phi_2(\theta,T)\} &= \frac{1}{s+\lambda_2} + \frac{\lambda_2}{s+\lambda_2}\mathcal{L}\{\phi_1(\theta,T)\}.
\end{aligned} \tag{2.41}$$

We now solve the pair of equations:

$$\mathcal{L}\{\phi_1(\theta,T)\} = \frac{1+\lambda_1\mathcal{L}\{\phi_2(\theta,T)\}}{s+\lambda_1-i\theta}, \tag{2.42}$$

$$\mathcal{L}\{\phi_2(\theta,T)\} = \frac{1+\lambda_2\mathcal{L}\{\phi_1(\theta,T)\}}{s+\lambda_2}. \tag{2.43}$$

Substituting (2.43) in (2.42) to find $\mathcal{L}\{\phi_1(\theta,T)\}$ and (2.42) to (2.43) in order to find $\mathcal{L}\{\phi_2(\theta,T)\}$, we have

$$\begin{aligned}
(s+\lambda_1-i\theta)\mathcal{L}\{\phi_2(\theta,T)\} &= (1+\lambda_1)\frac{1+\lambda_2\mathcal{L}\{\phi_2(\theta,T)\}}{s+\lambda_2} \\
&= \frac{s+\lambda_1-i\theta+\lambda_2+\lambda_1\lambda_2\mathcal{L}\{\phi_2(\theta,T)\}}{s+\lambda_1-i\theta},
\end{aligned}$$

$$\begin{aligned}
(s+\lambda_2)\mathcal{L}\{\phi_2(\theta,T)\} &= (1+\lambda_2)\frac{1+\lambda_1\mathcal{L}\{\phi_2(\theta,T)\}}{s+\lambda_1-i\theta} \\
&= \frac{s+\lambda_1-i\theta+\lambda_2+\lambda_1\lambda_2\mathcal{L}\{\phi_2(\theta,T)\}}{s+\lambda_1-i\theta}.
\end{aligned}$$

Solving the above pair of equations for $\mathcal{L}\{\phi_1(\theta,T)\}$ and $\mathcal{L}\{\phi_2(\theta,T)\}$ implies

$$
\begin{aligned}
&(s^2+\lambda_1 s+\lambda_2 s-i\theta s-i\theta\lambda_2)\mathcal{L}\{\phi_1(\theta,T)\}-\lambda_2-\lambda_1-s=0,\\
&(s^2+\lambda_1 s+\lambda_2 s-i\theta s-i\theta\lambda_2)\mathcal{L}\{\phi_2(\theta,T)\}-\lambda_2-\lambda_1-s+i\theta=0.
\end{aligned}
$$

Solving the pair of equations finally yields

$$
\begin{aligned}
\mathcal{L}\{\phi_1(\theta,T)\} &= \frac{s+\lambda_1+\lambda_2}{s^2+(\lambda_1+\lambda_2-i\theta)s-i\theta\lambda_2},\\
\mathcal{L}\{\phi_2(\theta,T)\} &= \frac{s+\lambda_1+\lambda_2-i\theta}{s^2+(\lambda_1+\lambda_2-i\theta)s-i\theta\lambda_2}.
\end{aligned}
\tag{2.44}
$$

Now let's take inverse Laplace transform to find $\mathcal{L}\{\phi_1(\theta,T)\}$ and $\mathcal{L}\{\phi_2(\theta,T)\}$. But first, assume s_1 and s_2 are the two roots of the equation

$$
s^2+(\lambda_1+\lambda_2-i\theta)s-i\theta\lambda_2=0. \tag{2.45}
$$

Hence

$$
\begin{aligned}
\phi_1(\theta,T) &= \frac{s+\lambda_1+\lambda_2}{(s-s_1)(s-s_2)} = \frac{(s+\lambda_1+\lambda_2)(s_1-s_2)}{(s-s_1)(s-s_2)(s_1-s_2)}\\
&= \frac{ss_1-ss_2+(\lambda_1+\lambda_2)s_1-(\lambda_1+\lambda_2)s_2\pm s_1s_2\pm(\lambda_1+\lambda_2)s}{(s-s_1)(s-s_2)(s_1-s_2)}\\
&= \frac{s_1(s-s_2)+(\lambda_1+\lambda_2)(s-s_2)+(s_1-s)s_2+(s_1-s)(\lambda_1+\lambda_2)}{(s-s_1)(s-s_2)(s_1-s_2)}\\
&= \frac{(s_1+\lambda_1+\lambda_2)(s-s_2)-(s-s_1)(s_2+\lambda_1+\lambda_2)}{(s-s_1)(s-s_2)(s_1-s_2)}\\
&= \frac{s_1+\lambda_1+\lambda_2}{(s_1-s_2)}\times\frac{1}{s-s_1}-\frac{s_2+\lambda_1+\lambda_2}{(s_1-s_2)}\times\frac{1}{s-s_2}\\
&= \frac{1}{s_1-s_2}\Big((s_1+\lambda_1+\lambda_2)\mathcal{L}^{-1}\{\mathcal{L}\{\mathrm{e}^{s_1T}\}\}-(s_2+\lambda_1+\lambda_2)\mathcal{L}^{-1}\{\mathcal{L}\{\mathrm{e}^{s_2}T)\}\}\Big),
\end{aligned}
$$

$$
\phi_2(\theta,T) = \frac{s+\lambda_1+\lambda_2-i\theta}{(s-s_1)(s-s_2)} = \frac{(s+\lambda_1+\lambda_2-i\theta)(s_1-s_2)}{(s-s_1)(s-s_2)(s_1-s_2)}
$$

$$
\begin{aligned}
&= \frac{ss_1 - ss_2 + (\lambda_1 + \lambda_2 - i\theta)s_1 - (\lambda_1 + \lambda_2 - i\theta)s_2 \pm s_1 s_2 \pm (\lambda_1 + \lambda_2 - i\theta)s}{(s - s_1)(s - s_2)(s_1 - s_2)} \\
&= \frac{s_1(s - s_2) + (\lambda_1 + \lambda_2 - i\theta)(s - s_2) + (s_1 - s)s_2 + (s_1 - s)(\lambda_1 + \lambda_2 - i\theta)}{(s - s_1)(s - s_2)(s_1 - s_2)} \\
&= \frac{(s_1 + \lambda_1 + \lambda_2 - i\theta)(s - s_2) - (s - s_1)(s_2 + \lambda_1 + \lambda_2 - i\theta)}{(s - s_1)(s - s_2)(s_1 - s_2)} \\
&= \frac{s_1 + \lambda_1 + \lambda_2 - i\theta}{(s_1 - s_2)} \times \frac{1}{s - s_1} - \frac{s_2 + \lambda_1 + \lambda_2 - i\theta}{(s_1 - s_2)} \times \frac{1}{s - s_2} \\
&= \frac{1}{s_1 - s_2}\Bigg((s_1 + \lambda_1 + \lambda_2 - i\theta)\mathcal{L}^{-1}\{\mathcal{L}\{\mathrm{e}^{s_1 T}\}\} \\
&\qquad - (s_2 + \lambda_1 + \lambda_2 - i\theta)\mathcal{L}^{-1}\{\mathcal{L}\{\mathrm{e}^{s_2} T)\}\}\Bigg).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\phi_1(\theta, T) &= \frac{1}{s_1 - s_2}((s_1 + \lambda_1 + \lambda_2)\mathrm{e}^{s_1 T} - (s + \lambda_1 + \lambda_2)\mathrm{e}^{s_2} T), \\
\phi_2(\theta, T) &= \frac{1}{s_1 - s_2}((s_1 + \lambda_1 + \lambda_2 - i\theta)\mathrm{e}^{s_1 T} - (s + \lambda_1 + \lambda_2 - i\theta)\mathrm{e}^{s_2} T). \qquad (2.46)
\end{aligned}
$$

The Fourier transform (2.18) in this case is given by

$$
\hat{c}(u) = \frac{\exp(B(u)T)\phi_{j_0}(A(u), T)}{(\rho - iu)(1 + \rho - iu)}, \qquad (2.47)
$$

where

$$
\begin{aligned}
A(u) = A(u, 1) = &-u\Big[\big(\mu(1) - \mu(2)\big) + \big(\frac{1}{2} + \rho\big)\big(\sigma^2(1) - \sigma^2(2)\big)\Big] \\
&+ \frac{1}{2}u^2\big(\sigma^2(1) - \sigma^2(2)\big)i + \Big[\big(r(1) - r(2)\big) - (1 + \rho)\big(\mu(1) - \mu(2)\big) \\
&- \frac{1}{2}\rho(1 + \rho)\big(\sigma^2(1) - \sigma^2(2)\big)\Big]i,
\end{aligned}
$$

$$
\begin{aligned}
B(u) = &-iu\Big[\mu(2) + (\frac{1}{2} + \rho)\sigma^2(2)\Big] - \frac{1}{2}u^2\sigma^2(2) \\
&+ (1 + \rho)\mu(2) - r(2) + \frac{1}{2}\rho(1 + \rho)\sigma^2(2). \qquad (2.48)
\end{aligned}
$$

2.4 FFT algorithm for option pricing

We adopt the approach introduced by Carr and Madan [3].There are many ways to define the discrete Fourier transform (DFT), varying in the sign of the exponent, normalization, etc. Since we are going to use Python implementation, therfore we are following the same definition for DFT given in Python package numpy. The DFT is defined in Python is

$$A_k = \sum_{m=0}^{n-1} a_m \exp\left\{ -2\pi i \frac{mk}{n} \right\}, \quad k = 0, ..., n-1. \tag{2.49}$$

The inverse DFT is defined as

$$a_m = \frac{1}{n} \sum_{k=0}^{n-1} A_k \exp\left\{ 2\pi i \frac{mk}{n} \right\} \quad m = 0, ..., n-1.$$

It differs from the Fourier transform by the sign of the exponential argument and the default normalization by $\frac{1}{n}$.

Given the transform function $\hat{c}(u)$, the modified option price $c(k)$ can be obtained by the inverse Fourier transform as described in (2.9)

$$C(k) = \frac{\exp(-\rho k)}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\nu k} \hat{c}(u) \mathrm{d}u = \frac{\exp(-\rho k)}{\pi} \int_{0}^{\infty} \mathrm{e}^{iuk} \hat{c}(u) \mathrm{d}u,$$

and the option price is, in view of (2.7), $C(k) = \mathrm{e}^{-\rho k} S_0 c(k)$, for $-\infty < k < \infty$.

Set $u_j = j\triangle_u$, $j = 0, 1, ..., N-1$, where $\triangle u$ is the grid size in the variable u. Then (2.9) can be approximated by the following summation:

$$c(k) \approx \frac{1}{\pi} \sum_{j=0}^{N-1} \mathrm{e}^{iu_j k} \hat{c}(u_j) \triangle_u. \tag{2.50}$$

Next, let $\triangle_k$ be the grid size in k and choose a grid along the modified log strike

k as below:

$$k_l = (l - \frac{N}{2})\triangle_k, \quad l = 0, 1, ..., N-1. \tag{2.51}$$

Then

$$\begin{aligned} c(k_l) &\approx \frac{1}{\pi}\sum_{j=0}^{N-1} \mathrm{e}^{iu_jk_l}\hat{c}(u_j)\triangle_u, \quad l = 0, 1, ..., N-1 \\ &= \frac{1}{\pi}\sum_{j=0}^{N-1} \mathrm{e}^{ij\triangle_u(l-\frac{N}{2})\triangle_k}\hat{c}(u_j)\triangle_u \\ &= \frac{1}{\pi}\sum_{j=0}^{N-1} \mathrm{e}^{ij\triangle_u l\triangle_k}\mathrm{e}^{-ij\triangle_u\frac{N}{2}\triangle_k}\hat{c}(u_j)\triangle_u. \end{aligned} \tag{2.52}$$

If we set

$$\triangle_u\triangle_k = \frac{2\pi}{N}, \tag{2.53}$$

then we have

$$c(k_l) \approx \frac{1}{\pi}\sum_{j=0}^{N-1} \mathrm{e}^{ijl\frac{2\pi}{N}}\mathrm{e}^{-ij\pi}\hat{c}(u_j)\triangle_u, \quad l = 0, 1, ..., N-1. \tag{2.54}$$

Using Simpson's rule for numerical integration, define a sequence of weighting factors by

$$w(j) = \begin{cases} \frac{1}{3}, & \text{if } j = 0, \\ \frac{4}{3}, & \text{if } j \text{ is odd}, \\ \frac{2}{3}, & \text{if } j \text{ is even}. \end{cases}$$

Then

$$c(k_l) \approx \frac{\triangle_u}{\pi N} \sum_{j=0}^{N-1} \mathrm{e}^{ijl\frac{2\pi}{N}} \mathrm{e}^{-ij\pi} \hat{c}(u_j) w(j) N, \quad l = 0, 1, ..., N-1. \tag{2.55}$$

Comparing (2.51) with (2.45), it is easily seen that $\Big\{c(k)\Big\}$ can be obtained by taking the Fourier transform of the sequence $\Big\{\mathrm{e}^{-ij\pi}\hat{c}(u_j)w(j)N\Big\}$, $j = 0, 1, ..., N-1$. Details for the code is provided in Appendix A.2.

2.5 Monte Carlo and Semi-Monte Carlo algorithm for option pricing

Monte Carlo simulations are frequently used when closed-form solutions are not available for complex stochastic problems. A Monte Carlo algorithm frequently serves as a benchmark for the "true value" used for testing other numerical methods. The benchmark value is obtained by running a great number of sample paths in simulating the underlying stochastic dynamics. It is very time consuming and therefore not feasible for most practical use in real time.

We employ Monte Carlo simulations (MCS) to price options in our two-state Markov-modulated model. To price non-path dependent options such as European call options, we only require the final stock price at maturity time S_T, not the stock price trajectory $\{S_t\}_{t\in[0,T]}$. Therefore we need to be creative on how to simulate the stock price trajectory in order to be able to implement MCS. As a first step, we consider simulating the trajectory of the Markov chain $\{\alpha(t)\}_{t\in[0,T]}$, conditional on $\alpha(0) = 1$. Hence we may consider distribution of the first jump from state 1 to state 2. The exponential distribution is often concerned with the amount of time until some specific event occurs. For example, the amount of time (beginning $t = 0$) until Markov chain leaves state 1 has an exponential distribution. Let's call the first time jumping from state 1 to state 2 J, therefore $J \sim \exp(\lambda_1)$ and $\mathrm{P}(J > t) = \mathrm{e}^{-\lambda_1 t}$ for any $t \in [0, T]$. We draw a uniform random variable on interval $[0, 1]$ to simulate the probability $\mathrm{P}(J > t)$. Hence our exponential random variable is $t = \frac{-\ln(p)}{\lambda_1}$. We then calculate the occupation time T_1 of state 1, and use independent increments property of Brownian motion as well as sum of two independent normally distributed random variables is normal, with its mean being the sum of the two means, and its variance being the sum of the two to simply observe that, given $T_1 = \tau_1 \in [0, T]$,

$$\begin{aligned}
\Big(\mathrm{Ln}(S_T)|\mathcal{F}_T\Big) &\approx \mathrm{Ln}S_0 + \int_0^{\tau_1} (\mu_1 - \frac{1}{2}\sigma_1^2)\mathrm{d}s + \int_{\tau_1}^{T} (\mu_2 - \frac{1}{2}\sigma_2^2)\mathrm{d}s \\
&+ \int_0^{\tau_1} \sigma_1 \mathrm{d}W_s + \int_{\tau_1}^{T} \sigma_2 \mathrm{d}W_s \\
&= \mathrm{Ln}S_0 + (\mu_1 - \frac{1}{2}\sigma_1^2)\tau_1 + (\mu_2 - \frac{1}{2}\sigma_2^2)(T - \tau_1) \\
&+ \sigma_1 W(\tau_1) + \sigma_2(W(T) - W(\tau_1)) \\
&\overset{\text{dist}}{\approx} \mathcal{N}\Big(\mathrm{Ln}S_0 + (\mu_1 - \frac{1}{2}\sigma_1^2)\tau_1 + (\mu_2 - \frac{1}{2}\sigma_2^2)(T - \tau_1), \sigma_1^2\tau_1 + \sigma_2^2(T - \tau_1)\Big).
\end{aligned} \tag{2.56}$$

By using the procedure above and (2.56), we only require one pseudo-random sample from the standard normal distribution, which minimizes the required computational time. To price options using the Monte Carlo algorithm above, let N be the number of replications. For $n = 1, ..., N$,

1. Obtain the nth sample path of $\alpha(t)$, $t \in [0, T]$.
2. Find the occupation time spent at given initial state (In our case, state 1. Note than we only need to find occupation time in one state as we only have two states).
3. Use equation (2.56) to simulate the log of terminal of stock price.
4. Calculate $C_n(K)$ for the nth sample path.
5. Calculate the average $C(K) = \dfrac{1}{N} \sum_{n=1}^{N} C_n(K)$.

A Monte Carlo base algorithm is presented by Liu el. at [1] that can be also used as benchmark values in numerical experiments. It's called semi-Monte Carlo simulation algorithm. As noted by Buffington and Elliott [11], for a given realization of the Markov chain $\alpha(\cdot) = \{\alpha(t) : 0 \le t \le T\}$, the European call option price whose underlying asset is governed by regime-switching GBM can be calculated by the usual Black-Scholes formula in which the volatility and the interest rate are replaced by the sample path values. Semi-Monte Carlo simulation approach only takes random sampling of the Markov chain and then takes advantage of the availability of analytical formula of the conditional price. Recall that from Section 2, $\mathcal{F}_T$ denotes the σ-algebra generated by the Markov chain $\alpha(t)$, $0 \le t \le T$. Then the call option price can be calculated by

$$\begin{aligned} C(K) &= \mathbb{E}\left\{\exp\left(-\int_0^T r(\alpha(t))\mathrm{d}t\right)(S(T)-K)^+\right\} \\ &= \mathbb{E}\left\{\mathbb{E}\left\{\mathrm{e}^{-R_T}(S(T)-K)^+ | \mathcal{F}_T\right\}\right\}. \end{aligned}$$

The conditional expectation is given by the Black-Scholes formula, that is,

$$\mathbb{E}\left\{\mathrm{e}^{-R_T}(S(T)-K)^+ | \mathcal{F}_T\right\} = S_0\mathrm{e}^{-(R_T-L_T)}\mathcal{N}\Big(d_1(L_T,V_T)\Big) - K\mathrm{e}^{-R_T}\mathcal{N}\Big(d_2(L_T,V_T)\Big)$$

where

$$d_1(L_T,V_T) = \frac{\ln(\frac{S_0}{K}) + L_T + \frac{1}{2}V_T}{\sqrt{V_T}}, \quad d_2(L_T,V_T) = d_1(L_T,V_T) - \sqrt{V_T}$$

and $\mathcal{N}(\cdot)$is the cumulative standard normal distribution function.

Detail of the code is provided in Appendix A.3. We consider a two-state ($m = 2$) example. When the underlying Markov chain $\alpha(\cdot)$ has only two states, an analytical formula in terms of an integral with respect to the Bessel function is developed by

Guo [9] for the European call option prices. Fuh, Hu, Ho and Wang [13] considered a specific example and compared various methods (binomial tree, Monte Carlo, and an approximation approach presented in their paper) with the analytical results (equation(7)). Here we consider the same example to compare the Monte Carlo algorithm, Semi-MC simulations with the analytical results. Details of the code for MC, analytical results and Semi-MC are provided in Appendix A.3, Appendix A.4 and Appendix A.5 respectively.

2.6 Numerical experiments using MC and analytical method

The parameters used, in this example, are $S_0 = 100$, $K = 90$, $\lambda_1 = \lambda_2 = 1.0$, $\mu_1 = \mu_2 = r_1 = r_2 = 0.1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$. The initial state is $\alpha(0) = 1$ and $N = 100000$ replications are used in the Monte Carlo(MC) simulations. Table 2.1 lists the numerical results for a range of option expiry times. We then change the initial state from 1 to 2 with the same given parameters and the results are provide in Table 2.2.

It is clear from Table 2.1 and Table 2.2 that the Monte Carlo simulation converges to real prices. All the errors show a clear indication of high accuracy. It's worth to mention that Fuh and Wang [13] used 50 000 000 replications in their Monte Carlo simulations to obtain the numbers. We only used 100000 replications (1/50 of theirs) in the Monte Carlo simulations but achieved a much higher degree of accuracy in our Semi-MC simulation.

