ESTIMATES FOR THE NUMBER OF EIGENVALUES OF NON-SELF-ADJOINT OPERATORS

by

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A dissertation submitted to the faculty of
The University of North Carolina at Charlotte
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in
Applied Mathematics

Charlotte

2018

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ABSTRACT

ARTEM HULKO. Estimates For the Number of Eigenvalues of Non-self-adjoint Operators. (Under the direction of DR. OLEG SAFRONOV)

In this dissertation we find estimates for the total number of eigenvalues of non-self-adjoint operators. We consider five different operators, three of them discrete and two continuous. Discrete operators are as follows: Schrödinger operator defined on $\mathbb{Z}_+$ with a complex potential, Schrödinger operator defined on $\mathbb{Z}$ with a complex potential, and a Dirac operator defined on $\mathbb{Z}$, also with a complex potential. The latter of which we will also define in this dissertation, as, to the best of our knowledge, it has not yet been defined. Then we also consider a continuous Biharmonic operator on $\mathbb{R}^3$, and then a Polyharmonic operator of order $2l$ on $\mathbb{R}^d$, both perturbed by a complex potential. For each of these operators we will find uniform bounds for the total number of eigenvalues located outside of their continuous spectrums. By 'uniform bounds' we mean bounds which depend on the potential only through some simple quantities like L^p norms.

ACKNOWLEDGMENTS

I would like to express my gratitude to all those without whom this dissertation would not be possible.

First and foremost, I would like to express my deepest gratitude to my adviser, Dr. Oleg Safronov. I would like to thank him for all the hours he has spent passing on to me his knowledge and expertise. Without his help and guidance I would not be able to accomplish the intensive task of writing this dissertation.

I also would like to thank all the members of my dissertation committee, Dr. Boris Vainberg, Dr. Kevin McGoff, and Dr. Donald Jacobs, for their willingness to allocate time to serve on this committee. Moreover, their suggestions, correction and support meant a lot to me during the completion of this dissertation.

Furthermore, I would like to thank my mom, Vira Hulko, for instilling in me a love for education and hard work. I would also like to thank my family and friends at UNC Charlotte for providing their support and encouragement and abundance of heartwarming memories.

I would like to also thank the Department of Mathematics and Statistics here at the UNC Charlotte for the financial support through GASP and for such a great opportunity to be able to get my education here, to prepare for my future endeavors in the field of mathematics.

But above all I would would like to give all the glory to God who made all of this possible!

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CHAPTER 1: INTRODUCTION

In this chapter, we will discuss the general ideas of spectral theory and its applications. We will not go into details, but rather just give the reader some understanding of the basic background of the field.

1.1 Historical note

When we say the word 'spectrum' to regular people (people who are not experts in the fields of Physics, Mathematics or Chemistry) the first thing that usually comes to their minds is one of two things. They either think "a band of colors, as seen in a rainbow, produced by separation of the components of light by their different degrees of refraction according to wavelength" or "wide range" of something. However, when mathematicians hear the word quite often they start thinking of Hilbert spaces, and operators on them. Very often the first thing that comes to their minds is eigenvalues. It is very common for a mathematician to study such concepts separately, without thinking about their physical interpretations. As a result, some students studying these topics, or maybe even some mathematicians might have a false impression that such concepts are completely unrelated to the common meaning of the term that everyone uses. At some point in time it might have been the case, as the study of a vibrating string seems to not have anything to do with the spectrum of light. However, deeper study reveals that they are, in fact, related.

As L.A. Steen once said: "Not least because such different objects as atoms, operators and algebras possess spectra, the evolution of spectral theory is one of the most informative chapters in the history of contemporary mathematics." When we study the origins of the word ‘spectrum’ in mathematics we realize that indeed it does have to do with the spectrum of light. The word “spectrum” means “vision” in Latin. Newton used this word to describe the colors of the rainbow produced on white paper when a beam of light is dispersed by a glass prism. Later, in nineteenth century, infrared and ultra-violet extensions of the spectrum were discovered. By the end of nineteenth century scientists were able to tell, based on the spectrum of the sun and stars (dark rings around them) many of their properties, such as the composition of their atmosphere. This led to a discovery of the chemical unity of the Universe. Similar observations in spectroscopy also appear on a much smaller level in study of atoms and molecules. This gave scientists an ability to find some very interesting results pertaining to atoms and their properties [22].