Table 2.1: Comparison of Analytical prices, MC and Semi-MC Simulations at state $\alpha(0) = 1$

T(year)	Analytical $\alpha(0) = 1$	MC(error) $\alpha(0) = 1$	Semi-MC(error) $\alpha(0) = 1$
0.1	10.999	10.999 (0.000)	10.993 (0.006)
0.2	12.194	12.186 (0.008)	12.165 (0.029)
0.5	15.754	15.671 (0.083)	15.615 (-0.421)
1.0	21.075	20.970 (0.105)	20.723 (-0.648)
2.0	29.943	29.970 (-0.027)	29.289 (0.654)
3.0	37.246	37.422 (-0.176)	36.473 (0.773)

Table 2.2: Comparison of Analytical prices, MC and Semi-MC Simulations at state $\alpha(0) = 2$

T(year)	Analytical $\alpha(0) = 2$	MC(error) $\alpha(0) = 2$	Semi-MC(error) $\alpha(0) = 2$
0.1	11.367	11.370 (-0.003)	11.361 (0.006)
0.2	12.914	12.906 (0.008)	12.889 (0.25)
0.5	16.813	16.730 (0.083)	16.720 (0.93)
1.0	22.003	21.749 (0.254)	21.820 (0.183)
2.0	30.383	29.944 (0.439)	30.087 (0.296)
3.0	37.421	37.135 (0.286)	37.062 (0.359)

2.7 Numerical experiments using FFT

In this section, we report numerical results of using FFT for option pricing developed in this chapter. In implementing the FFT, we choose the number of grid points $N = 4096(2^{12})$. That is, we invoke the FFT procedure to calculate 4096 option prices simultaneously. The grid size along the log strike price k is set to be $\triangle_k = 0.01$.

Consequently, $\triangle_u = 0.1534$ by (2.53). We choose the damping factor ρ to be $\rho = 1.0$. All options considered in the examples have maturity $T = 1$ (year). The initial asset price $S_0 = \$100$. We consider a two-state Markov chain model. The parameters are given by $\lambda_1 = 20.0$, $\lambda_2 = 30.0$, $\mu_1 = r_1 = 0.05$, $\mu_2 = r_2 = 0.1$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.3$. Note that, unlike previous example, in this model, the parameters μ, σ and r all vary with different states. Large jump rates λ_1 and λ_2 are chosen so that the system switches frequently during the life of the options. Table 2.3 and Table 2.4 report the results for seven call options with different strike prices (from deep-in-the-money to at-the-money and to deep-out-of-money) obtained using FFT, MC and Semi-MC simulations. Column one in both tables lists the log strike (the strike) for the options. Columns two, three and four in both tables list the FFT, MC and Semi-MC prices for both $\alpha(0) = 1$ and $\alpha(0) = 2$.

In each case ($\alpha(0) = 1$ and $\alpha(0) = 2$), a single run of FFT algorithm produces 4096 option prices (each one with a different strike price and all other parameters are the same). It took only 0.069 seconds, to run the FFT algorithm. This shows the clear advantage of the FFT.

Table 2.3: Comparison of FFT, MC and Semi-MC simulations at state $\alpha(0) = 1$

$\ln(K/S_0)(K)$	FFT $\alpha(0) = 1$	MC $\alpha(0) = 1$	Semi-MC $\alpha(0) = 1$
-0.3 (74.082)	34.774	34.708	34.773
-0.2 (81.873)	29.696	29.682	29.695
-0.1 (90.484)	24.763	24.766	24.763
0 (100)	20.116	20.095	20.116
0.1 (110.517)	15.881	15.888	15.881
0.2 (122.140)	12.157	12.162	12.155
0.3 (134.986)	9.006	9.019	9.004

Table 2.4: Comparison of FFT, MC and Semi-MC simulations at state $\alpha(0) = 2$

$\ln(K/S_0)(K)$	FFT $\alpha(0) = 2$	MC $\alpha(0) = 2$	Semi-MC $\alpha(0) = 2$
-0.3 (74.082)	34.742	34.661	34.741
-0.2 (81.873)	29.642	29.426	29.642
-0.1 (90.484)	24.688	24.744	24.688
0 (100)	20.022	20.060	20.024
0.1 (110.517)	15.774	15.767	15.772
0.2 (122.140)	12.043	12.086	12.043
0.3 (134.986)	8.893	8.876	8.893

2.8 Concluding remarks

The fast Fourier transform (FFT) has been used for calculating option prices for a wide range of asset price models. In this chapter, we extended the technique to the class of regime-switching diffusion models and developed the FFT scheme. When the number of states of the driving Markov chain in the model is very large, the calculation of the characteristic function involved in the FFT approach becomes computationally intensive. The speedup of FFT along with acceptable accuracy is a promising direction for future research.

In fact, we have illustrated how the calculation of the call price via the Carr-Madan formula can be done fast and accurately using the fast Fourier Transform. Typically, N is a power of 2 (where N is the number of discretization steps). The number of operations of the FFT algorithm is of the order $\mathcal{O}(N\log(N))$ and this is in contrast to the straightforward evaluation of the sums which give rise to $\mathcal{O}(N^2)$ number of operations.

The methodology cannot only be applied when it's not possible to get an explicit form for characteristic function. However, the fact remains that the FFT is the most fast and efficiently method for options price.

CHAPTER 3: FFT APPROACH FOR VALUATION OF COMMODITY AND FUTURES OPTIONS UNDER A REGIME SWITCHING MODEL

In this chapter, we use the fast Fourier transform (FFT) approach to value commodity and futures options price, where the log of the underlying commodity spot price is governed by a regime-switching Ornstein-Uhlenbeck. This Chapter is organized as follows. In the first section, we state our motivation. Section 2 presents the model dynamics. In Section 3, we first derive the valuation of commodity options via inverse Fourier transform approach. Then the value of futures option price is derived. The final section makes concluding remarks.

3.1 Motivitation

In the previous chapter, we described how one can price, very fast and efficiently, European call options where the underlying asset price is governed by a regime-switching geometric Brownian motion using the theory of characteristic functions and the fast Fourier transforms. We have developed a solid understanding of the current frameworks for pricing European call options using FFT, and we have provided the mathematical and practical background necessary to apply and implement the technique. The fast Fourier transform method is particularly interesting in case of advanced equity models, like the future contracts, its stochastic volatility extension, and many other models like the Heston model, where no closed-form solutions for call options exist.

An important advantage of the method is that we only need as input the characteristic function of the dynamics of the underlying model. If one wants to switch to another model, only the corresponding characteristic functions needs to be changed and the actual pricing algorithm remains untouched.

Macroeconomic conditions could have significant impacts on commodity prices. An early work which highlights the link between business cycles and commodity prices was attributed to the paper by Fama and French [14]. From a practical perspective, it is of interest to model and investigate the impacts of structural changes in macroeconomic conditions on commodity prices. The basic idea of regime-switching models is that one set of model parameters is in force at a time depending on the state of the underlying state process at that time. The state process is usually described by a Markov chain.

In this chapter, we consider an observable Markovian regime-switching Ornstein-Uhlenbeck model (MRSOU) for evaluating European-style commodity options and futures options. The main feature of our model is that model parameters, the mean-reverting level and the volatility of the commodity spot price, are governed by an observable continuous-time, finite-state, Markov chain. We then discuss the valuation of commodity options and then the valuation of commodity futures options using inverse Fourier transform. in the final section, we provide concluding remarks. All proofs in this chapter are standard and involve the use of standard mathematical techniques.

3.2 Model dynamics

We consider a continuous-time economy with a finite time horizon $\tau = [0, T^*]$, where $T^* < \infty$. Uncertainties in the economy are described by a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is the real-world probability measure. Let $\mathbf{X} = \{X(t) | t \in \tau\}$ be an observable continuous-time, finite-state Markov chain with state space $\varepsilon = \{e_1, e_2, ..., e_N\} \subset \mathbb{R}^N$, where e_i is the unit vector in $\mathbb{R}^N$ with one in the i-th position and zero elsewhere. In particular, there could be just two states for $\mathbf{X}$, write $X(t) =$

$(1,0)^T$ and $X(t) = (0,1)^T$ for any $t \in \tau$, where $(1,0)^T$ denotes the transpose of $(1,0)$. State 1 and State 2 represent a "good" economy and a "bad" one, respectively. From Elliott et al.[15] in appendix B, a semi-martingale representation for the chain is given by

$$X(t) = X(0) + \int_0^t A\,X(u)\mathrm{d}u + M(t), \quad t \in \tau \tag{3.1}$$

where $A := [a_{ij}]_{N\times N}$ is a constant rate matrix of the chain and $\{M(t)|t \in \tau\}$ is a martingale under $\mathcal{P}$. The element a_{ij} in A is the constant intensity of the transition of the chain $\mathbf{X}$ from State e_j to State e_i.

We consider commodities that are not investment assets, such as agricultural products, oil or metals. Evidence from futures prices highlights that the spot prices of these commodities follow mean reverting processes. Hence the main model feature is mean reversion in commodity prices, indeed mean reversion toward a Regime-Switching mean price level. We define the commodity spot price as $S = e^x$. The process for x is assumed to be

$$\mathrm{d}x(t) = \beta\Big(\theta(t) - x(t)\Big)\mathrm{d}t + \sigma(t)\mathrm{d}W(t) \tag{3.2}$$

where $W(t)$ is a Wiener process under the risk-neutral probability measure. $\theta(t)$ is the long term mean level. All future trajectories of $x(t)$ will evolve around a mean level $\theta(t)$ in the long run; and let β be the parameter controlling the speed of mean reversion for the logarithmic commodity price process, where $\beta > 0$. The instantaneous volatility of the model is $\sigma(t)$. It measures instant by instant the amplitude of randomness entering the system.

We assume that $\{\theta(t)|t \in \tau\}$ changes over time according to the state process of

$\{X(t)|t \in \tau\}$ as follows:

$$\theta(t) = \langle \theta, X(t) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the scalar product $\mathbb{R}^N$. Here $\theta = (\theta_1, \theta_2, ..., \theta_N)^T \in \mathbb{R}^N$ with $\theta_i > 0$, for each $i = 1, 2, ..., N$. In particular, θ_i is the mean-reversion level of the commodity process corresponding to the i-th state of the economic condition.

Define $\{\sigma(t)|t \in \tau\}$ as the volatility of the commodity price. Again we suppose that this volatility changes over time according to the state process of the economy as follows:

$$\sigma(t) = \langle \sigma, X(t) \rangle$$

where $\sigma = (\sigma_1, \sigma_2, ..., \sigma_N) \in \mathbb{R}^N$ with $\sigma_i > 0$.

The solution to the stochastic differential equation (3.2) is given by the following equation:

$$x(T) = e^{-\beta(T-t)}x(t) + \beta \int_t^T \theta(u)e^{-\beta(T-u)}\mathrm{d}u + \int_t^T \sigma(u)e^{-\beta(T-u)}\mathrm{d}W(u) \tag{3.3}$$

3.3 Valuation of commodity futures and options

We now present an observable Markovian Regime-Switching Ornstein-Uhlenbeck model proposed by Schwartz [16] for a commodity futures pricing. In the proposed model by Schwartz [16] θ and σ are constant, which seems restrictive. Our goal in this section is to evaluate the prices of commodity options and futures options at time 0, denoted by $C(0,T)$ and $C_f(0,T,U)$, respectively. The valuation of the two products at an arbitrary time $t \in [0,T]$, where $T < T^*$ can be conducted in a similar fashion. The mathematical results presented in this section are similar to those in, for example, Fan et al. [17], Section 4 therein. In Fan et al. [17], the FFT approach was used to value European call options in a Markovian regime-switching stochastic interest rate environment. Hence, the expressions of the characteristic functions and the option prices presented here are not exactly the same as those in Fan et al. [17].

Under the risk-neutral probability measure $\mathcal{P}$, the prices of a T-maturity futures contract and a T-maturity European-style commodity option at time 0 are given as follows:

$$F(0,T) = \mathbb{E}\{S(T)\} \tag{3.4}$$

and

$$C(0,T) = \mathbb{E}\Big\{\mathrm{e}^{-rT}\Big(S(T) - K\Big)^+\Big\}, \tag{3.5}$$

where $S(T)$ is the terminal commodity price; K is the strike price of the commodity option; $\mathbb{E}$ is the expectation with respect to the risk-neutral probability measure $\mathcal{P}$. Consider a European-style futures option with a strike price K_f, the terminal payoff function at the maturity time T of the option is $(F(T,U) - K_f)^+$, where $F(T,U)$ represents the futures price with maturity time U at time T. Then the price of the futures option at time 0 is given by

$$C_f(0,T,U) = \mathbb{E}\Big\{\mathrm{e}^{-rT}\Big(F(T,U) - K_f\Big)^+\Big\}. \tag{3.6}$$

3.3.1 Valuation of commodity options

Following the notation in the previous chapter, write $\kappa = \ln(K)$, the dampened commodity option price is given by

$$c(\kappa) = e^{\alpha\kappa}C(O,T) \tag{3.7}$$

where α is called the dampening coefficient and assumed to be positive. To obtain a square integrable function, the dampening coefficient α is selected and the dampened commodity pricing formula is defined. We derive an explicit formula for the Fourier transform of $c(\kappa)$ next. Let $\mathcal{F}_T$ be the σ-algebra generated by $\{X(t), t \in \tau\}$, that is, $\mathcal{F}_T = \sigma\{\mathcal{F}^X(t), t \in \tau\}$. Let $f_{\mathcal{F}_T}(x)$ be the conditional density function of $x(T)$ given $\mathcal{F}_T$. Then, the dampened commodity Fourier transform is given by

$$\hat{c}(u) = \frac{\mathrm{e}^{-rT}\, \mathbb{E}\Big\{ \phi_{\mathcal{F}_T}(-i(1+\alpha) - u) \Big\}}{(\alpha - iu)(1 + \alpha - iu)} \tag{3.8}$$

where

$$\phi_{\mathcal{F}_T}(\nu) = \mathbb{E}\{\mathrm{e}^{i\nu x(T)} | \mathcal{F}_T\} = \int_{-\infty}^{\infty} \mathrm{e}^{i\nu x} f_{\mathcal{F}_T}(x) \mathrm{d}x \tag{3.9}$$

is the conditional characteristic function of $x(T)$ given $\mathcal{F}_T$. See chapter 1 section 1.3.2 for details of calculation in (3.8).

Note that given $\mathcal{F}_T$, $x(T)$ has Gaussian distribution with mean

$$\mathbb{E}\{x(T)|\mathcal{F}_T\} = e^{-\beta T} x(0) + \beta \int_0^T \theta(u)\, e^{-\beta(T-u)}\, \mathrm{d}u \tag{3.10}$$

and variance

$$\mathrm{Var}(x(T)|\mathcal{F}_T) = \int_0^T \sigma^2(u)\, e^{-2\beta(T-u)}\, \mathrm{d}u. \tag{3.11}$$

It follows that

$$\begin{aligned} \phi_{\mathcal{F}_T}(\nu) &= \exp\Big(i\nu \mathbb{E}\{x(T)|\mathcal{F}_T\} - \frac{1}{2}\, \nu^2 Var(x(T)|\mathcal{F}_T) \Big) \\ &= \exp\Big(i\nu \big(e^{-\beta T} x(0) + \beta \int_0^T \theta(u) e^{-\beta(T-u)} \mathrm{d}u \big) - \frac{1}{2}\, \nu^2 \int_0^T \sigma^2(u)\, e^{-2\beta(T-u)}\, \mathrm{d}u \Big). \end{aligned}$$

For each $t \in [0, T]$ and $\nu \in \mathbb{R}$, let $G(t, \nu) = \big(g_1(t,\nu), g_2(t,\nu), ..., g_N(t,\nu) \big)$, where

$g_j(t,\nu)$ for each $j = 1, 2, ..., N$ is

$$g_j(t,\nu) = i\nu\,\beta\,\theta_j\,e^{-\beta(T-t)} - \frac{1}{2}\,\nu^2\,\sigma_j^2\,e^{-2\beta(T-t)}.$$

Therefore,

$$\phi_{\mathcal{F}_T}(\nu) = \exp\left(i\nu\,e^{-\beta T}x(0) + \int_0^T \langle G(t,\nu), X(t)\rangle \mathrm{d}t\right). \tag{3.12}$$

Substituting (3.12) in (3.8) implies

$$\hat{c}(u) = \frac{\mathrm{e}^{-rT}\,\mathbb{E}\left\{\exp\left(i\nu\,e^{-\beta T}x(0) + \int_0^T \langle G(t,\nu), X(t)\rangle \mathrm{d}t\right)\right\}}{(\alpha - iu)(1+\alpha - iu)}, \quad \nu = -(u + i(1+\alpha)). \tag{3.13}$$

In order to derive an explicit formula for $\hat{c}(u)$, it's necessary to calculate the expectation given in (3.13). To this end, we use a modification to the proof of Lemma A_1 in Buffington and Elliott [11]. Let $\mathbf{diag}(G(t,\nu))$ denote the diagonal matrix with diagonal elements given by the components of $G(t,\nu)$, $\mathbf{1} = (1, 1, ..., 1)^T$ and $\mathbf{I}$ denote the $(n \times n)$-identity matrix. Let's define

$$W(t,\nu) := \exp\left(\int_0^t \langle G(s,\nu), X(s)\rangle \mathrm{d}s\right) X(t), \quad W(0,\nu) = X(0). \tag{3.14}$$

Consequently,

$$\mathrm{d}W(t,\nu) = \langle G(t,\nu), X(t)\rangle\,W(t,\nu)\,\mathrm{d}t + \exp\left(\int_0^t \langle G(s,\nu), X(s)\rangle \mathrm{d}s\right)\mathrm{d}X(t).$$

Note that under $\mathcal{P}$,

$$\mathrm{d}X(t) = A^T\,X(t)\mathrm{d}t + \mathrm{d}M(t),$$

and that

$$\langle G(t,\nu), X(t)\rangle\, W(t,\nu) = \mathbf{diag}(G(t,\nu))\, W(t,\nu), \quad \forall t \in \tau.$$

Then

$$\begin{aligned}
\mathrm{d}W(t,\nu) &= \langle G(t,\nu), X(t)\rangle\, W(t,\nu)\,\mathrm{d}t + \exp\Big(\int_0^t \langle G(s,\nu), X(s)\rangle \mathrm{d}s\Big)\, A^T X(t)\mathrm{d}t \\
&\qquad + \exp\left(\int_0^t \langle G(s,\nu), X(s)\rangle\, \mathrm{d}s\right) \mathrm{d}M(t) \\
&= \mathbf{diag}(G(t,\nu))\, W(t,\nu)\,\mathrm{d}t + A^T\, W(t,\nu)\,\mathrm{d}t \\
&\qquad + \exp\left(\int_0^t \langle G(s,\nu), X(s)\rangle\, \mathrm{d}s\right) \mathrm{d}M(t) \\
&= \left[\mathbf{diag}(G(t,\nu)) + A^T\right] W(t,\nu)\,\mathrm{d}t + \exp\left(\int_0^t \langle G(s,\nu), X(s)\rangle\, \mathrm{d}s\right) \mathrm{d}M(t).
\end{aligned}$$

By taking integration from both sides, we have

$$\begin{aligned}
W(t,\nu) &= W(0,\nu) + \int_0^t \left[\mathbf{diag}(G(s,\nu)) + A^T\right] W(s,\nu)\,\mathrm{d}s \\
&\qquad + \int_0^t \exp\left(\int_0^s \langle G(s,\nu), X(s)\rangle\, \mathrm{d}s\right) \mathrm{d}M(s) \\
&= X(0) + \int_0^t \left[\mathbf{diag}(G(s,\nu)) + A^T\right] W(s,\nu)\,\mathrm{d}s \\
&\qquad + \int_0^t \exp\left(\int_0^s \langle G(s,\nu), X(s)\rangle\, \mathrm{d}s\right) \mathrm{d}M(s).
\end{aligned}$$

Taking expectation under $\mathcal{P}$ gives:

$$\mathbb{E}\{W(t,\nu)\} = X(0) + \int_0^t \left[\mathbf{diag}(G(s,\nu)) + A^T\right] \mathbb{E}\{W(s,\nu)\}\,\mathrm{d}s.$$

Hence $\mathbb{E}\{W(t,\nu)\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N:

$$\frac{\mathrm{d}\mathbb{E}\{W(t,\nu)\}}{\mathrm{d}t} = \left[\mathbf{diag}(G(t,\nu)) + A^T\right] \mathbb{E}\{W(t,\nu)\}, \quad \mathbb{E}\{W(0,\nu)\} = X(0). \tag{3.15}$$

Suppose $\Phi(t,\nu)$ denotes the fundamental matrix solution of

$$\frac{\mathrm{d}\Phi(t,\nu)}{\mathrm{d}t} = \left[\mathbf{diag}(G(t,\nu)) + A^T\right]\Phi(t,\nu), \quad \Phi(0,\nu) = \mathbf{I}.$$

If $\left[\mathbf{diag}(G(t,\nu)) + A^T\right] = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Phi(t,\nu)$ is

$$\Phi(t,\nu) = \exp(\Delta\, t).$$

In general, there exists a unique fundamental matrix solution $\Phi(t,\nu)$ of the linear matrix differential Eq. (3.15). Now, $\mathbb{E}\{W(t,\nu)\}$ can be represented in terms of the fundamental matrix solution $\Phi(t,\nu)$ as below:

$$\mathbb{E}\{W(t,\nu)\} = \Phi(t,\nu)\, X(0).$$

Now

$$\begin{aligned}
\mathbb{E}\left\{\exp\left(\int_0^T \langle G(t,\nu), X(t)\rangle \mathrm{d}t\right)\right\} &= \mathbb{E}\left\{\exp\left(\int_0^T \langle G(t,\nu), X(t)\rangle \mathrm{d}t\right)\langle X(T), \mathbf{1}\rangle\right\} \\
&= \mathbb{E}\left\{\left\langle \exp\left(\int_0^T \langle G(t,\nu), X(t)\rangle \mathrm{d}t\right) X(T), \mathbf{1}\right\rangle\right\} \\
&= \left\langle \mathbb{E}\left\{\exp\left(\int_0^T \langle G(t,\nu), X(t)\rangle \mathrm{d}t\right) X(T)\right\}, \mathbf{1}\right\rangle \\
&= \left\langle \mathbb{E}\{W(T,\nu)\}, \mathbf{1}\right\rangle \\
&= \left\langle \Phi(T,\nu)\, X(0), \mathbf{1}\right\rangle.
\end{aligned}$$

Therefore under Markovian regime-switching Ornstein-Uhlenbeck model, the price of the commodity option is given by:

$$\hat{c}(u) = \frac{\exp\left(-rT + i\nu\, e^{-\beta T} x(0)\right)\left\langle \Phi(T,\nu)\, X(0), \mathbf{1}\right\rangle}{(\alpha - iu)(1+\alpha - iu)}, \quad \nu = -(u + i(1+\alpha)). \quad (3.16)$$

3.3.2 Valuation of futures options

In this subsection we consider the valuation of commodity futures options. We wish to evaluate the time-zero value of a standard European call option on the future price $F(T,U)$ with strike price K_f and maturity at time T. That is to evaluate:

$$C_f(0,T,U) = \mathbb{E}\left\{ \mathrm{e}^{-rT} \left(F(T,U) - K_f \right)^+ \right\}.$$

As seen before, the dampened commodity futures options price is given by:

$$c_f(\kappa_f) = \mathrm{e}^{\alpha_f \kappa_f} C_f(0,T,U), \tag{3.17}$$

where $\kappa_f = \ln(K_f)$. Let's define $Y(t) := \ln\big(F(t,U)\big)$ for each $t \in \tau$. Now we derive an explit formula for the Fourier transform of the dampened commodity futures options.

$$
\begin{aligned}
\hat{c_f}(u) &= \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa_f}\, c(\kappa_f)\,\mathrm{d}\kappa_f \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa_f}\, \mathrm{e}^{\alpha_f\kappa_f}\, \mathbb{E}\Big\{\mathrm{e}^{-rT}(\mathrm{e}^{Y(T)} - \mathrm{e}^{\kappa_f})^+\Big\}\,\mathrm{d}\kappa_f \\
&= \mathbb{E}\Bigg\{\int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa_f}\, \mathrm{e}^{\alpha_f\kappa_f}\, \mathbb{E}\Big\{\mathrm{e}^{-rT}\,(\mathrm{e}^{Y(T)} - \mathrm{e}^{\kappa_f})^+\Big|\mathcal{F}_T\Big\}\,\mathrm{d}\kappa_f\Bigg\} \\
&= \mathbb{E}\Bigg\{\int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa_f}\, \mathrm{e}^{\alpha_f\kappa_f}\, \mathrm{e}^{-rT} \int_{\kappa_f}^{\infty} \big(\mathrm{e}^{y} - \mathrm{e}^{\kappa_f}\big)\, f_{\mathcal{F}_T}(y)\,\mathrm{d}y\,\mathrm{d}\kappa_f\Bigg\} \\
&= \mathrm{e}^{-rT}\,\mathbb{E}\Bigg\{\int_{-\infty}^{\infty} f_{\mathcal{F}_T}(y) \int_{-\infty}^{y} \big(\mathrm{e}^{y}\,\mathrm{e}^{(\alpha_f - iu)\kappa_f} - \mathrm{e}^{(1+\alpha_f - iu)\kappa_f}\big)\,\mathrm{d}\kappa_f\,\mathrm{d}y\Bigg\} \\
&= \mathrm{e}^{-rT}\,\mathbb{E}\Bigg\{\int_{-\infty}^{\infty} f_{\mathcal{F}_T}(y)\left(\frac{\mathrm{e}^{(1+\alpha_f - iu)y}}{(\alpha_f - iu)} - \frac{\mathrm{e}^{(1+\alpha_f - iu)y}}{(1+\alpha_f - iu)}\right)\mathrm{d}y\Bigg\} \\
&= \mathrm{e}^{-rT}\,\mathbb{E}\Bigg\{\left(\frac{\psi_{\mathcal{F}_T}(-i(1+\alpha_f) - u)}{(\alpha_f - iu)} - \frac{\psi_{\mathcal{F}_T}(-i(1+\alpha_f) - u)}{(1+\alpha_f - iu)}\right)\Bigg\} \\
&= \frac{\mathrm{e}^{-rT}\,\mathbb{E}\Big\{\psi_{\mathcal{F}_T}(-i(1+\alpha_f) - u)\Big\}}{(\alpha_f - iu)(1+\alpha_f - iu)}
\end{aligned}
\tag{3.18}
$$

where

$$
\psi_{\mathcal{F}_T}(\nu) = \mathbb{E}\{\mathrm{e}^{i\nu Y(T)}|\mathcal{F}_T\} = \int_{-\infty}^{\infty} \mathrm{e}^{i\nu y} f_{\mathcal{F}_T}(y)\mathrm{d}y \tag{3.19}
$$

is the conditional characteristic function of $Y(T)$ given $\mathcal{F}_T$. In order to derive an explicit formula for $\hat{c_f}(u)$, we need to derive an analytical formula of the characteristic function of the logarithmic commodity futures price, in other words $\ln\big(F(T,U)\big)$.

To do so, first we need to derive the time-t price of a T-maturity futures contract. In other words,

$$
F(t,T) = \mathbb{E}\{S(T)|\mathcal{F}_t\} = \mathbb{E}\big\{\mathbb{E}\{S(T)|\mathcal{F}_T\}|\mathcal{F}_t\big\}
$$

Since $S(T) = \mathrm{e}^{x(T)}$ and given the initial condition at time T, $x(T)$ has a Gaussian distribution with mean

$$\mathbb{E}\{x(T)|\mathcal{F}_T\} = e^{-\beta(T-t)}x(t) + \beta \int_t^T \theta(u)\, e^{-\beta(T-u)}\, \mathrm{d}u, \tag{3.20}$$

and variance

$$\mathrm{Var}\big(x(T)|\mathcal{F}_T\big) = \int_t^T \sigma^2(u)\, e^{-2\beta(T-u)}\, \mathrm{d}u, \tag{3.21}$$

then, it's easy to see that

$$\begin{aligned}
\mathbb{E}\{S(T)|\mathcal{F}_T\} &= \exp\bigg(\mathbb{E}\{x(T)|\mathcal{F}_T\} + \frac{1}{2}\,\mathrm{Var}\big(x(T)|\mathcal{F}_T\big)\bigg) \\
&= \exp\bigg(e^{-\beta(T-t)}x(t) + \beta \int_t^T \theta(u)\, e^{-\beta(T-u)}\, \mathrm{d}u \\
&\qquad + \frac{1}{2}\int_t^T \sigma^2(u)\, e^{-2\beta(T-u)}\, \mathrm{d}u\bigg).
\end{aligned}$$

Then the time-t price of a T-maturity futures contract is given by

$$\begin{aligned}
F(t,T) &= \mathbb{E}\{S(T)|\mathcal{F}_t\} = \mathbb{E}\big\{\mathbb{E}\{S(T)|\mathcal{F}_T\}|\mathcal{F}_t\big\} \\
&= \mathbb{E}\bigg\{\exp\bigg(e^{-\beta(T-t)}x(t) + \beta \int_t^T \theta(u)\, e^{-\beta(T-u)}\, \mathrm{d}u \\
&\qquad + \frac{1}{2}\int_t^T \sigma^2(u)\, e^{-2\beta(T-u)}\, \mathrm{d}u\bigg)\bigg|\mathcal{F}_t\bigg\}.
\end{aligned}$$

Let's define $N(s) = \big(n_1(s), n_2(s), ..., n_N(s)\big)$ for each $s \in [0,T]$, where for each $j = 1, 2, ..., N$,

$$n_j(s) = \beta\, \theta_j\, e^{-\beta(T-s)} + \frac{1}{2}\,\sigma_j^2\, e^{-2\beta(T-s)}.$$

Therefore

$$F(t,T) = \mathbb{E}\Big\{\exp\Big(e^{-\beta(T-t)}x(t) + \int_t^T \langle N(s), X(s)\rangle \,\mathrm{d}s\Big)\Big|\mathcal{F}_t\Big\}. \tag{3.22}$$

In order to derive an explicit formula for $F(t,T)$, It's necessary to calculate the expectation given in (3.22). To do this end, we adopt the same methodology used in previous subsection. Let $\mathbf{diag}(N(t))$ denote the diagonal matrix with diagonal elements given by the components of $N(t)$. Let's define

$$H(t,u) = \exp\Big(\int_t^u \langle N(s), X(s)\rangle \mathrm{d}s\Big)X(u), \quad H(t,t) = X(t).$$

Consequently,

$$\mathrm{d}H(t,u) = \langle N(u), X(u)\rangle\, H(t,u)\,\mathrm{d}u + \exp\Big(\int_t^u \langle N(s), X(s)\rangle \mathrm{d}s\Big)\mathrm{d}X(u).$$

Note that under $\mathcal{P}$,

$$\mathrm{d}X(u) = A^T\, X(u)\mathrm{d}u + \mathrm{d}M(u),$$

and that

$$\langle N(u), X(u)\rangle\, H(t,u) = \mathbf{diag}(N(u))\, H(t,u), \quad \forall t \in \tau.$$

Then

$$
\begin{aligned}
\mathrm{d}H(t,u) &= \langle N(u), X(u)\rangle\, H(t,u)\,\mathrm{d}u + \exp\biggl(\int_t^u \langle N(s), X(s)\rangle \mathrm{d}s\biggr)\, A^T\, X(u)\mathrm{d}u \\
&\qquad + \exp\biggl(\int_t^u \langle N(s), X(s)\rangle\,\mathrm{d}s\biggr)\,\mathrm{d}M(u) \\
&= \mathbf{diag}(N(u))\, H(t,u)\,\mathrm{d}u + \ A^T\, H(t,u)\,\mathrm{d}u \\
&\qquad + \ \exp\biggl(\int_t^u \langle N(s), X(s)\rangle\,\mathrm{d}s\biggr)\,\mathrm{d}M(u) \\
&= \Bigl[\mathbf{diag}(N(u)) + A^T\Bigr]\, H(t,u)\,\mathrm{d}u + \exp\biggl(\int_t^u \langle N(s), X(s)\rangle\,\mathrm{d}s\biggr)\mathrm{d}M(u).
\end{aligned}
$$

By taking integration from both sides, we have

$$
\begin{aligned}
H(t,u) &= H(t,t) + \int_t^u \ \Bigl[\mathbf{diag}(N(s)) + \ A^T\Bigr]\, H(t,s)\,\mathrm{d}s \\
&\quad + \int_t^u \ \exp\biggl(\int_t^u \langle N(s), X(s)\rangle\,\mathrm{d}s\biggr)\,\mathrm{d}M(s) \\
&= X(t) + \int_t^u \ \Bigl[\mathbf{diag}(N(s)) + \ A^T\Bigr]\, H(t,s)\,\mathrm{d}s \\
&\quad + \int_t^u \ \exp\biggl(\int_t^u \langle N(s), X(s)\rangle\,\mathrm{d}s\biggr)\,\mathrm{d}M(s).
\end{aligned}
$$

Taking expectation under $\mathcal{P}$ given $\mathcal{F}_t$ gives:

$$
\mathbb{E}\{H(t,u)|\mathcal{F}_t\} = X(t) + \int_t^u \ \bigl[\mathbf{diag}(N(s)) + \ A^T\,\bigr]\, \mathbb{E}\{H(t,s)|\mathcal{F}_t\}\,\mathrm{d}s.
$$

Hence $\mathbb{E}\{H(t,u)|\mathcal{F}_t\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N:

$$
\frac{\mathrm{d}\mathbb{E}\{H(t,u)|\mathcal{F}_t\}}{\mathrm{d}u} = \bigl[\mathbf{diag}(N(u)) + \ A^T\,\bigr]\, \mathbb{E}\{H(t,u)|\mathcal{F}_t\}, \quad \mathbb{E}\{H(t,t)|\mathcal{F}_t\} = X(t).
\tag{3.23}
$$

Suppose $\Psi(t,u)$ denotes the fundamental matrix solution of

$$\frac{\mathrm{d}\Psi(t,u)}{\mathrm{d}u} = \left[\mathbf{diag}(N(u)) + A^T\right]\Psi(t,u), \qquad \Psi(t,t) = \mathbf{I}.$$

If $\left[\mathbf{diag}(N(u)) + A^T\right] = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Psi(t,u)$ is

$$\Psi(t,u) = \exp(\Delta\,(u-t)).$$

In general, there exists a unique fundamental matrix solution $\Psi(t,u)$ of the linear matrix differential Eq. (3.23). Now, $\mathbb{E}\{H(t,u)|\mathcal{F}_t\}$ can be represented in terms of the fundamental matrix solution $\Psi(t,u)$ as below:

$$\mathbb{E}\{H(t,u)|\mathcal{F}_t\} = \Psi(t,u)\,X(t).$$

Now

$$\begin{aligned}
F(t,T) &= \mathbb{E}\left\{\exp\left(\mathrm{e}^{-\beta(T-t)}x(t) + \int_t^T \langle N(u), X(u)\rangle\,\mathrm{d}u\right)\middle|\mathcal{F}_t\right\}\\
&= \exp\left(\mathrm{e}^{-\beta(T-t)}x(t)\right)\mathbb{E}\left\{\exp\left(\int_0^T \langle N(t), X(t)\rangle\,\mathrm{d}u\right)\middle|\mathcal{F}_t\right\}\\
&= \exp\left(\mathrm{e}^{-\beta(T-t)}x(t)\right)\mathbb{E}\left\{\exp\left(\int_t^T \langle N(u), X(u)\rangle \mathrm{d}u\right)\langle X(T), \mathbf{1}\rangle\middle|\mathcal{F}_t\right\}\\
&= \exp\left(\mathrm{e}^{-\beta(T-t)}x(t)\right)\mathbb{E}\left\{\left\langle\exp\left(\int_t^T \langle N(u), X(u)\rangle \mathrm{d}u\right)X(T), \mathbf{1}\right\rangle\middle|\mathcal{F}_t\right\}\\
&= \exp\left(e^{-\beta(T-t)}x(t)\right)\left\langle\mathbb{E}\left\{\exp\left(\int_t^T \langle N(u), X(u)\rangle \mathrm{d}u\right)X(T)\middle|\mathcal{F}_t\right\}, \mathbf{1}\right\rangle\\
&= \exp\left(\mathrm{e}^{-\beta(T-t)}x(t)\right)\left\langle\mathbb{E}\{H(t,T)|\mathcal{F}_t\}, \mathbf{1}\right\rangle\\
&= \exp\left(\mathrm{e}^{-\beta(T-t)}x(t)\right)\left\langle\Psi(t,T)\,X(t), \mathbf{1}\right\rangle.
\end{aligned}$$

Then under the risk-neutral probability measure $\mathcal{P}$, the time-T price of a U-

maturity futures contract is given by:

$$
\begin{aligned}
F(T,U) = \mathbb{E}\{S(U)|\mathcal{F}_T\} &= \exp\Big(\mathrm{e}^{-\beta(U-T)}x(T)\Big) \times \big\langle \Psi(T,U)\,X(T), \mathbf{1}\big\rangle \\
&= \exp\Big(\mathrm{e}^{-\beta(U-T)}\Big[\mathrm{e}^{-\beta(T)}x(0) + \beta\int_0^T \theta(u)\mathrm{e}^{-\beta(T-u)}\mathrm{d}u \\
&\qquad + \int_0^T \sigma(u)\mathrm{e}^{-\beta(T-u)}\mathrm{d}W(u)\Big]\Big) \times \big\langle \Psi(T,U)\,X(T), \mathbf{1}\big\rangle \\
&= \exp\Big(\mathrm{e}^{-\beta U}\,x(0) + \mathrm{e}^{-\beta U}\,\beta\int_0^T \theta(u)\mathrm{e}^{\beta\,u}\mathrm{d}u \\
&\qquad + \mathrm{e}^{-\beta U}\int_0^T \sigma(u)\mathrm{e}^{\beta\,u}\mathrm{d}W(u)\Big) \times \big\langle \Psi(T,U)\,X(T), \mathbf{1}\big\rangle.
\end{aligned}
$$

where $T < U < T^*$.