In the beginning of twentieth century many scientists started deriving results in physics, using some mathematical approaches. In 1930 Norbert Wiener developed a mathematical approach to analyze the spectrum of white light. With the development of quantum mechanics it became evident that the two ‘different spectrums’ are indeed connected. Quantum theory gave a huge push to the development of the spectral theory for unbounded linear operators. In the past one hundred years the study of spectrum by mathematicians and physicists working hand-in-hand became essential to the development of Quantum mechanics and had a huge impact on many other aspects of our lives.

1.2 Spectrum in Mathematics and Known Results

As discussed in the previous section, spectral theory was very intensively studied in the past 80-90 years. In the words of Dr. Stanislav Molchanov “spectral analysis is a very well respected field in mathematics.” This is partly due to the vast amount of applications of the results in the fields of physics and chemistry as well as some others. M. Zworski once said: “Eigenvalues of self-adjoint operators describe, among other things, the energies of bound states, states that exist forever if unperturbed. These do exist in real life[...]. In most situations however, states do not exist for ever, and a more accurate model is given by a decaying state that oscillates at some rate.[...] Eigenvalues are yet another expression of humanity's narcissistic desire for immortality." [22]

Self-adjoint operators have been widely studied in the past 50 years or so. Consequently, some very interesting results have been obtained. The following two nice results are well known and pertain to the continuous self-adjoint case.

Theorem 1.1 (Lieb-Thirring). *Consider $-\Delta + V$ where the potential is real valued. If one of the following is satisfied:*

$$\gamma \geq \frac{1}{2},\, n = 1$$

$$\gamma > 0,\, n = 2$$

$$\gamma \geq 0,\, n \geq 3$$

then one can find a constant $L_{\gamma,n}$ such that

$$\sum_{j\geq 1} |\lambda_j|^\gamma \leq L_{\gamma,n} \int_{\mathbb{R}^n} [V_-(x)]^{\gamma+\frac{n}{2}}\, dx$$

where $V_-(x) := max(-V(x), 0)$.

Theorem 1.2 (Cwikel-Lieb-Rozenblum). *The number of negative eigenvalues, counting multiplicities, of the operator $-\Delta + V$ in $L^2(\mathbb{R}^d)$, $d \geq 3$ satisfies*

$$N(V) \leq C_d \int_{\mathbb{R}^d} [V_-(x)]^{\frac{d}{2}}\, dx$$

for some C_d which depends only on the dimension d.

The non-self-adjoint case, however, has not been studied as much, nor has been the discrete case. As mentioned above, the eigenvalues of the self-adjoint (also called hermitian) operators describe energies of bound states of a quantum system which remain forever if unperturbed, also called physical observables. For example in quantum mechanics, the Schrödinger (Hamiltonian) operator $H = -\Delta + V(x)$ specifies the energy levels and time evolution of a quantum system. While such systems exist in the real word and are most often considered in experiments, due to having real eigenvalues, non-hermitian operators also exist and are also useful. For example in the scattering theory describing an effect of common non-elastic processes on the elastic channel (the optic potential approach). Another thing they are useful for is in a number of pure theoretical investigations of different relations between the physical quantities (e.g., at the conclusion of the Heisenberg uncertainty relations from the basic notions of Quantum Mechanics).

Most of the cases considered by scientists and results obtained deal with the continuous cases. However, as Dr. S. Molchanov stated, presently scientists start realizing that discrete models reflect our universe better than the continuous models, as everything in our universe is finite and discrete. Also, as Dr. Donald Jacobs stated, many times physicists try to come up with a continuous model to solve a problem, but then they have to discretize it in order to plug it into computers, and then try to convert results back to continuous case. As a result many scientists get more and more interested in the discrete operators to create simpler and more accurate models. In this dissertation we will consider both, some discrete and some continuous operators.