Note that given $\mathcal{F}_T$, $Y(T) = \ln(F(T,U))$ has Gaussian distribution with mean

$$
\mathbb{E}\{Y(T)|\mathcal{F}_T\} = \mathrm{e}^{-\beta\,U}x(0) + \beta\int_0^T \theta(u)\,\mathrm{e}^{-\beta(U-u)}\,\mathrm{d}u + \ln\langle\Psi(T,U)\,X(T), \mathbf{1}\rangle \tag{3.24}
$$

and variance

$$
\mathrm{Var}(Y(T))|\mathcal{F}_T) = \int_0^T \sigma^2(u)\,e^{-2\beta(U-u)}\,\mathrm{d}u. \tag{3.25}
$$

It follows from (3.24) and (3.25) that

$$
\begin{aligned}
\psi_{\mathcal{F}_T}(\nu) &= \exp\Big(i\,\nu\,\mathbb{E}\{Y(T)|\mathcal{F}_T\} - \frac{1}{2}\,\nu^2\,\mathrm{Var}(Y(T)|\mathcal{F}_T)\Big) \\
&= \exp\Big(i\,\nu\,\mathrm{e}^{-\beta\,U}x(0) + \beta\int_0^T \theta(u)\,\mathrm{e}^{-\beta(U-u)}\,\mathrm{d}u\Big) \times \langle\Psi(T,U)\,X(T), \mathbf{1}\rangle^{i\,\nu} \\
&\times \exp\Big(-\frac{1}{2}\,\nu^2\int_0^T \sigma^2(u)\,\mathrm{e}^{-2\beta(U-u)}\,\mathrm{d}u\Big).
\end{aligned} \tag{3.26}
$$

Define $Z(\nu) = (Z_1(\nu), Z_2(\nu), ..., Z_N(\nu))$ and $z(t,\nu) = (z_1(t,\nu), z_2(t,\nu), ..., z_N(t,\nu))$

for each $\nu \in \mathbb{R}$ and $t \in \tau$, where $Z_j(\nu)$ and $z_j(t,\nu)$ for each $j = 1, 2, ..., N$ are

$$Z_j(\nu) = Z(\mathrm{e}_j, \nu) = \exp\Big(i\,\nu\,\mathrm{e}^{-\beta U} x(0)\Big) \times \langle \Psi(T,U)\,\mathrm{e}_j, \mathbf{1}\rangle^{i\,\nu} \tag{3.27}$$

and

$$z_j(t,\nu) = z(t, \mathrm{e}_j, \nu) = i\,\nu\,\beta \mathrm{e}^{-\beta(U-t)}\theta_j - \frac{1}{2}\,\nu^2 \mathrm{e}^{-2\beta(U-t)}\sigma_j^2. \tag{3.28}$$

Therefore,

$$\psi_{\mathcal{F}_T}(\nu) = Z(X(T),\nu)\exp\bigg(\int_0^T \Big\langle z(t, X(t), \nu), X(t)\Big\rangle \mathrm{d}t\bigg). \tag{3.29}$$

Substituting (3.29) in (3.18) implies

$$\hat{c}_f(u) = \frac{\mathrm{e}^{-rT}\,Z(X(T),\nu)\,\mathbb{E}\Big\{\exp\Big(\int_0^T \Big\langle z(t,X(t),\nu), X(t)\Big\rangle \mathrm{d}t\Big)\Big\}}{(\alpha - iu)(1+\alpha-iu)}, \qquad \nu = -(u + i(1+\alpha)). \tag{3.30}$$

As we have seen before, in order to derive an explicit formula for $\hat{c}_f(u)$, It's necessary to calculate the expectation given in (3.30). Let $\mathbf{diag}(z(t, X(t), \nu))$ denote the diagonal matrix with diagonal elements given by the components of $z(t, X(t), \nu)$. Let's define

$$\Gamma(t,\nu) = \exp\bigg(\int_0^t \Big\langle z(s, X(s), \nu), X(s)\Big\rangle \mathrm{d}s\bigg) X(t), \quad \Gamma(0,\nu) = X(0). \tag{3.31}$$

Applying Itô differentiation rule to $\Gamma(t,\nu)$ gives

$$\mathrm{d}\Gamma(t,\nu) = \Big\langle z(t, X(t), \nu), X(t)\Big\rangle\,\Gamma(t,\nu)\,\mathrm{d}t + \exp\bigg(\int_0^t \Big\langle z(s,X(s),\nu), X(s)\Big\rangle \mathrm{d}s\bigg)\mathrm{d}X(t).$$

Note that under $\mathcal{P}$,

$$\mathrm{d}X(t) = A^T X(t)\mathrm{d}t + \mathrm{d}M(t),$$

and that

$$\Big\langle z(t, X(t), \nu), X(t) \Big\rangle \, \Gamma(t, \nu) = \mathbf{diag}(z(t, X(t), \nu)) \, \Gamma(t, \nu), \quad \forall t \in \tau.$$

Then

$$\begin{aligned}
\mathrm{d}\Gamma(t, \nu) &= \Big\langle z(t, X(t), \nu), X(t) \Big\rangle \, \Gamma(t, \nu) \, \mathrm{d}t + \exp\bigg(\int_0^t \Big\langle z(s, X(s), \nu), X(s) \Big\rangle \mathrm{d}s \bigg) \, A^T X(t)\mathrm{d}t \\
&\quad + \exp\bigg(\int_0^t \Big\langle z(s, X(s), \nu), X(s) \Big\rangle \, \mathrm{d}s \bigg) \, \mathrm{d}M(t) \\
&= \mathbf{diag}(z(t, X(t), \nu)) \, \Gamma(t, \nu) \, \mathrm{d}t + A^T \, \Gamma(t, \nu) \, \mathrm{d}t \\
&\quad + \exp\bigg(\int_0^t \Big\langle z(s, X(s), \nu), X(s) \Big\rangle \, \mathrm{d}s \bigg) \, \mathrm{d}M(t) \\
&= \Big[\mathbf{diag}(z(t, X(t), \nu)) + \, A^T \Big] \, \Gamma(t, \nu) \, \mathrm{d}t \, + \, \exp\bigg(\int_0^t \Big\langle z(s, X(s), \nu), X(s) \Big\rangle \, \mathrm{d}s \bigg) \, \mathrm{d}M(t).
\end{aligned}$$

By taking integration from both sides, we have

$$\begin{aligned}
\Gamma(t, \nu) &= \Gamma(0, \nu) + \int_0^t \, \Big[\mathbf{diag}(z(s, X(s), \nu)) + \, A^T \Big] \, \Gamma(s, \nu) \, \mathrm{d}s \\
&\quad + \int_0^t \exp\bigg(\int_0^t \Big\langle z(s, X(s), \nu), X(s) \Big\rangle \, \mathrm{d}s \bigg) \, \mathrm{d}M(s) \\
&= X(0) + \int_0^t \, \Big[\mathbf{diag}(z(s, X(s), \nu)) + \, A^T \Big] \, \Gamma(s, \nu) \, \mathrm{d}s \\
&\quad + \int_0^t \exp\bigg(\int_0^t \Big\langle G(s, \nu), X(s) \Big\rangle \, \mathrm{d}s \bigg) \, \mathrm{d}M(s).
\end{aligned}$$

Taking expectation under $\mathcal{P}$ gives:

$$\mathbb{E}\{\Gamma(t, \nu)\} = X(0) + \int_0^t \, \big[\mathbf{diag}(z(s, X(s), \nu)) + \, A^T \, \big] \, \mathbb{E}\{\Gamma(s, \nu)\} \, \mathrm{d}s.$$

Hence $\mathbb{E}\{\Gamma(t,\nu)\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N:

$$\frac{\mathrm{d}\mathbb{E}\{\Gamma(t,\nu)\}}{\mathrm{d}t} = \left[\mathbf{diag}(z(t,X(t),\nu)) + A^T\right]\mathbb{E}\{\Gamma(t,\nu)\}, \quad \mathbb{E}\{\Gamma(0,\nu)\} = X(0). \quad (3.32)$$

Suppose $\Upsilon(t,\nu)$ denotes the fundamental matrix solution of

$$\frac{\mathrm{d}\Upsilon(t,\nu)}{\mathrm{d}t} = \left[\mathbf{diag}(z(t,X(t),\nu)) + A^T\right]\Upsilon(t,\nu), \qquad \Upsilon(0,\nu) = \mathbf{I}.$$

If $\left[\mathbf{diag}(z(t,X(t),\nu)) + A^T\right] = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Phi(t,\nu)$ is

$$\Upsilon(t,\nu) = \exp(\Delta\, t).$$

In general, there exists a unique fundamental matrix solution $\Upsilon(t,\nu)$ of the linear matrix differential Eq. (3.32). Now, $\mathbb{E}\{\Gamma(t,\nu)\}$ can be represented in terms of the fundamental matrix solution $\Upsilon(t,\nu)$ as below:

$$\mathbb{E}\{\Gamma(t,\nu)\} = \Upsilon(t,\nu)\, X(0).$$

Now

$$\begin{aligned}
\mathbb{E}\left\{\exp\left(\int_0^T \left\langle z(t,X(t),\nu), X(t)\right\rangle \mathrm{d}t\right)\right\} &= \mathbb{E}\left\{\exp\left(\int_0^T \left\langle z(t,X(t),\nu), X(t)\right\rangle \mathrm{d}t\right)\langle X(T), \mathbf{1}\rangle\right\} \\
&= \mathbb{E}\left\{\left\langle \exp\left(\int_0^T \langle z(t,X(t),\nu), X(t)\rangle \mathrm{d}t\right) X(T), \mathbf{1}\right\rangle\right\} \\
&= \left\langle \mathbb{E}\left\{\exp\left(\int_0^T \langle z(t,X(t),\nu), X(t)\rangle \mathrm{d}t\right) X(T)\right\}, \mathbf{1}\right\rangle \\
&= \left\langle \mathbb{E}\{\Gamma(T,\nu)\}, \mathbf{1}\right\rangle \\
&= \left\langle \Upsilon(T,\nu)\, X(0), \mathbf{1}\right\rangle.
\end{aligned}$$

Therefore under Markovian regime-switching Ornstein-Uhlenbeck model, the price of the futures option is given by:

$$\hat{c}_f(u) = \frac{\mathrm{e}^{-rT}\, Z(X(T),\nu) \left\langle \Upsilon(T,\nu)\, X(0), \mathbf{1} \right\rangle}{(\alpha - iu)(1+\alpha-iu)}, \quad \nu = -(u + i(1+\alpha)). \tag{3.33}$$

3.4 Conclusions

In this chapter, We discussed the valuation of European-style call options on commodity spot price and futures price in a Markovian regime-switching Ornstein-Uhlenbeck model. The model parameters were assumed to be modulated by an observable, finite-state Markov chain, whose states represent the states of an economy. The main feature of our study is that the regime-switching effect is emphasized, i.e., the structural changes of macroeconomic conditions could be incorporated in the model. We applied the inverse Fourier transform to evaluate the prices of commodity options and futures options. We my further our work to investigate the valuation of American-style options under Markovian regime-switching Ornstein-Uhlenbeck model, since as we know, most of commodity options traded in NYMEX/CME are American-style options.

CHAPTER 4: FFT APPROACH FOR PRICING COMMODITY AND FUTURES OPTIONS UNDER A REGIME SWITCHING STOCHASTIC INTEREST RATE MODEL

In this chapter, we investigate the pricing of European-style commodity and futures options under a Markovian regime-switching Ornstien-Ohlenbec model with a Markovian regime-switching Hull-White interest rate model. The model parameters, including the mean reversion level, the volatility of the stochastic interest rate, and the volatility of the commodity price process are modulated by an observable, continuous-time, finite-state Markov chain. We employ the concept of stochastic flows to derive an exponential affine form of the price of a zero-coupon bond. Then, we represent the exponential affine form of the bond price in terms of fundamental matrix solutions of linear matrix differential equations. Furthermore, we give the forward measure when taking the zero-coupon bond as the numéraire. Then we adopt similar methodology to find a closed-form expression for the characteristic function of the logarithmic terminal commodity and futures price.

4.1 Motivitation

Option valuation has been an important problem in the theory and practice of financial economics. A major breakthrough in this area was made by Fischer Black, Myron Scholes, and Robert Merton [22]. Despite the practical importance of the Black–Scholes–Merton model, its underlying assumptions, including the constant interest rate and volatility, are not consistent with empirical observations. It is phenomenal that interest rates have become volatile in the past few decades. Many stochastic interest rate models have been introduced in the literature. Some popular short rate models include those proposed by Vasicek [23], Cox et al. [24], Hull and White [25], amongst others. The main feature of these models is that the short rate process, commonly described as a diffusion process, is mean-reverting. This means that the short rate process will eventually revert to a long-term value. This property is a “stylized” fact of the empirical behavior of interest rates.

Structural changes in economic conditions affect stochastic evolution of interest rates over time. Regime-switching models may be used to describe such impacts. This class of models was popularized by Hamilton [26] in financial econometrics. There has been some interest in pricing bonds and related options in Markovian regime-switching stochastic interest rate models. Elliott and Mamon [27] considered a Vasicek model, with the mean-reverting level being modulated by a continuous-time, finite-state Markov chain, while a regime-switching Hull-White model was considered in Elliott and Wilson [28].

Using the concept of stochastic flows, Elliott and Siu [18] discussed a bond valuation problem under a regime-switching Hull–White short rate model and a regime-switching Cox–Ingersoll–Ross model. Siu [19] proposed a general short rate model incorporating jumps of the interest rate due to some extraordinary market events or economic cycles. More specifically, Siu [19] derived a bond pricing formula under a jump-augmented Vasicek model, a kind of jump-diffusion-type short rate models, using techniques in stochastic flows. Shen and Siu [20] employed a partial differential equation approach to derive exponential-affine formulas for a zero-coupon bond and a longevity bond, respectively, while Shen and Siu [21] considered the valuation of a bond option under a regime-switching Hull–White model. Elliott and Siu [29] considered the valuation of bond options in a Markovian regime-switching Heath–Jarrow–Morton (HJM) model and derived semi-analytical formulas for pricing bond options using the Fourier transform space.

4.2 The model dynamics

As described in the previous chapter, we consider a continuous-time economy with a finite time horizon $\mathcal{T}$, i.e., $\mathcal{T} := [0, T]$, where $T < \infty$. Suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space, where $\mathcal{P}$ is a risk-neutral probability measure. Here, we start with a risk-neutral probability as in some literature on stochastic interest rate models. We assume the state of an economy is modeled by a continuous-time, finite-state, observable Markov Chain $\mathbf{X} := \{X(t)|t \in \mathcal{T}\}$. The state space of the chain is denoted by $S := \{s_1, s_2, ..., s_N\}$, representing N different observable states of an economy. Without loss of generality, using the convention in Elliott et al. [15], we identify the state space of the chain with a finite set of standard unit vectors $\varepsilon := \{e_1, e_2, ..., e_N\} \subset \mathbb{R}^N$, where the j-th component of e_i is the Kronecker delta δ_{ij}, for each $i, j = 1, 2, ..., N$. Let $Q := [q_{ij}]_{i,j=1,2,...,N}$ denote the generator or rate matrix of the chain $\mathbf{X}$. Then, Elliott et al. [15] in Lemma 1.1 Appendix B obtained the following semi-martingale dynamics for the chain $\mathbf{X}$:

$$X(t) = X(0) + \int_0^t Q\,X(u)\mathrm{d}u + M(t). \quad t \in \mathcal{T} \tag{4.1}$$

Here $\{M(t)|t \in \mathcal{T}\}$ is an $\mathbb{R}^N$-valued martingale with respect to the filtration generated by $\mathbf{X}$ under the measure $\mathcal{P}$.

We now present the Markovian regime-switching models for the dynamics of the underlying logarithmic commodity spot price and the stochastic interest rate. Let y^T be the transpose of a vector or a matrix y. Denote $\{\alpha(t)|t \in \mathcal{T}\}$ and $\{\gamma(t)|t \in \mathcal{T}\}$ as the mean reversion level and the volatility of the short rate process, respectively. Suppose that

$$\alpha(t) = \langle \alpha, X(t) \rangle$$

and

$$\gamma(t) = \langle \gamma, X(t) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the scalar production $\mathbb{R}^N$. Here $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)^T \in \mathbb{R}^N$ with $\alpha_i > 0$, and $\gamma = (\gamma_1, \gamma_2, ..., \gamma_N)^T \in \mathbb{R}^N$ with $\gamma_i > 0$ for each $i = 1, 2, ..., N$. The mean reversion coefficient η describing the speed of mean reversion is assumed to be a positive constant.

Let $\sigma(t)$ and $\theta(t)$ be the volatility and the mean reversion level of the underlying logaritthmic commodity spot price at time t. Again suppose that

$$\sigma(t) = \langle \sigma, X(t) \rangle$$

and

$$\theta(t) = \langle \theta, X(t) \rangle$$

where $\sigma = (\sigma_1, \sigma_2, ..., \sigma_N)^T \in \mathbb{R}^N$ with $\sigma_i > 0$ and $\theta = (\theta_1, \theta_2, ..., \theta_N)^T \in \mathbb{R}^N$ with $\theta_i > 0$ for each $i = 1, 2, ..., N$. In particular, θ_i is the mean reversion level of the commodity process corresponding to the i-th state of the hidden economic condition for each $i = 1, 2, ..., N$. Let β be the parameter controlling the speed of mean reversion for the logaritthmic commodity price process, where $\beta > 0$.

We define the commodity spot price as $S = \mathrm{e}^x$. Then, we assume that under the risk-neutral probability measure $\mathcal{P}$, the dynamics of the underlying logaritthmic commodity spot price and the short rate are given by

$$\mathrm{d}x(t) = \beta\Big(\theta(t) - x(t)\Big)\mathrm{d}t + \sigma(t)\mathrm{d}W_S(t) \tag{4.2}$$

$$\mathrm{d}r(t) = \eta\Big(\alpha(t) - r(t)\Big)\mathrm{d}t + \gamma(t)\mathrm{d}W_r(t) \tag{4.3}$$

where $W_S := \{W_S(t)|t \in \mathcal{T}\}$ and $W_r := \{W_r(t)|t \in \mathcal{T}\}$ are two standard Brownian motions with respect to their right-continuous, $\mathcal{P}$-complete, natural filtrations under $\mathcal{P}$. Furthermore, we suppose that the two Brownian motions W_S and W_r are correlated, and the instantaneous correlation coefficient $\rho(t)$ at time t is given by

$$\rho(t) = \langle W_S, W_r \rangle = \int_0^t \rho(s) \mathrm{d}s,$$

where $\rho(t) = \langle \rho, \mathbf{X(t)} \rangle$ and $\rho := (\rho_{\mathbf{1}}, \rho_{\mathbf{2}}, ..., \rho_{\mathbf{N}}) \in \mathbb{R}^{\mathbf{N}}$ with $-1 < \rho_i < 1$.

$\{\langle W_S, W_r \rangle(t)|t \in \mathcal{T}\}$ is the (predictable) quadratic covariation between W_S and W_r. Consequently, the correlation coefficient between the spot price and the short rate depends on the state of an economy.

Let $\mathcal{F}^X = \{\mathcal{F}^X(t)|t \in \mathcal{T}\}$, $\mathcal{F}^S = \{\mathcal{F}^S(t)|t \in \mathcal{T}\}$ and $\mathcal{F}^r = \{\mathcal{F}^r(t)|t \in \mathcal{T}\}$ be the natural filtrations generated by $\{\mathbf{X}(t)|t \in \mathcal{T}\}$, $\{S(t)|t \in \mathcal{T}\}$ and $\{r(t)|t \in \mathcal{T}\}$ respectively. As usual, we assume that the filtrations given above are right-continuous and $\mathcal{P}$-complete. Define two enlarged filtrations $\mathcal{G} = \{\mathcal{G}(t)|t \in \mathcal{T}\}$ and $\mathcal{H} = \{\mathcal{H}(t)|t \in \mathcal{T}\}$ by letting

$$\mathcal{G}(t) := \mathcal{F}^r(t) \vee \mathcal{F}^X(t)$$

and

$$\mathcal{H}(t) := \mathcal{F}^r(t) \vee \mathcal{F}^S(t) \vee \mathcal{F}^X(t).$$

Here $\mathcal{A} \vee \mathcal{B}$ represents the minimal σ-field containing both the σ-fields $\mathcal{A}$ and $\mathcal{B}$.

4.3 Bond pricing and the forward measure

In this section, we shall employ the concept of stochastic flows to derive an exponential affine form of the price of a zero-coupon bond. Then, we shall represent the exponential affine form of the bond price in terms of fundamental matrix solutions of linear matrix differential equations. Furthermore, we give the forward measure when taking the zero-coupon bond as the numéraire. To do this end, we adopt the same methodology used in Elliott and Siu [18], Shen and Siu [20 & 21] and Siu [19].

4.3.1 Stochastic flows and bond prices

Let $r_{t,s}(r)$ be a version of the process $r_{t,s}$, $s \geq t$, with initial condition $r_{t,t}(r) = r \in \mathbb{R}$. Then, from (4.3),

$$r_{t,s}(r) = r + \int_t^s \eta\big(\alpha(u) - r_{t,u}(r)\big)\mathrm{d}u + \int_t^s \gamma(u)\mathrm{d}W_r(u) \tag{4.4}$$

Write

$$D_{t,s} = \frac{\partial r_{r,u}(r)}{\partial r}$$

for the derivative of the map $r \longrightarrow r_{t,u}(r)$. Differentiating (4.4) with respect to r gives

$$D_{t,s} = 1 - \eta \int_t^s D_{t,u}\mathrm{d}u,$$

with initial condition $D_{t,t} = 1$.

So,

$$D_{t,s} = \mathrm{e}^{-\eta(s-t)}.$$

Here, $D_{t,s}$ is a deterministic real-valued process.

The price at time $t \in \mathcal{T}$ of any contingent claim $V \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ is

$$P(t) = \mathbb{E}\left[\exp\left(-\int_t^T r_{t,u}(r)\mathrm{d}u\right)V\middle|\mathcal{G}(t)\right]$$

Here, $\mathbb{E}[]$ represents an expectation with respect to the risk-neutral measure $\mathcal{P}$. Letting $V(\omega) = 1$, for each $\omega \in \Omega$, the price of a zero-coupon bond at time t with maturity at time T is:

$$P(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r_{t,u}(r)\mathrm{d}u\right)\middle|\mathcal{G}(t)\right].$$

Since $(r, \mathbf{X})$ is a two-dimensional Markov process with respect to the enlarged filtration $\mathcal{G}(t)$, given that $r(t) = r$ and $X(t) = x$,

$$P(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r_{t,u}(r)\mathrm{d}u\right)\middle|X(t) = x, r(t) = r\right] = P(t,T,r,x).$$

Define $B(t,T)$ as the following path integral:

$$B(t,T) = \int_t^T D_{t,u}\mathrm{d}(u) = \frac{1}{\eta}(1 - \mathrm{e}^{-\eta(T-t)}),$$

so it is a real-valued deterministic process.

Since the exponential is bounded,

$$\begin{aligned}\frac{\partial P(t,T,r,x)}{\partial r} &= \left(-\int_t^T D_{t,u}\mathrm{d}u\right)\mathbb{E}\left[\exp\left(-\int_t^T r_{t,u}(r)\mathrm{d}u\right)\middle|X(t) = x, r(t) = r\right] \\ &= -B(t,T)P(t,T,r,x). \qquad (4.5)\end{aligned}$$

Integrating (4.5) in r gives

$$P(t,T,r,x) = \tilde{A}(t,T,x)\exp(-B(t,T)\,r) = \exp\Big(A(t,T,x) - B(t,T)\,r\Big),$$

where $A(t,T,x) = \ln[\tilde{A}(t,T,x)]$.

Consider the discounted bond price back to time zero:

$$\tilde{P}(t,T,r,x) = \exp\bigg(-\int_0^t r_{0,u}(r_0)\mathrm{d}u\bigg)P(t,T,r,x) = \mathbb{E}\bigg[\exp\bigg(-\int_0^T r_{t,u}(r_0)\mathrm{d}u\bigg)\bigg|\mathcal{G}(t)\bigg].$$

Here, $\tilde{P}(t,T,r,x)$ is a $(\mathcal{G},\mathcal{P})$-martingale.

Write $\tilde{P}_i = \tilde{P}(t,T,r,\mathrm{e}_i)$ for $i = 1,2,...,N$ and $\tilde{\mathbf{P}} = (\tilde{P}_1,\tilde{P}_2,...,\tilde{P}_N) \in \mathbb{R}^N$. Applying Itô's differentiation rule to $\tilde{P}(t,T,r,x)$

$$\begin{aligned}
\tilde{P}(t,T,r,x) &= \tilde{P}(0,T,r_0,x_0) + \int_0^t \frac{\partial \tilde{P}}{\partial u}\,\mathrm{d}u + \int_0^t \frac{\partial \tilde{P}}{\partial r}\,\eta\left(\alpha(u) - r(u-)\right)\mathrm{d}u \\
&\quad + \int_0^t \frac{\partial \tilde{P}}{\partial r}\,\gamma(u)\,\mathrm{d}W_r(u) + \int_0^t \big\langle \tilde{\mathbf{P}}, Q\,X(u)\big\rangle \mathrm{d}u + \int_0^t \big\langle \tilde{\mathbf{P}}, \mathrm{d}M(u)\big\rangle \\
&= \tilde{P}(0,T,r_0,x_0) + \int_0^t \frac{\partial \tilde{P}}{\partial r}\,\gamma(u)\,\mathrm{d}W_r(u) + \int_0^t \big\langle \tilde{\mathbf{P}}, \mathrm{d}M(u)\big\rangle \\
&\quad + \int_0^t \bigg\{\frac{\partial \tilde{P}}{\partial u} + \frac{\partial \tilde{P}}{\partial r}\,\eta\big(\alpha(u) - r(u-)\big) + \langle \tilde{\mathbf{P}}, Q\,X(u)\rangle\bigg\}\mathrm{d}u
\end{aligned}$$

Note that $\tilde{P}(t,T,r,x)$ is a $(\mathcal{G},\mathcal{P})$-martingale. So, the bounded variation terms, which are not martingales, in the above stochastic integral representation for $\tilde{P}(t,T,r,x)$ must sum to zero. Therefore,

$$\frac{\partial \tilde{P}}{\partial t} + \frac{\partial \tilde{P}}{\partial r}\,\eta\left(\alpha(t) - r(t-)\right) + \frac{1}{2}\frac{\partial^2 \tilde{P}}{\partial r^2}\gamma^2(t) + \langle \tilde{\mathbf{P}}, Q\,x\rangle = 0$$

Write, for each $i = 1,2,...,N$, $P_i = P(t,T,r,e_i)$ and $\mathbf{P} = (P_1,P_2,...,P_N)^T \in \mathbb{R}^N$. Then,

$$\begin{aligned}
\exp\bigg(-\int_0^t r_{0,u}(r)\mathrm{d}u\bigg)\bigg\{&\frac{\partial P}{\partial t} - r(t-)P + \frac{\partial P}{\partial r}\,\eta\left(\alpha(t) - r(t-)\right) \\
&+ \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\gamma^2(t) + \langle \mathbf{P}, Q\,x\rangle\bigg\} = 0
\end{aligned}$$

So, we have the following regime-switching partial differential equation (PDE) for

$P(t, T, r, x)$

$$\frac{\partial P}{\partial t} - r(t-)P + \frac{\partial P}{\partial r}\,\eta\left(\alpha(t) - r(t-)\right) + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\gamma^2(t) + \langle \mathbf{P}, Q\, x\rangle = 0$$

with terminal condition

$$P(T, T, r(T), X(T)) = 1.$$

Equivalently, the vector of bond prices P satisfies the following system of N coupled PDEs

$$\frac{\partial P_i}{\partial t} - r(t-)P_i + \frac{\partial P_i}{\partial r}\,\eta\left(\alpha_i - r(t-)\right) + \frac{1}{2}\frac{\partial^2 P_i}{\partial r^2}\gamma_i^2 + \langle \mathbf{P}, Q\, e_i\rangle = 0$$

with terminal condition

$$P(T, T, r(T), e_i) = 1, \quad i = 1, 2, ..., N.$$

Note that the bond price has the following Markovian regime-switching exponential affine form

$$P(t, T, r, x) = \exp\bigl(A(t, T, x) - B(t, T)\, r\bigr).$$

Recall that $\tilde{A}(t, T, x) = \exp\bigl(A(t, T, x)\bigr)$. Let $\tilde{A}_i = \tilde{A}(t, T, e_i)$ for $i = 1, 2, ..., N$ and $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_N)^T \in \mathbb{R}^N$. Note that

$$\begin{aligned}
\frac{\partial P}{\partial t} &= P\biggl(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\biggr),\\
\frac{\partial P}{\partial r} &= -B\, P,\\
\frac{\partial^2 P}{\partial r^2} &= B^2 P.
\end{aligned}$$

Then, $A(t, T, x)$ satisfies the following Markovian regime-switching ordinary differential equation (ODE)

$$\frac{\mathrm{d}A}{\mathrm{d}t} - \alpha(t)\big(1 - \mathrm{e}^{-\eta(T-t)}\big) + \frac{1}{2\eta^2}\gamma^2(t)\big(1 - \mathrm{e}^{-\eta(T-t)}\big)^2 + \mathrm{e}^{-A}\langle \tilde{\mathbf{A}}, Q\,x\rangle = 0$$

with $A(T, T, X(T)) = 0$.

Write, for each $i = 1, 2, ..., N$, $A_i = A(t, T, e_i)$ and $\mathbf{A} = (A_1, A_2, ..., A_N)^T \in \mathbb{R}^N$. Then, the vector of coefficients $\mathbf{A}$ satisfies the following system of N coupled ODEs

$$\frac{\mathrm{d}A_i}{\mathrm{d}t} - \alpha_i\big(1 - \mathrm{e}^{-\eta(T-t)}\big) + \frac{1}{2\eta^2}\gamma_i^2\big(1 - \mathrm{e}^{-\eta(T-t)}\big)^2 + \mathrm{e}^{-A_i}\langle \tilde{\mathbf{A}}, Q\,e_i\rangle = 0 \qquad (4.6)$$

with $A(T, T, e_i) = 0$ for each $i = 1, 2, ..., N$.

Write, for each $i = 1, 2, ..., N$,

$$F_i(t) = \alpha_i\big(1 - \mathrm{e}^{-\eta(T-t)}\big) + \frac{1}{2\eta^2}\gamma_i^2\big(1 - \mathrm{e}^{-\eta(T-t)}\big)^2.$$

Consider the following diagonal matrix

$$\mathrm{diag}\big(F(t)\big) = \mathrm{diag}\big(F_1(t), F_2(t), ..., F_N(t)\big).$$

Substituting $\tilde{A}_i = \exp(A_i)$ for each $i = 1, 2, ..., N$ into (4.6), $\tilde{\mathbf{A}}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N

$$\frac{\mathrm{d}\tilde{\mathbf{A}}(t)}{\mathrm{d}t} = \Big[\mathrm{diag}\big(F(t)\big) - Q^T\Big]\tilde{\mathbf{A}}(t), \quad \tilde{\mathbf{A}}(0) = \mathbf{1}$$

where $\mathbf{1} = (1, 1, ..., 1)^T \in \mathbb{R}^N$.

Define

$$\Delta(t) = \mathrm{diag}\big(F(t)\big) - Q^T$$

If we let $\tau = T - t$, then $\mathrm{d}\tau = -\mathrm{d}t$. So

$$\frac{\mathrm{d}\tilde{\mathbf{A}}(\tau)}{\mathrm{d}\tau} = -\Delta(\tau)\,\tilde{\mathbf{A}}(\tau), \quad \tilde{\mathbf{A}}(0) = 1.$$

Suppose $\Phi(t)$ denotes the fundamental matrix solution of

$$\frac{\mathrm{d}\Phi(\tau)}{\mathrm{d}\tau} = -\Delta(\tau)\,\Phi(\tau), \quad \Phi(0) = \mathbf{I} \tag{4.7}$$

where $\mathbf{I}$ is the $n \times n$-identity matrix.

If $\Delta(t) = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Phi(T)$ is $\Phi(t) = \exp(\Delta\, t)$.

In general, there exists a unique fundamental matrix solution $\Phi(t)$ of the linear matrix differential Eq. (4.7). Now, $\tilde{\mathbf{A}}$ can be represented in terms of the fundamental matrix solution $\Phi(t)$ as below.

$$\tilde{\mathbf{A}}(t) = \Phi(t)\,\tilde{\mathbf{A}}(0) = \Phi(t)\mathbf{1}.$$

So,

$$A(t,T,x) = \sum_{i=1}^{N} \ln\big(\langle \Phi(t)\mathbf{1}, \mathrm{e}_i\rangle\big)\langle x, \mathrm{e}_i\rangle.$$

Therefore, the bond price is represented as the following Markovian regime-switching exponential affine form

$$P(t,T,r,x) = \exp\left(\sum_{i=1}^{N} \ln\big(\langle \Phi(t)\mathbf{1}, \mathrm{e}_i\rangle\big)\langle x, \mathrm{e}_i\rangle - B(t,T)\right).$$

So far, we employed the concept of stochastic flows to derive an exponential affine form of the bond price when the short rate process is governed by a Markovian regime-switching Hull-White model. Our model allowed the market parameters, including the mean-reversion level and the volatility rate to switch over time according to a continuous-time, finite-state Markov chain. We provided a representation to the exponential affine form of the bond price in terms of fundamental matrix solutions of linear matrix differential equations.

4.3.2 Bond pricing and the forward measure

The following lemma was given in Shen and Siu [17, 20 & 21] and gives the dynamics of the underlying commodity spot price, the interest rate, and the Markov chain under a forward measure $\mathcal{P}^T$ to be defined below.

Lemma 4.1. *Let $\Lambda(T)$ denote the Radon–Nikodym derivative defined by*

$$\Lambda(T) = \frac{\mathrm{d}\mathcal{P}^T}{\mathrm{d}\mathcal{P}}\bigg|_{\mathcal{G}(\mathcal{T})} = \frac{exp\big(-\int_0^T r(t)\,\mathrm{d}t\big)}{\mathbb{E}\big[exp\big(-\int_0^T r(t)\mathrm{d}t\big)\big]} \tag{4.8}$$

Under the following assumptions, we have

1. *The Novikov condition is satisfied.*

$$\mathbb{E}\big[exp\big(\frac{1}{2}\int_0^T \gamma^2(t)B^2(t,T)\mathrm{d}t\big)\big] < \infty.$$

2. *$\tilde{A}(t,T,x)$ is a suitable function in the sense that*

$$\frac{\tilde{A}(t,T,X(t)}{\tilde{A}(0,T,X(0)}exp\bigg(-\int_0^t \frac{\frac{\partial\tilde{A}}{\partial s}+Q\,\tilde{A}(s,T,X(s))}{\tilde{A}(s,T,X(s))}\mathrm{d}s\bigg), \quad t\in\mathcal{T}$$

is a $(\mathcal{G},\mathcal{P})$-martingale.

Then, under the forward probability measure $\mathcal{P}$, the following results hold:

1. *The dynamics of the underlying logarithmic of commodity spot price and the short rate are given by*

$$\mathrm{d}x(t) = \Big(\beta\big(\theta(t) - x(t)\big) - \rho(t)\gamma(t)\sigma(t)B(t,T)\Big)\mathrm{d}t + \sigma(t)\mathrm{d}W_S^T(t) \tag{4.9}$$

$$\mathrm{d}r(t) = \Big(\eta\big(\alpha(t) - r(t)\big) - \gamma^2(t)B(t,T)\Big)\mathrm{d}t + \gamma(t)\mathrm{d}W_r^T(t) \tag{4.10}$$

where

$$W_S^T(t) = W_S(t) + \int_0^t \rho(s)\gamma(s)B(s,T)\mathrm{d}s, \quad t \in \mathcal{T},$$

and

$$W_r^T(t) = W_r(t) + \int_0^t \gamma(s)B(s,T)\mathrm{d}s, \quad t \in \mathcal{T},$$

are $\mathcal{P}^T$-standard Brownian motions with instantaneous correlation coefficient $\rho(t)$ at time t, i. e., $\langle W_S^T, W_r^T\rangle = \int_0^t \rho(s)\mathrm{d}s$.

2. *The rate matrix of the chain $\mathbf{X}$ is $Q^T(t) = [q_{ij}^T(t)]_{i,j=1,2,\ldots,N}$*

$$q_{ij}^T(t) = \begin{cases} q_{ij}\dfrac{\tilde{A}(t,T,e_j)}{\tilde{A}(t,T,e_i)} & i \neq j \\ -\sum\limits_{k\neq i} q_{ik}\dfrac{\tilde{A}(t,T,e_k)}{\tilde{A}(t,T,e_i)} & i = j \end{cases}$$

and the semimartingale dynamics of the chain is given by

$$X(t) = X(0) + \int_0^t Q^T(s)X(s)\mathrm{d}s + M^T(t), \quad t \in \mathcal{T},$$

where $\{M^T(t) | t \in \mathcal{T}\}$ is an $\mathbb{R}^N$-valued, $(\mathcal{F}^X, \mathcal{P}^T)$-martingale.

Proof. The proof is given in Lemma 3.3 of Fan et al. [17] and follows the same arguments in Lemma 3.2 in Shen and Siu [20]. So we do not repeat it again here. $\square$

4.4 Valuation of commodity futures and options

In this section, we derive the price of a European-style commodity option and futures option under the regime-switching stochastic interest rate model at time 0, denoted by $C(0,T)$ and $C_f(0,T,U)$, respectively. Under the risk-neutral probability measure $\mathcal{P}$, the prices of a T-maturity futures contract and a T-maturity European-style commodity option at time 0 are given as follows:

$$F(0,T) = \mathbb{E}\{S(T)\} \tag{4.11}$$

and

$$C(0,T) = \mathbb{E}\bigg\{\Big(\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\Big)\big(S(T) - K\big)^+\bigg\}, \tag{4.12}$$

where $S(T)$ is the terminal commodity price; K is the strike price of the commodity option; $\mathbb{E}$ is the expectation with respect to the risk-neutral probability measure $\mathcal{P}$. Consider a European-style futures option with a strike price K_f, the terminal payoff function at the maturity time T of the option is $(F(T,U) - K_f)^+$, where $F(T,U)$ represents the futures price with maturity time U at time T. Then the price of the futures option at time 0 is given by

$$C_f(0,T,U) = \mathbb{E}\bigg\{\Big(\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\Big)\big(F(T,U) - K_f\big)^+\bigg\}. \tag{4.13}$$

By change of measures defined in the earlier section, (4.12) and (4.13) become

$$C(0,T) = P(0,T)\mathbb{E}^T\big\{\big(S(T) - K\big)^+\big\}, \tag{4.14}$$

and

$$C_f(0,T,U) = P(0,T)\mathbb{E}^T\{(F(T,U) - K_f)^+\} \tag{4.15}$$

where $\mathbb{E}^T$ is an expectation under the forward measure $\mathcal{P}^T$.

4.4.1 Valuation of commodity options

From now on, al the calculations are pretty similar to chapter 3 section 3.

Following the notation in previous chapter, write $\kappa = \ln(K)$, the dampened commodity option price is given by

$$c(\kappa) = e^{\alpha\kappa}C(O,T) \tag{4.16}$$

where α is called the dampening coefficient and assumed to be positive. To obtain a square integrable function, the dampening coefficient α is selected and the dampened commodity pricing formula is defined. The problem how to choose the value of the coefficient α is completely explained in chapter 2. We derive an explicit formula for the Fourier transform of $c(\kappa)$ next. Let $f_{\mathcal{H}_T}(x)$ be the conditional density function of $x(T)$ given $\mathcal{H}(T)$. Then, the dampened commodity Fourier transform is given by

$$\begin{aligned}
\hat{c}(u) &= \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa}\, c(\kappa)\,\mathrm{d}\kappa \\
&= P(0,T)\int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa}\,\mathrm{e}^{\alpha\kappa}\,\mathbb{E}^T\Big\{(\mathrm{e}^{x(T)} - \mathrm{e}^{\kappa})^+\Big\}\,\mathrm{d}\kappa \\
&= P(0,T)\mathbb{E}^T\bigg\{\int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa}\,\mathrm{e}^{\alpha\kappa}\,\mathbb{E}^T\Big\{(\mathrm{e}^{x(T)} - \mathrm{e}^{\kappa})^+\Big|\mathcal{H}(T)\Big\}\,\mathrm{d}\kappa\bigg\} \\
&= P(0,T)\mathbb{E}^T\bigg\{\int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa}\,\mathrm{e}^{\alpha\kappa}\int_{\kappa}^{\infty}\left(\mathrm{e}^{x} - \mathrm{e}^{\kappa}\right) f_{\mathcal{H}_T}(x)\,\mathrm{d}x\,\mathrm{d}\kappa\bigg\}
\end{aligned}$$

$$= P(0,T)\,\mathbb{E}^T\bigg\{\int_{-\infty}^{\infty} f_{\mathcal{H}_T}(x)\int_{-\infty}^{x}\left(\mathrm{e}^{x}\,\mathrm{e}^{(\alpha-iu)\kappa}-\mathrm{e}^{(1+\alpha-iu)\kappa}\right)\mathrm{d}\kappa\,\mathrm{d}x\bigg\}$$

$$= P(0,T)\,\mathbb{E}^T\bigg\{\int_{-\infty}^{\infty} f_{\mathcal{H}_T}(x)\left(\frac{\mathrm{e}^{(1+\alpha-iu)x}}{(\alpha-iu)}-\frac{\mathrm{e}^{(1+\alpha-iu)x}}{(1+\alpha-iu)}\right)\mathrm{d}x\bigg\}$$

$$= P(0,T)\,\mathbb{E}^T\bigg\{\left(\frac{\phi_{\mathcal{H}_T}(-i(1+\alpha)-u)}{(\alpha-iu)}-\frac{\phi_{\mathcal{H}_T}(-i(1+\alpha)-u)}{(1+\alpha-iu)}\right)\bigg\}$$

$$= \frac{P(0,T)\,\mathbb{E}^T\Big\{\phi_{\mathcal{H}_T}(-i(1+\alpha)-u)\Big\}}{(\alpha-iu)(1+\alpha-iu)} \tag{4.17}$$

where

$$\phi_{\mathcal{H}_T}(\nu)=\mathbb{E}^T\{\mathrm{e}^{i\nu x(T)}|\mathcal{H}_T\}=\int_{-\infty}^{\infty}\mathrm{e}^{i\nu x}f_{\mathcal{H}_T}(x)\mathrm{d}x \tag{4.18}$$

is the conditional characteristic function of $x(T)$ given $\mathcal{H}(T)$. The second equality in (4.17) holds by the well-known property of conditional expectations ($\mathbb{E}\{\mathbb{E}\{X|Y\}\} = \mathbb{E}\{X\}$), and fifth equality holds by Fubini's theorem since the modified commodity price is bounded.