1.3 Main influence for our work

In this section I would like to note a result obtained in 2016 by R. L. Frank, A. Laptev, and O. Safronov [15] in which they estimated the total number of eigenvalues of the Shcrödinger operator with a complex potential on $\mathbb{R}^n$ $(n - odd)$. They proved the following two nice theorems:

Theorem 1.3. *The number N of eigenvalues of $-\frac{d^2}{dx^2}+V$ in $L^2(\mathbb{R}_+)$ with a Dirichlet boundary condition, counting algebraic multiplicities, satisfies, for any $\varepsilon > 0$,*

$$N \leq \frac{1}{\varepsilon^2}\left(\int_0^\infty e^{\varepsilon x}|V(x)|dx\right)^2$$

Theorem 1.4. *Let $d \geq 3$ be odd. Then the number N of eigenvalues of $-\Delta + V$ in $L^2(\mathbb{R}^d)$ counting algebraic multiplicities, satisfies, for any $\varepsilon > 0$,*

$$N \leq \frac{C_d}{\varepsilon^2}\left(\int_{\mathbb{R}^d} e^{\varepsilon|x|}|V(x)|^{(d+1)/2}dx\right)^2$$

with a constant C_d depending only on d.

This paper played a huge part in our decision to study this topic. We decided to adopt the same trace formula approach used by them and use it to study other operators. As a result, we get similar nice bounds for two discrete and two continuous cases, which depend on the potential only through some simple quantities.

CHAPTER 2: SOME GENERAL CONCEPTS WHICH WILL BE USED THROUGHOUT THE DISSERTATION

2.1 Some Interesting Results

In 1966 B. Pavlov [25] used the notion of quasi-analyticity to prove that the operator $-d^2/dx^2 + V(x)$ on the half-line $[0, \infty)$ has finitely many eigenvalues if $|V| \leq C \exp(-c\sqrt{x})$ for some C, $c > 0$. . It was established that the eigenvalues cannot accumulate to a point of the positive half- line, which is enough to conclude that the set of all eigenvalues is finite.

On the other hand, In 1967 Pavlov proved another remarkable result (see [25], [26]), which says that, for any $0 < p < 1/2$, there exists a complex-valued potential V satisfying $|V| \leq C \exp(-c|x|^p)$ and a complex number θ, such that the operator $-d^2/dx^2 + V(x)$ with the boundary condition $\psi'(0) = \theta\psi(0)$ has infinitely many eigenvalues. Another interesting result was recently established by Bögli [2]. It was shown that there exists a potential for which the eigenvalues accumulate to every point on $[0, \infty)$.

2.2 Blaschke Product

Here we would like to very briefly introduce the notion of the Blaschke product, which we will be using throughout the following chapters.

Later on in this dissertation we would like to consider a function $\ln(a(k))$. However, the function $a(k)$ contains zeros at some sequence of points a_j. So before we can consider the logarithmic function we need to remove all the zeros of $a(k)$. This can be done with the use of, so called, Blaschke Product. In 1915 Austrian mathematician Wilhelm Johann Eugen Blaschke proved a result that allowed to create a bounded analytic function in open unit disc with a specific set of zeros. Namely he proved the following: If the sequence $\{a_n\}$ satisfies the condition $\sum_n (1-a_n) < \infty$ (Blaschke Condition), then the function $B(z) = \prod_n B(a_n, z)$, where

$$B(a_n, z) = \begin{cases} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - a_n z} & if\ a_n \neq 0 \\ z & if\ a_n = 0 \end{cases}$$

is bounded and analytic in the unit disc, and has zeros (counting multiplicities) at the points a_n exactly. This result can be extended to a disc of any radius R. Furthermore, it can be generalized to any simply connected domain, since it is conformally equivalent to a disc. So in the later chapters we will apply this result to come up with a Blaschke Product whose zeros coincide with the zeros of our function $a(k)$. As a result, after we divide $a(k)$ by $B(k)$, the quotient will be analytic and nonzero everywhere in our circle of radius R.