Note that given $\mathcal{H}(T)$, $x(T)$ has Gaussian distribution with mean

$$\mathbb{E}^T\{x(T)|\mathcal{H}_T\}=e^{-\beta T}x(0)+\int_0^T\big(\beta\,\theta(u)-\rho(u)\gamma(u)\sigma(u)B(u,T)\big)\,e^{-\beta(T-u)}\,\mathrm{d}u \tag{4.19}$$

and variance

$$\mathrm{Var}(x(T)|\mathcal{H}_T)=\int_0^T\sigma^2(u)\,e^{-2\beta(T-u)}\,\mathrm{d}u. \tag{4.20}$$

It follows that

$$\phi_{\mathcal{H}_T}(\nu) = \exp\Big(i\nu\mathbb{E}^T\{x(T)|\mathcal{H}_T\} - \frac{1}{2}\,\nu^2 Var(x(T)|\mathcal{F}_T)\Big)$$
$$= \exp\Big(i\nu\big(e^{-\beta T}x(0) + \int_0^T \big(\beta\,\theta(u) - \rho(u)\gamma(u)\sigma(u)B(u,T)\big)\,e^{-\beta(T-u)}\,\mathrm{d}u\big)$$
$$- \frac{1}{2}\,\nu^2\int_0^T \sigma^2(u)\,e^{-2\beta(T-u)}\,\mathrm{d}u\Big).$$

For each $t \in [0,T]$ and $\nu \in \mathbb{R}$, let $G(t,\nu) = \big(g_1(t,\nu), g_2(t,\nu), ..., g_N(t,\nu)\big)$, where $g_j(t,\nu)$ for each $j = 1, 2, ..., N$ is

$$g_j(t,\nu) = i\nu\,e^{-\beta(T-t)}(\,\beta\,\theta_j\; - \rho_j\,\gamma_j\,\sigma_j\,B(t,T))\; - \frac{1}{2}\,\nu^2\,\sigma_j^2\,e^{-2\beta(T-t)}.$$

Therefore,

$$\phi_{\mathcal{H}_T}(\nu) = \exp\Big(i\nu\,e^{-\beta T}x(0) + \int_0^T \langle G(t,\nu), X(t)\rangle \mathrm{d}t\Big). \tag{4.21}$$

Substituting (4.21) in (4.17) implies

$$\hat{c}(u) = \frac{P(0,T)\,\mathbb{E}^T\Big\{\exp\Big(i\nu\,e^{-\beta T}x(0) + \int_0^T\langle G(t,\nu), X(t)\rangle \mathrm{d}t\Big)\Big\}}{(\alpha - iu)(1+\alpha-iu)},$$
$$\nu = -(u + i(1+\alpha)). \tag{4.22}$$

In order to derive an explicit formula for $\hat{c}(u)$, It's necessary to calculate the expectation given in (4.22). To do this end, we use a modification of proof of lemma 1 in Buffington and Elliott [11]. Let $\mathbf{diag}(G(t,\nu))$ denote the diagonal matrix with diagonal elements given by the components of $G(t,\nu)$, $\mathbf{1} = (1,1,...,1)^T$ and $\mathbf{I}$ denote the $(n \times n)$-identity matrix. Let's define

$$W(t,\nu) = \exp\biggl(\int_0^t \langle G(s,\nu), X(s) \rangle \mathrm{d}s \biggr) X(t), \quad W(0,\nu) = X(0). \tag{4.23}$$

Consequently,

$$\mathrm{d}W(t,\nu) = \langle G(t,\nu), X(t) \rangle \, W(t,\nu) \, \mathrm{d}t + \exp\biggl(\int_0^t \langle G(s,\nu), X(s) \rangle \mathrm{d}s \biggr) \mathrm{d}X(t).$$

Note that under $\mathcal{P}^T$,

$$\mathrm{d}X(t) = Q^T \, X(t) \mathrm{d}t + \mathrm{d}M(t),$$

and that

$$\langle G(t,\nu), X(t) \rangle \, W(t,\nu) = \mathbf{diag}(G(t,\nu)) \, W(t,\nu), \quad \forall t \in \tau.$$

Then

$$\begin{aligned}
\mathrm{d}W(t,\nu) &= \langle G(t,\nu), X(t) \rangle \, W(t,\nu) \, \mathrm{d}t + \exp\biggl(\int_0^t \langle G(s,\nu), X(s) \rangle \mathrm{d}s \biggr) \, Q^T \, X(t) \mathrm{d}t \\
&\qquad + \exp\biggl(\int_0^t \langle G(s,\nu), X(s) \rangle \, \mathrm{d}s \biggr) \, \mathrm{d}M(t) \\
&= \mathbf{diag}(G(t,\nu)) \, W(t,\nu) \, \mathrm{d}t + Q^T \, W(t,\nu) \, \mathrm{d}t \\
&\qquad + \exp\biggl(\int_0^t \langle G(s,\nu), X(s) \rangle \, \mathrm{d}s \biggr) \, \mathrm{d}M(t) \\
&= \Bigl[\mathbf{diag}(G(t,\nu)) + \, Q^T \Bigr] \, W(t,\nu) \, \mathrm{d}t \, + \, \exp\biggl(\int_0^t \langle G(s,\nu), X(s) \rangle \, \mathrm{d}s \biggr) \, \mathrm{d}M(t).
\end{aligned}$$

By taking integration from both sides, we have

$$\begin{aligned}
W(t,\nu) &= W(0,\nu) + \int_0^t \Big[\mathbf{diag}(G(s,\nu)) + Q^T\Big] W(s,\nu)\,\mathrm{d}s \\
&\quad + \int_0^t \exp\left(\int_0^t \langle G(s,\nu), X(s)\rangle\,\mathrm{d}s\right)\mathrm{d}M(s) \\
&= X(0) + \int_0^t \Big[\mathbf{diag}(G(s,\nu)) + Q^T\Big] W(s,\nu)\,\mathrm{d}s \\
&\quad + \int_0^t \exp\left(\int_0^t \langle G(s,\nu), X(s)\rangle\,\mathrm{d}s\right)\mathrm{d}M(s).
\end{aligned}$$

Taking expectation under forward measure $\mathcal{P}^T$ gives:

$$\mathbb{E}^T\{W(t,\nu)\} = X(0) + \int_0^t \big[\mathbf{diag}(G(s,\nu)) + Q^T\big]\,\mathbb{E}^T\{W(s,\nu)\}\,\mathrm{d}s.$$

Hence $\mathbb{E}^T\{W(t,\nu)\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N:

$$\frac{\mathrm{d}\mathbb{E}^T\{W(t,\nu)\}}{\mathrm{d}t} = \big[\mathbf{diag}(G(t,\nu)) + Q^T\big]\,\mathbb{E}^T\{W(t,\nu)\}, \quad \mathbb{E}^T\{W(0,\nu)\} = X(0). \tag{4.24}$$

Suppose $\Phi(t,\nu)$ denotes the fundamental matrix solution of

$$\frac{\mathrm{d}\Phi(t,\nu)}{\mathrm{d}t} = \big[\mathbf{diag}(G(t,\nu)) + Q^T\big]\,\Phi(t,\nu), \quad \Phi(0,\nu) = \mathbf{I}.$$

If $\big[\mathbf{diag}(G(t,\nu)) + Q^T\big] = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Phi(t,\nu)$ is

$$\Phi(t,\nu) = \exp(\Delta\, t).$$

In general, there exists a unique fundamental matrix solution $\Phi(t,\nu)$ of the linear matrix differential Eq. (3.15). Now, $\mathbb{E}^T\{W(t,\nu)\}$ can be represented in terms of the fundamental matrix solution $\Phi(t,\nu)$ as below:

$$\mathbb{E}^T\{W(t,\nu)\} = \Phi(t,\nu)\,X(0).$$

Now

$$\begin{aligned}
\mathbb{E}^T\Big\{\exp\Big(\int_0^T \langle G(t,\nu), X(t)\rangle \mathrm{d}t\Big)\Big\} &= \mathbb{E}^T\Big\{\exp\Big(\int_0^T \langle G(t,\nu), X(t)\rangle \mathrm{d}t\Big)\langle X(T), \mathbf{1}\rangle\Big\} \\
&= \mathbb{E}^T\Big\{\Big\langle \exp\Big(\int_0^T \langle G(t,\nu), X(t)\rangle \mathrm{d}t\Big) X(T), \mathbf{1}\Big\rangle\Big\} \\
&= \Big\langle \mathbb{E}^T\Big\{\exp\Big(\int_0^T \langle G(t,\nu), X(t)\rangle \mathrm{d}t\Big) X(T)\Big\}, \mathbf{1}\Big\rangle \\
&= \big\langle \mathbb{E}^T\{W(T,\nu)\}, \mathbf{1}\big\rangle \\
&= \big\langle \Phi(T,\nu)\,X(0), \mathbf{1}\big\rangle.
\end{aligned}$$

Therefore under Markovian regime-switching Ornstein-Uhlenbeck model, the price of the commodity option is given by:

$$\hat{c}(u) = \frac{P(0,T)\exp\Big(i\nu\, e^{-\beta T}x(0)\Big)\big\langle \Phi(T,\nu)\,X(0), \mathbf{1}\big\rangle}{(\alpha - iu)(1+\alpha-iu)}, \quad \nu = -(u + i(1+\alpha)). \quad (4.25)$$

4.4.2 Valuation of futures options

In this subsection we consider the valuation of commodity futures options. We wish to evaluate the time-zero value of a standard European call option on the future price $F(T,U)$ with strike price K_f and maturity at time T. That is to evaluate:

$$C_f(0,T,U) = P(0,T)\,\mathbb{E}^T\Big\{\Big(F(T,U) - K_f\Big)^+\Big\}.$$

As seen before, the dampened commodity futures options price is given by:

$$c_f(\kappa_f) = \mathrm{e}^{\alpha_f\,\kappa_f}\,C_f(0,T,U), \quad (4.26)$$

where $\kappa_f = \ln(K_f)$. Let's define $Y(t) := \ln\big(F(t,U)\big)$ for each $t \in \tau$. Now we derive an explit formula for the Fourier transform of the dampened commodity futures options.

$$
\begin{aligned}
\hat{c_f}(u) &= \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa_f}\, c(\kappa_f)\, \mathrm{d}\kappa_f \\
&= P(0,T) \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa_f}\, \mathrm{e}^{\alpha_f\kappa_f}\, \mathbb{E}^T\Big\{(\mathrm{e}^{Y(T)} - \mathrm{e}^{\kappa_f})^+\Big\}\, \mathrm{d}\kappa_f \\
&= P(0,T)\, \mathbb{E}^T\Bigg\{ \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa_f}\, \mathrm{e}^{\alpha_f\kappa_f}\, \mathbb{E}^T\Big\{ (\mathrm{e}^{Y(T)} - \mathrm{e}^{\kappa_f})^+ \Big| \mathcal{H}_T \Big\}\, \mathrm{d}\kappa_f \Bigg\} \\
&= P(0,T)\, \mathbb{E}^T\Bigg\{ \int_{-\infty}^{\infty} \mathrm{e}^{-iu\kappa_f}\, \mathrm{e}^{\alpha_f\kappa_f} \int_{\kappa_f}^{\infty} \big(\mathrm{e}^{y} - \mathrm{e}^{\kappa_f}\big)\, f_{\mathcal{H}_T}(y)\, \mathrm{d}y\, \mathrm{d}\kappa_f \Bigg\} \\
&= P(0,T)\, \mathbb{E}^T\Bigg\{ \int_{-\infty}^{\infty} f_{\mathcal{H}_T}(y) \int_{-\infty}^{y} \big(\mathrm{e}^{y}\, \mathrm{e}^{(\alpha_f - iu)\kappa_f} - \mathrm{e}^{(1+\alpha_f - iu)\kappa_f}\big)\, \mathrm{d}\kappa_f\, \mathrm{d}y \Bigg\} \\
&= P(0,T)\, \mathbb{E}^T\Bigg\{ \int_{-\infty}^{\infty} f_{\mathcal{H}_T}(y) \left(\frac{\mathrm{e}^{(1+\alpha_f - iu)y}}{(\alpha_f - iu)} - \frac{\mathrm{e}^{(1+\alpha_f - iu)y}}{(1+\alpha_f - iu)} \right) \mathrm{d}y \Bigg\} \\
&= P(0,T)\, \mathbb{E}^T\Bigg\{ \left(\frac{\psi_{\mathcal{H}_T}(-i(1+\alpha_f) - u)}{(\alpha_f - iu)} - \frac{\psi_{\mathcal{H}_T}(-i(1+\alpha_f) - u)}{(1+\alpha_f - iu)} \right) \Bigg\} \\
\hat{c_f}(u) &= \frac{P(0,T)\, \mathbb{E}^T\Big\{ \psi_{\mathcal{H}_T}(-i(1+\alpha_f) - u) \Big\}}{(\alpha_f - iu)(1+\alpha_f - iu)}
\end{aligned}
\tag{4.27}
$$

where

$$
\psi_{\mathcal{H}_T}(\nu) = \mathbb{E}^T\{\mathrm{e}^{i\nu Y(T)} | \mathcal{H}_T\} = \int_{-\infty}^{\infty} \mathrm{e}^{i\nu y} f_{\mathcal{H}_T}(y) \mathrm{d}y \tag{4.28}
$$

is the conditional characteristic function of $Y(T)$ given $\mathcal{H}_T$. In order to derive an explicit formula for $\hat{c_f}(u)$, we need to derive an analytical formula of the characteristic function of the logarithmic commodity futures price, in other words $\ln\big(F(T,U)\big)$.

To do so, first we need to derive the time-t price of a T-maturity futures contract.

In other words,

$$F(t,T) = \mathbb{E}^T\{S(T)|\mathcal{H}_t\} = \mathbb{E}^T\big\{\mathbb{E}^T\{S(T)|\mathcal{H}_T\}|\mathcal{H}_t\big\}$$

Since $S(T) = \mathrm{e}^{x(T)}$ and under forward measure $\mathcal{P}^T$, the conditional distribution of $x(T)$ given $\mathcal{H}_T$ is a Gaussian distribution with mean

$$\mathbb{E}^T\{x(T)|\mathcal{H}_T\} = e^{-\beta(T-t)}x(t) + \int_t^T \big(\beta\,\theta(u) - \rho(u)\gamma(u)\sigma(u)B(u,T)\big)\, e^{-\beta(T-u)}\,\mathrm{d}u \tag{4.29}$$

and variance

$$\mathrm{Var}(x(T)|\mathcal{H}_T) = \int_t^T \sigma^2(u)\, e^{-2\beta(T-u)}\,\mathrm{d}u, \tag{4.30}$$

then, it's easy to see that

$$\begin{aligned}
\mathbb{E}^T\{S(T)|\mathcal{H}_T\} &= \exp\Big(\mathbb{E}^T\{x(T)|\mathcal{H}_T\} + \frac{1}{2}\,\mathrm{Var}\big(x(T)|\mathcal{H}_T\big)\Big)\\
&= \exp\Big(e^{-\beta(T-t)}x(t) + \int_t^T \big(\beta\,\theta(u) - \rho(u)\gamma(u)\sigma(u)B(u,T)\big)\, e^{-\beta(T-u)}\,\mathrm{d}u\\
&\qquad + \frac{1}{2}\int_t^T \sigma^2(u)\, e^{-2\beta(T-u)}\,\mathrm{d}u\Big).
\end{aligned}$$

Then the time-t price of a T-maturity futures contract is given by

$$\begin{aligned}
F(t,T) &= \mathbb{E}^T\{S(T)|\mathcal{H}_t\} = \mathbb{E}^T\big\{\mathbb{E}^T\{S(T)|\mathcal{H}_T\}|\mathcal{H}_t\big\}\\
&= \mathbb{E}^T\Big\{\exp\Big(e^{-\beta(T-t)}x(t) + \int_t^T \big(\beta\,\theta(u) - \rho(u)\gamma(u)\sigma(u)B(u,T)\big)\, e^{-\beta(T-u)}\,\mathrm{d}u\\
&\qquad + \frac{1}{2}\int_t^T \sigma^2(u)\, e^{-2\beta(T-u)}\,\mathrm{d}u\Big)\Big|\mathcal{H}_t\Big\}.
\end{aligned}$$

Let's define $N(s) = \big(n_1(s), n_2(s), ..., n_N(s)\big)$ for each $s \in [0,T]$, where for each

$j = 1, 2, ..., N$,

$$n_j(s) = (\beta\,\theta_j - \rho_j\gamma_j\sigma_j B(s,T))\, e^{-\beta(T-s)} + \frac{1}{2}\,\sigma_j^2\, e^{-2\beta(T-s)}.$$

Therefore

$$F(t,T) = \mathbb{E}^T\left\{\exp\left(e^{-\beta(T-t)}x(t) + \int_t^T \langle N(s), X(s)\rangle\,\mathrm{d}s\right)\middle|\mathcal{F}_t\right\}. \tag{4.31}$$

In order to derive an explicit formula for $F(t,T)$, It's necessary to calculate the expectation given in (4.38). To do this end, we adopt the same methodology used in previous subsection. Let $\mathbf{diag}(N(t))$ denote the diagonal matrix with diagonal elements given by the components of $N(t)$. Let's define

$$H(t,u) = \exp\left(\int_t^u \langle N(s), X(s)\rangle \mathrm{d}s\right) X(u), \quad H(t,t) = X(t).$$

Consequently,

$$\mathrm{d}H(t,u) = \langle N(u), X(u)\rangle\, H(t,u)\,\mathrm{d}u + \exp\left(\int_t^u \langle N(s), X(s)\rangle \mathrm{d}s\right)\mathrm{d}X(u).$$

Note that under $\mathcal{P}^T$,

$$\mathrm{d}X(u) = Q^T\, X(u)\mathrm{d}u + \mathrm{d}M(u),$$

and that

$$\langle N(u), X(u)\rangle\, H(t,u) = \mathbf{diag}(N(u))\, H(t,u), \quad \forall t \in \tau.$$

Then

$$
\begin{aligned}
\mathrm{d}H(t,u) &= \langle N(u), X(u)\rangle\, H(t,u)\,\mathrm{d}u + \exp\biggl(\int_t^u \langle N(s), X(s)\rangle \mathrm{d}s\biggr)\, Q^T\, X(u)\mathrm{d}u \\
&\qquad + \exp\biggl(\int_t^u \langle N(s), X(s)\rangle\, \mathrm{d}s\biggr)\, \mathrm{d}M(u) \\
&= \mathbf{diag}(N(u))\, H(t,u)\,\mathrm{d}u + \ Q^T\, H(t,u)\,\mathrm{d}u \\
&\qquad + \ \exp\biggl(\int_t^u \langle N(s), X(s)\rangle\, \mathrm{d}s\biggr)\, \mathrm{d}M(u) \\
&= \Bigl[\mathbf{diag}(N(u)) + Q^T\Bigr]\, H(t,u)\,\mathrm{d}u + \exp\biggl(\int_t^u \langle N(s), X(s)\rangle\, \mathrm{d}s\biggr)\mathrm{d}M(u).
\end{aligned}
$$

By taking integration from both sides, we have

$$
\begin{aligned}
H(t,u) &= H(t,t) + \int_t^u \Bigl[\mathbf{diag}(N(s)) + \ Q^T\Bigr]\, H(t,s)\,\mathrm{d}s \\
&\quad + \int_t^u \exp\biggl(\int_t^u \langle N(s), X(s)\rangle\, \mathrm{d}s\biggr)\, \mathrm{d}M(s) \\
&= X(t) + \int_t^u \Bigl[\mathbf{diag}(N(s)) + \ Q^T\Bigr]\, H(t,s)\,\mathrm{d}s \\
&\quad + \int_t^u \exp\biggl(\int_t^u \langle N(s), X(s)\rangle\, \mathrm{d}s\biggr)\, \mathrm{d}M(s).
\end{aligned}
$$

Taking expectation under $\mathcal{P}^T$ given $\mathcal{H}_t$ gives:

$$
\mathbb{E}^T\{H(t,u)|\mathcal{H}_t\} = X(t) + \int_t^u \bigl[\mathbf{diag}(N(s)) + \ Q^T\,\bigr]\, \mathbb{E}^T\{H(t,s)|\mathcal{H}_t\}\,\mathrm{d}s.
$$

Hence $\mathbb{E}^T\{H(t,u)|\mathcal{H}_t\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N:

$$
\frac{\mathrm{d}\mathbb{E}^T\{H(t,u)|\mathcal{H}_t\}}{\mathrm{d}u} = \bigl[\mathbf{diag}(N(u)) + \ Q^T\,\bigr]\, \mathbb{E}^T\{H(t,u)|\mathcal{H}_t\}, \quad \mathbb{E}^T\{H(t,t)|\mathcal{H}_t\} = X(t). \tag{4.32}
$$

Suppose $\Psi(t,u)$ denotes the fundamental matrix solution of

$$\frac{\mathrm{d}\Psi(t,u)}{\mathrm{d}u} = \left[\mathbf{diag}(N(u)) + Q^T\right]\Psi(t,u), \qquad \Psi(t,t) = \mathbf{I}.$$

If $\left[\mathbf{diag}(N(u)) + Q^T\right] = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Psi(t,u)$ is

$$\Psi(t,u) = \exp(\Delta\,(u-t)).$$

In general, there exists a unique fundamental matrix solution $\Psi(t,u)$ of the linear matrix differential Eq. (4.32). Now, $\mathbb{E}^T\{H(t,u)|\mathcal{H}_t\}$ can be represented in terms of the fundamental matrix solution $\Psi(t,u)$ as below:

$$\mathbb{E}^T\{H(t,u)|\mathcal{H}_t\} = \Psi(t,u)\,X(t).$$

Now

$$\begin{aligned}
F(t,T) &= \mathbb{E}^T\left\{\exp\left(\mathrm{e}^{-\beta(T-t)}x(t) + \int_t^T \langle N(u), X(u)\rangle\,\mathrm{d}u\right)\middle|\mathcal{H}_t\right\} \\
&= \exp\left(\mathrm{e}^{-\beta(T-t)}x(t)\right)\mathbb{E}^T\left\{\exp\left(\int_0^T \langle N(t), X(t)\rangle\,\mathrm{d}u\right)\middle|\mathcal{H}_t\right\} \\
&= \exp\left(\mathrm{e}^{-\beta(T-t)}x(t)\right)\mathbb{E}^T\left\{\exp\left(\int_t^T \langle N(u), X(u)\rangle \mathrm{d}u\right)\langle X(T), \mathbf{1}\rangle\middle|\mathcal{H}_t\right\} \\
&= \exp\left(\mathrm{e}^{-\beta(T-t)}x(t)\right)\mathbb{E}^T\left\{\left\langle\exp\left(\int_t^T \langle N(u), X(u)\rangle \mathrm{d}u\right)X(T), \mathbf{1}\right\rangle\middle|\mathcal{H}_t\right\} \\
&= \exp\left(e^{-\beta(T-t)}x(t)\right)\left\langle\mathbb{E}^T\left\{\exp\left(\int_t^T \langle N(u), X(u)\rangle \mathrm{d}u\right)X(T)\middle|\mathcal{H}_t\right\}, \mathbf{1}\right\rangle \\
&= \exp\left(\mathrm{e}^{-\beta(T-t)}x(t)\right)\left\langle\mathbb{E}^T\{H(t,T)|\mathcal{H}_t\}, \mathbf{1}\right\rangle \\
&= \exp\left(\mathrm{e}^{-\beta(T-t)}x(t)\right)\left\langle\Psi(t,T)\,X(t), \mathbf{1}\right\rangle.
\end{aligned}$$

Then under the forward measure $\mathcal{P}^T$, the time-T price of a U-maturity futures contract is given by:

$$
\begin{aligned}
F(T,U) = \mathbb{E}^T\{S(U)|\mathcal{H}_T\} &= \exp\Big(\mathrm{e}^{-\beta(U-T)}x(T)\Big) \times \big\langle \Psi(T,U)\, X(T), \mathbf{1}\big\rangle \\
&= \exp\bigg(\mathrm{e}^{-\beta(U-T)}\bigg[\mathrm{e}^{-\beta(T)}x(0) + \int_0^T \mathrm{e}^{-\beta(T-u)}\big(\beta\theta(u) - \rho(u)\gamma(u)\sigma(u)B(u,T)\big)\mathrm{d}u \\
&\qquad + \int_0^T \sigma(u)\mathrm{e}^{-\beta(T-u)}\mathrm{d}W_S^T(u)\bigg]\bigg) \times \big\langle \Psi(T,U)\, X(T), \mathbf{1}\big\rangle \\
&= \exp\bigg(\mathrm{e}^{-\beta\, U}\, x(0) + \mathrm{e}^{-\beta U}\int_0^T \big(\beta\,\theta(u) - \rho(u)\gamma(u)\sigma(u)B(u,T)\big)\mathrm{e}^{\beta\, u}\mathrm{d}u \\
&\qquad + \mathrm{e}^{-\beta U}\int_0^T \sigma(u)\mathrm{e}^{\beta\, u}\mathrm{d}W_S^T(u)\bigg) \times \big\langle \Psi(T,U)\, X(T), \mathbf{1}\big\rangle.
\end{aligned}
$$

where $T < U < T^*$.

Note that given $\mathcal{H}_T$, $Y(T) = \ln(F(T,U))$ has Gaussian distribution with mean

$$
\begin{aligned}
\mathbb{E}^T\{Y(T)|\mathcal{H}_T\} = \mathrm{e}^{-\beta\, U}\, x(0) &+ \mathrm{e}^{-\beta U}\int_0^T \big(\beta\,\theta(u) - \rho(u)\gamma(u)\sigma(u)B(u,T)\big)\mathrm{e}^{\beta\, u}\mathrm{d}u \\
&+ \mathrm{e}^{-\beta U}\int_0^T \sigma(u)\mathrm{e}^{\beta\, u}\mathrm{d}W_S^T(u) + \ln\langle \Psi(T,U)\, X(T), \mathbf{1}\rangle
\end{aligned}
\tag{4.33}
$$

and variance

$$
\mathrm{Var}(Y(T)|\mathcal{H}_T) = \int_0^T \sigma^2(u)\, e^{-2\beta(U-u)}\,\mathrm{d}u. \tag{4.34}
$$

It follows from (4.33) and (4.34) that

$$
\begin{aligned}
\psi_{\mathcal{H}_T}(\nu) &= \exp\Big(i\,\nu\,\mathbb{E}\{Y(T)|\mathcal{H}_T\} - \frac{1}{2}\nu^2\,\mathrm{Var}(Y(T)|\mathcal{H}_T)\Big) \\
&= \exp\bigg(i\,\nu\,\mathrm{e}^{-\beta\, U}\, x(0) + i\,\nu\,\mathrm{e}^{-\beta U}\int_0^T \big(\beta\,\theta(u) - \rho(u)\gamma(u)\sigma(u)B(u,T)\big)\mathrm{e}^{\beta\, u}\mathrm{d}u \\
&\quad + i\,\nu\,\mathrm{e}^{-\beta U}\int_0^T \sigma(u)\mathrm{e}^{\beta\, u}\mathrm{d}W_S^T(u)\bigg) \times \langle \Psi(T,U)\, X(T), \mathbf{1}\rangle^{i\,\nu} \\
&\quad \times \exp\bigg(-\frac{1}{2}\nu^2\int_0^T \sigma^2(u)\,\mathrm{e}^{-2\beta(U-u)}\,\mathrm{d}u\bigg).
\end{aligned}
$$

Define $Z(\nu) = (Z_1(\nu), Z_2(\nu), ..., Z_N(\nu))$ and $z(t,\nu) = (z_1(t,\nu), z_2(t,\nu), ..., z_N(t,\nu))$

for each $\nu \in \mathbb{R}$ and $t \in \tau$, where $Z_j(\nu)$ and $z_j(t,\nu)$ for each $j = 1, 2, ..., N$ are

$$Z_j(\nu) = Z(\mathrm{e}_j, \nu) = \exp\Big(i\,\nu\,\mathrm{e}^{-\beta U} x(0)\Big) \times \langle \Psi(T,U)\,\mathrm{e}_j, \mathbf{1}\big\rangle^{i\,\nu} \tag{4.35}$$

and

$$z_j(t,\nu) = z(t, \mathrm{e}_j, \nu) = i\,\nu\,\mathrm{e}^{-\beta(U-t)}(\beta\,\theta_j - \rho_j\gamma_j\sigma_j B(t,T)) - \frac{1}{2}\,\nu^2\mathrm{e}^{-2\beta(U-t)}\sigma_j^2. \tag{4.36}$$

Therefore,

$$\psi_{\mathcal{H}_T}(\nu) = Z(X(T),\nu)\exp\bigg(\int_0^T \Big\langle z(t, X(t), \nu), X(t)\Big\rangle \mathrm{d}t\bigg). \tag{4.37}$$

Substituting (4.37) in (4.27) implies

$$\hat{c}_f(u) = \frac{P(0,T)\,\mathbb{E}\Big\{Z(X(T),\nu)\exp\Big(\int_0^T \Big\langle z(t,X(t),\nu),X(t)\Big\rangle \mathrm{d}t\Big)\Big\}}{(\alpha - iu)(1+\alpha - iu)},$$
$$\nu = -(u + i(1+\alpha)). \tag{4.38}$$

As we have seen before, in order to derive an explicit formula for $\hat{c}_f(u)$, It's necessary to calculate the expectation given in (4.38). Let $\mathbf{diag}(z(t, X(t), \nu))$ denote the diagonal matrix with diagonal elements given by the components of $z(t, X(t), \nu)$. Let's define

$$\Gamma(t,\nu) = \exp\bigg(\int_0^t \Big\langle z(s, X(s), \nu), X(s)\Big\rangle \mathrm{d}s\bigg) X(t), \quad \Gamma(0,\nu) = X(0). \tag{4.39}$$

Applying Itô differentiation rule to $\Gamma(t,\nu)$ gives

$$\mathrm{d}\Gamma(t,\nu) = \Big\langle z(t,X(t),\nu), X(t)\Big\rangle\,\Gamma(t,\nu)\,\mathrm{d}t + \exp\bigg(\int_0^t \Big\langle z(s,X(s),\nu), X(s)\Big\rangle \mathrm{d}s\bigg)\mathrm{d}X(t).$$

Note that under $\mathcal{P}^T$,

$$\mathrm{d}X(t) = A^T X(t)\mathrm{d}t + \mathrm{d}M(t),$$

and that

$$\Big\langle z(t, X(t), \nu), X(t) \Big\rangle \, \Gamma(t,\nu) = \mathbf{diag}(z(t, X(t), \nu)) \, \Gamma(t,\nu), \quad \forall t \in \tau.$$

Then

$$\begin{aligned}
\mathrm{d}\Gamma(t,\nu) &= \Big\langle z(t, X(t), \nu), X(t) \Big\rangle \, \Gamma(t,\nu) \, \mathrm{d}t + \exp\bigg(\int_0^t \Big\langle z(s, X(s), \nu), X(s) \Big\rangle \mathrm{d}s \bigg) \, Q^T X(t)\mathrm{d}t \\
&\quad + \exp\bigg(\int_0^t \Big\langle z(s, X(s), \nu), X(s) \Big\rangle \, \mathrm{d}s \bigg) \, \mathrm{d}M(t) \\
\mathrm{d}\Gamma(t,\nu) &= \mathbf{diag}(z(t, X(t), \nu)) \, \Gamma(t,\nu) \, \mathrm{d}t + Q^T \, \Gamma(t,\nu) \, \mathrm{d}t \\
&\quad + \exp\bigg(\int_0^t \Big\langle z(s, X(s), \nu), X(s) \Big\rangle \, \mathrm{d}s \bigg) \, \mathrm{d}M(t) \\
\mathrm{d}\Gamma(t,\nu) &= \Big[\mathbf{diag}(z(t, X(t), \nu)) + \, Q^T \Big] \, \Gamma(t,\nu) \, \mathrm{d}t \, + \, \exp\bigg(\int_0^t \Big\langle z(s, X(s), \nu), X(s) \Big\rangle \, \mathrm{d}s \bigg) \, \mathrm{d}M(t).
\end{aligned}$$

By taking integration from both sides, we have

$$\begin{aligned}
\Gamma(t,\nu) &= \Gamma(0,\nu) + \int_0^t \, \Big[\mathbf{diag}(z(s, X(s), \nu)) + \, Q^T \Big] \, \Gamma(s,\nu) \, \mathrm{d}s \\
&\quad + \int_0^t \exp\bigg(\int_0^t \Big\langle z(s, X(s), \nu), X(s) \Big\rangle \, \mathrm{d}s \bigg) \, \mathrm{d}M(s) \\
\Gamma(t,\nu) &= X(0) + \int_0^t \, \Big[\mathbf{diag}(z(s, X(s), \nu)) + \, Q^T \Big] \, \Gamma(s,\nu) \, \mathrm{d}s \\
&\quad + \int_0^t \exp\bigg(\int_0^t \Big\langle G(s,\nu), X(s) \Big\rangle \, \mathrm{d}s \bigg) \, \mathrm{d}M(s).
\end{aligned}$$

Taking expectation under $\mathcal{P}^T$ gives:

$$\mathbb{E}^T\{\Gamma(t,\nu)\} = X(0) + \int_0^t \, \big[\mathbf{diag}(z(s, X(s), \nu)) + \, Q^T \, \big] \, \mathbb{E}^T\{\Gamma(s,\nu)\} \, \mathrm{d}s.$$

Hence $\mathbb{E}^T\{\Gamma(t,\nu)\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N:

$$\frac{\mathrm{d}\mathbb{E}^T\{\Gamma(t,\nu)\}}{\mathrm{d}t} = \left[\mathbf{diag}(z(t,X(t),\nu)) + Q^T\right]\mathbb{E}^T\{\Gamma(t,\nu)\}, \quad \mathbb{E}^T\{\Gamma(0,\nu)\} = X(0). \tag{4.40}$$

Suppose $\Upsilon(t,\nu)$ denotes the fundamental matrix solution of

$$\frac{\mathrm{d}\Upsilon(t,\nu)}{\mathrm{d}t} = \left[\mathbf{diag}(z(t,X(t),\nu)) + Q^T\right]\Upsilon(t,\nu), \quad \Upsilon(0,\nu) = \mathbf{I}.$$

If $\left[\mathbf{diag}(z(t,X(t),\nu)) + Q^T\right] = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Phi(t,\nu)$ is

$$\Upsilon(t,\nu) = \exp(\Delta\, t).$$

In general, there exists a unique fundamental matrix solution $\Upsilon(t,\nu)$ of the linear matrix differential Eq. (3.32). Now, $\mathbb{E}^T\{\Gamma(t,\nu)\}$ can be represented in terms of the fundamental matrix solution $\Upsilon(t,\nu)$ as below:

$$\mathbb{E}\{\Gamma(t,\nu)\} = \Upsilon(t,\nu)\, X(0).$$

$$\begin{aligned}
\mathbb{E}^T\left\{\exp\left(\int_0^T \left\langle z(t,X(t),\nu), X(t)\right\rangle \mathrm{d}t\right)\right\} &= \mathbb{E}^T\left\{\exp\left(\int_0^T \left\langle z(t,X(t),\nu), X(t)\right\rangle \mathrm{d}t\right)\langle X(T), \mathbf{1}\rangle\right\} \\
&= \mathbb{E}^T\left\{\left\langle \exp\left(\int_0^T \langle z(t,X(t),\nu), X(t)\rangle \mathrm{d}t\right) X(T), \mathbf{1}\right\rangle\right\} \\
&= \left\langle \mathbb{E}^T\left\{\exp\left(\int_0^T \langle z(t,X(t),\nu), X(t)\rangle \mathrm{d}t\right) X(T)\right\}, \mathbf{1}\right\rangle \\
&= \left\langle \mathbb{E}^T\{\Gamma(T,\nu)\}, \mathbf{1}\right\rangle \\
&= \left\langle \Upsilon(T,\nu)\, X(0), \mathbf{1}\right\rangle.
\end{aligned}$$

Therefore under Markovian regime-switching Ornstein-Uhlenbeck model with stochastic interest rate, the price of the futures option is given by:

$$\hat{c}_f(u) = \frac{P(0,T)\, Z(X(T),\nu) \left\langle \Upsilon(T,\nu)\, X(0), \mathbf{1} \right\rangle}{(\alpha - iu)(1+\alpha-iu)}, \quad \nu = -(u + i(1+\alpha)). \tag{4.41}$$

CHAPTER 5: CONCLUSION AND FUTURE DIRECTIONS

This dissertation is concerned with pricing of European-style derivatives such as call, commodity, and futures options under different regime-switching models. Only regime-switching models are considered in this thesis. We assume the Markov chain is observable.

Possible furture research directions are as follows:

One can demonstrate the practicality of the model via numerical examples. It is also possible to apply the techniques developed here to American options. It is also possible to consider the hedging of these products under regime-switching models. Static hedging and dynamic hedging are two main types of hedging. We could investigate the static hedging and the dynamic hedging of standard options, exotic options and insurance products with embedded option features and provide comparisons of these hedging strategies.

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A.1: PROOF FOR THE PAIR OF EQUATIONS (2.40)

Recalling that for the initial state $\alpha(0) = j_0$ we define

$$\phi_{j_0}(\theta, T) = \mathbb{E}(\mathrm{e}^{i\theta T_1} | \alpha(0) = j_0), \quad j_0 = 1, 2.$$

Let $J \sim \exp(\lambda_1)$ be the first time jumping from state 1 to state 2. Then

$$\begin{aligned}
\phi_1(\theta, T) &= \mathbb{E}\left[\mathrm{e}^{i\theta T_1} \middle| \alpha(0) = 1\right] \\
&= \mathbb{E}\left[\mathrm{e}^{i\theta\left(J + \int_J^T \mathbb{I}_{\{\alpha(s)=1\}}\mathrm{d}s\right)} \mathbb{I}_{\{J<T\}} + \mathrm{e}^{i\theta T}\mathbb{I}_{\{J \geq T\}} \middle| \alpha(0) = 1\right] \\
&= \mathbb{E}\left[\mathrm{e}^{i\theta\left(J + \int_J^T \mathbb{I}_{\{\alpha(s)=1\}}\mathrm{d}s\right)} \mathbb{I}_{\{J<T\}} \middle| \alpha(0) = 1\right] + \mathrm{e}^{i\theta T}\mathbb{P}(J \geq T) \\
&= \int_0^T \mathbb{E}\left[\mathrm{e}^{i\theta\left(t + \int_t^T \mathbb{I}_{\{\alpha(s)=1\}}\mathrm{d}s\right)} \middle| \alpha(t) = 2\right] \lambda_1 \mathrm{e}^{-\lambda_1 t}\mathrm{d}t + \mathrm{e}^{i\theta T}\mathrm{e}^{-\lambda_1 T} \\
&= \int_0^T \mathrm{e}^{i\theta T}\lambda_1 \mathrm{e}^{-\lambda_1 t}\mathbb{E}\left[\mathrm{e}^{i\theta \int_t^T \mathbb{I}_{\{\alpha(s)=1\}}\mathrm{d}s} \middle| \alpha(t) = 2\right]\mathrm{d}t + \mathrm{e}^{i\theta T}\mathrm{e}^{-\lambda_1 T} \\
&= \int_0^T \mathrm{e}^{i\theta T}\lambda_1 \mathrm{e}^{-\lambda_1 t}\phi_2(\theta, T-t)\mathrm{d}t + \mathrm{e}^{i\theta T}\mathrm{e}^{-\lambda_1 T}
\end{aligned}$$

Note that since J is exponentially distributed with rate λ_1, hence

$$\mathbb{E}\left[\mathbb{I}_{\{J \geq T\}} \middle| \alpha(0) = 1\right] = \mathbb{P}(J \geq T) = \mathrm{e}^{-\lambda_1 T}.$$

The fourth equality holds by well know property of expectations.

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{i\theta\left(J + \int_J^T \mathbb{I}_{\{\alpha(s)=1\}}\mathrm{d}s\right)} \mathbb{I}_{\{J<T\}} \middle| \alpha(0) = 1\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{i\theta\left(J + \int_J^T \mathbb{I}_{\{\alpha(s)=1\}}\mathrm{d}s\right)} \mathbb{I}_{\{J<T\}} \middle| \alpha(J) = 2\right] \middle| \alpha(0) = 1\right]
\end{aligned}$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{i\theta\left(J+\int_J^T \mathbb{I}_{\{\alpha(s)=1\}}\mathrm{d}s\right)}\mathbb{I}_{\{J<T\}}\middle|\alpha(J)=2\right]\right]$$

$$= \int_0^T \mathbb{E}\left[\mathrm{e}^{i\theta\left(t+\int_t^T \mathbb{I}_{\{\alpha(s)=1\}}\mathrm{d}s\right)}\middle|\alpha(t)=2\right]\lambda_1\mathrm{e}^{-\lambda_1 t}\mathrm{d}t$$

Following the same methodology we will get the second equality for $\phi_2(\theta, T)$. Now let's assume $J \sim \exp(\lambda_2)$ be the first time jumping from state 2 to state 1. Then

$$\begin{aligned}
\phi_2(\theta, T) &= \mathbb{E}\left[\mathrm{e}^{i\theta T_1}\middle|\alpha(0)=2\right] \\
&= \mathbb{E}\left[\mathrm{e}^{i\theta\left(\int_J^T \mathbb{I}_{\{\alpha(s)=1\}}\mathrm{d}s\right)}\mathbb{I}_{\{J<T\}} + \mathrm{e}^{i\theta 0}\mathbb{I}_{\{J\geq T\}}\middle|\alpha(0)=2\right] \\
&= \mathbb{E}\left[\mathrm{e}^{i\theta\left(\int_J^T \mathbb{I}_{\{\alpha(s)=1\}}\mathrm{d}s\right)}\mathbb{I}_{\{J<T\}}\middle|\alpha(0)=2\right] + \mathbb{P}(J \geq T) \\
&= \int_0^T \mathbb{E}\left[\mathrm{e}^{i\theta\left(\int_t^T \mathbb{I}_{\{\alpha(s)=1\}}\mathrm{d}s\right)}\middle|\alpha(t)=1\right]\lambda_2\mathrm{e}^{-\lambda_2 t}\mathrm{d}t + \mathrm{e}^{-\lambda_2 T} \\
&= \int_0^T \lambda_2\mathrm{e}^{-\lambda_2 t}\mathbb{E}\left[\mathrm{e}^{i\theta\int_t^T \mathbb{I}_{\{\alpha(s)=1\}}\mathrm{d}s}\middle|\alpha(t)=1\right]\mathrm{d}t + \mathrm{e}^{-\lambda_2 T} \\
&= \int_0^T \lambda_2\mathrm{e}^{-\lambda_2 t}\phi_1(\theta, T-t)\mathrm{d}t + \mathrm{e}^{-\lambda_2 T}
\end{aligned}$$

Note that since J is exponentially distributed with rate λ_2, hence

$$\mathbb{E}[\mathbb{I}_{\{J\geq T\}}|\alpha(0)=2] = \mathbb{P}(J \geq T) = \mathrm{e}^{-\lambda_2 T}.$$

A.2: PYTHON CODE FOR FFT METHOD IN CHAPTER 2

The Python code used throughout was generated using Python 3.5.2 64bits, Qt 5.6.0, PyQt5 5.6 on Darwin through the Anaconda Navigator 1.3.1 distribution, available at `https://anaconda.org/anaconda/anaconda-navigator/files?version=1.3.1`. I hope this information will help any reader to replicate our result without any trouble.

The following is the code that generated the data seen in Table 2.3 and Tabel 2.4.

```
import numpy as np
from math import pi
from scipy.interpolate import interp1d
from time import time
t0 = time()
class FFT_Euro:
    def __init__(self, m1, m2, r1, r2, sig1, sig2, l1, l2, a, w, T):
        self.m1    = m1
        self.m2    = m2
        self.r1    = r1
        self.r2    = r2
        self.sig1  = sig1
        self.sig2  = sig2
        self.l1    = l1
        self.l2    = l2
        self.a     = a
        self.w     = w
        self.T     = T
    def A(self):
        m1, m2, r1, r2, sig1, sig2, a, w = self.m1, self.m2, \
        self.r1, self.r2, self.sig1, self.sig2, self.a, self.w
```

```
        value = ((m1 - m2) + (0.5 + a) * (sig1**2 - sig2**2)) * -w \
                + 0.5 * 1j * w**2 * (sig1**2 - sig2**2) \
                +  ((r1 - r2) - (1 + a) * (m1 - m2) \
                - 0.5 * a * (1 + a) * (sig1**2 - sig2**2)) * 1j
        return value
    def B(self):
        m2, r2, sig2, a, w = self.m2, self.r2, \
                self.sig2, self.a, self.w
        value = 1j * -w * (m2 + (0.5 + a) * sig2**2) -\
                0.5 * w**2 * sig2**2 + (1 + a) * m2 - \
                r2 + 0.5 * a * (1 + a) * sig2**2
        return value
    def price(self):
        l1, l2, T, a, w = self.l1, self.l2, self.T, self.a, self.w
        A = self.A
        B = self.B
        #-----------------------------------------------
        # Prepare phi function according to
        # equation 2.22-2.23 paper 2006
        #-----------------------------------------------
        s1 = 0.5 * ((1j * A() - l1 - l2) - \
                    np.sqrt((l1 + l2 - 1j * A())**2 \
                    + 4 * 1j * A() * l2))
        s2 = 0.5 * ((1j * A() - l1 - l2) + \
                    np.sqrt((l1 + l2 - 1j * A())**2 \
                    + 4 * 1j * A() * l2))
        phi0 = (1/(s1-s2)) * ((s1 + l1 + l2) * np.exp(s1 * T)\
```

```
        - (s2 + l1 + l2) * np.exp(s2 * T))
phi1 = (1/(s1-s2)) * ((s1 + l1 + l2 - 1j * A())\
  * np.exp(s1 * T) - (s2 + l1 + l2 - 1j * A())\
   * np.exp(s2 * T))
#---------------------------------------------------
# Prepare characteristic function of
# modified price equation 2.24 paper 2006
#---------------------------------------------------
D = np.exp(B()*T) / (a**2 + a - w**2 - (1j * (1 + 2 * a) * w))
q0 = D * phi0
q1 = D * phi1
#---------------------------------------------------
# Prepare a mesh in the frequency (w)
#  & space (k) domain
#---------------------------------------------------
S0 = 100.0
N = int(2**12)
h = 0.1534
dk = (2.0 * pi) / (h * N)
kmin = (-N/2) * dk
kmax = ((N/2)-1) * dk
wmax = (N-1) * h
k = np.linspace(kmin, kmax, N)
w = np.linspace(0, wmax, N)
dw = np.zeros(N)
dw[0] = 1/3
for j in range(1,N):
```

```
    if (j % 2 == 0):
        dw[j] = 2/3
    else:
        dw[j] = 4/3
#----------------------------------------------
# Prepare the A vector for
# Python's FFT implementation
#----------------------------------------------
I = np.zeros(N)
for i in range(N):
    I[i] = i
A_vector0 = np.exp(-1j * I * pi) * q0 * dw * N
A_vector1 = np.exp(-1j * I * pi) * q1 * dw * N
#-----------------------------------------------
# Compute the DFT of A_vector
#  and retrieve its real part
#----------------------------------------------
a_vector0 = np.fft.ifft(A_vector0)
a_vector1 = np.fft.ifft(A_vector1)
a_vector0 = np.real(a_vector0)
a_vector1 = np.real(a_vector1)
#---------------------------------------------
# Convert the a_vector into a
# value vector of European option
#--------------------------------------------
V_vector0 = (h * S0 / pi) * np.exp(-a * k) * a_vector0
V_vector1 = (h * S0 / pi) * np.exp(-a * k) * a_vector1
```

```
        #--------------------------------------------
        # Obtain a continuous value f
        # unction using interpolation
        #--------------------------------------------
        # This is linear interpolation.
        K_vector = S0 * np.exp(k)
        V0 = interp1d(K_vector, V_vector0)
        V1 = interp1d(K_vector, V_vector1)
        return V0, V1, K_vector
if __name__=='__main__':
    l1  = 20.0
    l2  = 30.0
    m1   =  r1  = 0.05
    m2   =  r2  = 0.1
    sig1  = 0.5
    sig2  = 0.3
    T    =  1.0
    S0   = 100.0
    K    = [74.082, 81.873, 90.484, 100.0, 110.517, 122.140, 134.986]
    N    = int(2**12)
    h    = 0.1534
    dk = (2.0 * pi) / (h * N)
    kmin = (-N/2) * dk
    kmax = ((N/2)-1) * dk
    wmax = (N-1)*h
    k = np.linspace(kmin, kmax, N)
    w = np.linspace(0, wmax, N)
```

```
    a = 1.0
    print ('\n   European Call Option Price via FFT method ')
    print ('\n   Strike Price      STATE 1        STATE 2 \n')
    for n in range(7):
        callprice = FFT_Euro(m1, m2, r1, r2, sig1, sig2,\
         l1, l2, a, w, T)
        V0, V1, K_vector = callprice.price()
        print ('%20f %20.10f %20.10f \n' % (K[n]  ,V0(K[n]), V1(K[n])))
tn = time()-t0
print ("\n Duration in seconds %7.3f \n" %tn)
```

A.3: PYTHON CODE FOR MONTE CARLO SIMULATIONS IN CHAPTER 2

The following is the codes that generated the data seen in Table 2.3, Tabel 2.4, Table 2.1 and Tabel 2.2. Monte Carlo algorithm relies on repeated random sampling to obtain numerical results. Its essential idea is using randomness to solve problems that might be deterministic in principleI. As we said before, it's very time consuming versus FFT is less than a second and it's not feasible in most practical use in real time. Other reason makes FFT more useful is that simulating of random variable in MC makes slightly different answers each try, while we always get the same results in FFT.

This code is created for MC simulations with different strike prices in Table 2.3 and Tabel 2.4.

```
import numpy as np
import random
from time import time
t0 = time()
# Given parameters
T = 1.0
S0 = 100.0
Initial_State = 1.0
sig1 = 0.5
sig2 = 0.3
r1 = mu1 = 0.05
r2 = mu2 = 0.1
n = 100000
Strikes = [74.082, 81.873, 90.484, 100.0, 110.517, 122.140, 134.986]
l1 = 20.0
l2 = 30.0
```

```
#np.random.seed(19)
print ('\n  European Call Option Price under RSGBM via MC method ')
print ('\n         Strike Price     STATE 1   \n')
for j in range(7):
    K = Strikes[j]
    SumofTermVals = 0.0
    TermValOneRun = 0.0
    for x  in range(n):
        LogStock = float(np.log(S0))
        Curr_Time = 0.0
        tau1 = 0.0
        Curr_State = Initial_State
        #Determine our occupation time of state 1
        while Curr_Time < T:
            #Determine time until next change of state
            # p(tau_i>t)=exp(-lambda_i * t)
            p = random.uniform(0, 1)
            if Curr_State==1:
                ExpRV = -1*np.log(p)/l1
            else:
                ExpRV = -1*np.log(p)/l2
            #If the next state change is before maturity,
            # increment tau
            if Curr_Time + ExpRV < T and Curr_State==1:
                tau1 = tau1 + ExpRV
            # Else there is no state change between now and maturity
            else:
```

```
                if Curr_State==1:
                    tau1 = tau1 + T - Curr_Time
            #Increment to next switch time
            Curr_Time = Curr_Time + ExpRV
            #Switch State
            if Curr_State==1:
                Curr_State = 2
            else:
                Curr_State = 1
        # Obtain a pseudo-random sample from
        # standard normal distribution
        SimRand = float(np.random.standard_normal(1))
        #Calculate our terminal log stock price
        LogStock = LogStock + (mu1 - 0.5 * sig1**2) * tau1 + \
                    (mu2 - 0.5 * sig2**2) * (T-tau1) \
                    + SimRand*np.sqrt(tau1*sig1**2 \
                 + (T-tau1)*sig2**2)
        #Calculate terminal option value
        TermValOneRun = np.maximum(0, np.exp(LogStock) - K)
        #Add terminal option value running total
        SumofTermVals = SumofTermVals + TermValOneRun
    callprice = (SumofTermVals / n) * np.exp(-1 * ((tau1 * r1 )+...
     r2*(T-tau1)))
    print ('%20f %20.10f \n' % (K  ,callprice))
tn = time()-t0
print ("\n Duration in seconds %7.3f \n" %tn)
```

This code is created for MC simulations with different maturities in Table 2.1 and Tabel 2.2.

```
import numpy as np
import random
from time import time
t0 = time()
# Given parameters
Maturities = [0.1, 0.2, 0.5, 1.0, 2.0, 3.0]
S0 = 100.0
Initial_State = 2.0
sig1 = 0.2
sig2 = 0.3
r1 = mu1 = 0.1
r2 = mu2 = 0.1
n = 100000
K = 90.0
l1 = 1.0
l2 = 1.0
#np.random.seed(19)
print ('\n  European Call Option Price under RSGBM via MC method ')
print ('\n         Maturity     STATE 2   \n')
for j in range(6):
    T = Maturities[j]
    SumofTermVals = 0.0
    TermValOneRun = 0.0
    for x  in range(n):
        LogStock = float(np.log(S0))
```

```
Curr_Time = 0.0

tau1 = 0.0

Curr_State = Initial_State

#Determine our occupation time of state 1

while Curr_Time < T:

    #Determine time until next change of state

    # p(tau_i>t)=exp(-lambda_i * t)

    p = random.uniform(0, 1)

    if Curr_State==1:

        ExpRV = -1*np.log(p)/l1

    else:

        ExpRV = -1*np.log(p)/l2


    #If the next state change is before maturity, increment tau

    if Curr_Time + ExpRV < T and Curr_State==1:

        tau1 = tau1 + ExpRV


    # Else there is no state change between now and maturity

    else:

        if Curr_State==1:

            tau1 = tau1 + T - Curr_Time

    #Increment to next switch time

    Curr_Time = Curr_Time + ExpRV

    #Switch State

    if Curr_State==1:

        Curr_State = 2

    else:
```

```
                Curr_State = 1
        # Obtain a pseudo-random sample from
        # standard normal distribution
        SimRand = float(np.random.standard_normal(1))
        #Calculate our terminal log stock price
        LogStock = LogStock + (mu1 - 0.5 * sig1**2) * tau1 + \
                    (mu2 - 0.5 * sig2**2) * (T-tau1) \
                    + SimRand*np.sqrt(tau1*sig1**2 \
                   + (T-tau1)*sig2**2)
        #Calculate terminal option value
        TermValOneRun = np.maximum(0, np.exp(LogStock) - K)
        #Add terminal option value running total
        SumofTermVals = SumofTermVals + TermValOneRun
    callprice = (SumofTermVals / n) * np.exp(-1 * ((tau1 * r1 )+...
     r2*(T-tau1)))
    print ('%20f %20.10f \n' % (T  ,callprice))
tn = time()-t0
print ("\n Duration in seconds %7.3f \n" %tn)
```

A.4: PYTHON CODE FOR ANALYTICAL PRICES IN CHAPTER 2

The following is the code that generated the data seen in Table 2.1 and Tabel 2.2.

```
# first page

# mu(100,0 , 0, 0.1, 0.2, 0.3, 1, 1/3) = 4.668503519321425

# mu(100,0 , 0, 0.1, 0.2, 0.3, 1, 1)   = 4.685170185988092

# mu(100,0 , 0, 0.1, 0.2, 0.3, 1, 0)   = 4.6601701859880915

# var(0.2, 0.3, 1, 1/3) = 0.07333333333333333

import numpy as np

from math import log, pi

def mu(S0, d0, d1, r, sig0, sig1, T, t):

    value = log(S0)+(d1 - d0 - 0.5 * (sig0**2 - sig1**2)) * t \

            + (r - d1 - 0.5 * sig1**2)*T

    return value

def var(sig0, sig1, T, t):

    value = (sig0**2 - sig1**2) * t + sig1**2 * T

    return value

# second page

# call0(100,0 , 0, 0.1, 0.2, 0.3, 1, 1, 1, 90) =21.075037242396174

# call1(100,0 , 0, 0.1, 0.2, 0.3, 1, 1, 1, 90) = 22.002645343515372

import numpy as np

from math import pi, log

from scipy.integrate import quad

from parameters import mu, var

def integrand0(y, S0, d0, d1, r, sig0, sig1, T, l0, l1, K):

    m0T = mu(S0, d0, d1, r, sig0, sig1, T, T/3)

    mT  = mu(S0, d0, d1, r, sig0, sig1, T, T)

    v0T = var(sig0, sig1, T, T/3)
```

```
    vT = var(sig0, sig1, T, T)
    a = (1/np.sqrt(2*pi*v0T)) * np.exp(-(log(y+K)-m0T)**2 / (2*v0T))
    b = (1/np.sqrt(2*pi*vT)) * np.exp(-(log(y+K)-mT)**2 / (2*vT))
    value =  (y/(y+K)) * (a * (1-np.exp(-l0*T)) +  b * np.exp(-l0*T))
    return value
def call0(S0, d0, d1, r, sig0, sig1, T, l0, l1, K):
    return np.exp(-r*T) * quad(integrand0, 0, np.inf, \
                args=(S0, d0, d1, r, sig0, sig1, T, l0, l1, K))[0]
def integrand1(y, S0, d0, d1, r, sig0, sig1, T, l0, l1, K):
    m0T = mu(S0, d0, d1, r, sig0, sig1, T, T/3)
    m0  = mu(S0, d0, d1, r, sig0, sig1, T, 0)
    v0T = var(sig0, sig1, T, T/3)
    v0  = var(sig0, sig1, T, 0)
    a = (1/np.sqrt(2*pi*v0T)) * np.exp(-(log(y+K)-m0T)**2 / (2*v0T))
    b = (1/np.sqrt(2*pi*v0)) * np.exp(-(log(y+K)-m0)**2 / (2*v0))
    value =  (y/(y+K)) * (a * (1-np.exp(-l1*T)) +  b * np.exp(-l1*T))
    return value
def call1(S0, d0, d1, r, sig0, sig1, T, l0, l1, K):
    return np.exp(-r*T) * quad(integrand1, 0, np.inf, \
                args=(S0, d0, d1, r, sig0, sig1, T, l0, l1, K))[0]
# third page
import numpy as np
from price import call0, call1
from time import time
t0 = time()
def main():
    d0   =  d1  =  0
```

```
    l0   =  l1  =  1.0
    sig0  = 0.2
    sig1  = 0.3
    T    =  [0.1, 0.2, 0.5, 1.0, 2.0, 3.0]
    S0   = 100.0
    K    = 90.0
    r    = 0.1
    V0   = np.zeros((6))
    V1   = np.zeros((6))
    print ('\n European Call Option Price under RS GBM- Analytical Solutions ')
    print ('\n   Strike Price         STATE 1              STATE 2 \n')
    for i in range(6):
        V0[i] = call0(S0, d0, d1, r, sig0, sig1, T[i], l0, l1, K)
        V1[i] = call1(S0, d0, d1, r, sig0, sig1, T[i], l0, l1, K)
        print ('%20f %20.10f %20.10f \n' % (T[i]  ,V0[i], V1[i]))
    tn = time()-t0
    print ("\n Duration in seconds %7.3f \n" %tn)
main()
```

A.5: PYTHON CODE FOR SEMI-MC SIMULATIONS IN CHAPTER 2

This approach only takes random sampling of the Markov chain and then takes advantage of the availability of analytical formula (therefore exact) of the conditional price. Thus semi-Monte Carlo simulation outperforms Monte Carlo method. As we can see the obtained results from FFT and semi-MC simulation are so closed to each other than the results obtained from MC simulations.

The following is the code that generated the data seen in Table 2.1 and Tabel 2.2 for for semi-MC simulations with different maturities. One can simply obtain the data generated in Table 2.3 and Tabel 2.4 for semi-MC simulations with different strike price.

```
from scipy import log, sqrt, exp

import random

from time import time

from scipy.stats import norm

t0 = time()

# Given parameters

Maturities = [0.1, 0.2, 0.5, 1.0, 2.0, 3.0]

S0 = 100.0

Initial_State = 2.0

sig1 = 0.2

sig2 = 0.3

r1 = mu1 = 0.1

r2 = mu2 = 0.1

n = 100000

K = 90.0

l1 = 1.0

l2 = 1.0
```

```
#np.random.seed(19)
print ('\n  European Call Option Price under RSGBM via Semi-MC method ')
print ('\n         Maturity     STATE 2   \n')
for j in range(6):
    T = Maturities[j]
    SumofTermVals = 0.0
    TermValOneRun = 0.0
    for x  in range(n):
        LogStock = float(log(S0))
        Curr_Time = 0.0
        tau1 = 0.0
        Curr_State = Initial_State
        #Determine our occupation time of state 1
        while Curr_Time < T:
            #Determine time until next change of state
            # p(tau_i>t)=exp(-lambda_i * t)
            p = random.uniform(0, 1)
            if Curr_State==1:
                ExpRV = -1*log(p)/l1
            else:
                ExpRV = -1*log(p)/l2
            #If the next state change is before maturity,
            # increment tau
            if Curr_Time + ExpRV < T and Curr_State==1:
                tau1 = tau1 + ExpRV
            # Else there is no state change between now and maturity
            else:
```

```
                if Curr_State==1:
                    tau1 = tau1 + T - Curr_Time
            #Increment to next switch time
            Curr_Time = Curr_Time + ExpRV
           #Switch State
            if Curr_State==1:
                Curr_State = 2
            else:
                Curr_State = 1
        LT =  mu1 * tau1 + mu2 * (T-tau1)
        VT =  sig1**2 * tau1 + sig2**2 * (T-tau1)
        RT =  tau1 * r1 + r2 * (T-tau1)
        d1 = (log(S0/K) + LT + (0.5 * VT)) / sqrt(VT)
        d2 =  d1 - sqrt(VT)
        TermValOneRun = S0 * exp(-(RT - LT)) * norm.cdf(d1)   \
       - K * exp(-RT) * norm.cdf(d2)
        SumofTermVals = SumofTermVals + TermValOneRun
    callprice = (SumofTermVals / n)
    print ('%20f %20.10f \n' % (T  ,callprice))
tn = time()-t0
print ("\n Duration in seconds %7.3f \n" %tn)
```