2.3 Birman–Schwinger Principle

We will state the Birman–Schwinger Principle in the case where H_0 is a bounded self-adjoint operator and $V = G^* G_0$. We will also assume that G_0 and G are compact operators. Now, set

$$H = H_0 + V.$$

The Birman–Schwinger principle states that $z \in \rho(H_0)$ is an eigenvalue of H if and only if -1 is an eigenvalue of the Birman–Schwinger operator $G_0(H_0 - z)^{-1} G^*$. Moreover, the corresponding geometric multiplicities coincide. And the operator $G_0(H_0 - \zeta)^{-1} G^*$ is called Birman–Schwinger operator.

The following lemma says that even the algebraic multiplicities of eigenvalues of H

can be characterizes in terms of a quantity related to the Birman–Schwinger operator. Though before we do that, we would like to define the $n-th$ regularized determinant

$$\det{}_n(I+K(k)), \quad k \notin \sigma(H_0)$$

for $n \geq 2$. The standard way to describe $\det{}_n(I+K)$ in terms of eigenvalues z_j of a compact operator $K \in \mathfrak{S}_n$ is to define it as

$$\det{}_n(I+K) = \prod_j (1+z_j)\exp\Big(\sum_{m=1}^{n-1}\frac{(-1)^m z_j^m}{m}\Big), \qquad n \geq 2;$$

$$\det(I+K) = \prod_j (1+z_j), \qquad n = 1.$$

(Note that the geometric multiplicity of an eigenvalue λ corresponds to the dimension of the eigenspace corresponding to λ, whereas the algebraic multiplicity is the number that eigenvalue repeats and corresponds to the dimension of the invariant subspace corresponding to λ.)

Lemma 2.1. *Let $n \in \mathbb{N}$. Assume that $G_0(H_0-\zeta)^{-1}G^* \in \mathfrak{S}_n$ for all $\zeta \in \rho(H_0)$. Then the function $\zeta \mapsto \det_n(1+G_0(H_0-\zeta)^{-1}G^*)$ is analytic in $\rho(H_0)$. A point $z \in \rho(H_0)$ is an eigenvalue of H if and only if $\det_n(1+G_0(H_0-z)^{-1}G^*) = 0$. Moreover, the order of the zero coincides with the algebraic multiplicity of the corresponding eigenvalue.*

Analyticity of the function $\zeta \mapsto \det_n(1+G_0(H_0-\zeta)^{-1}G^*)$ is well-known (see, e.g., [29]), as well as the result about the algebraic multiplicity in the case $n = 1$. The result for the general n is essentially due to [20]; you may also refer to [13] for an extension of the proof to the present setting.

2.4 Classes of compact operators and determinants

Let $1 \le p < \infty$. We say that a compact operator T belongs to the Schatten class $\mathfrak{S}_p$ if its singular values $s_j(T)$ satisfy

$$\|T\|_{\mathfrak{S}_p}^p := \sum_j s_j^p(T) < \infty.$$

The functional $\|\cdot\|_{\mathfrak{S}_p}$ is the norm on $\mathfrak{S}_p$.

The following property is well-known, but we include a proof for the sake of completeness.

Lemma 2.2. *Let $n \in \mathbb{N}$ and let $K \in \mathfrak{S}_n$. Then*

$$\ln|\det_n(1+K)| \le \Gamma_n \|K\|_{\mathfrak{S}_n}^n ,$$

where Γ_n is a positive constant independent of K. In particular,

$$\Gamma_1 = 1 \quad \textit{and} \quad \Gamma_2 = 1/2 . \tag{2.1}$$

Proof. To prove the lemma, let $f(z) := (1+z)\exp\left(\sum_{m=1}^{n-1} \frac{(-1)^m}{m} z^m\right)$. Then $\ln|f(z)|$ can be bounded by a constant times $|z|^n$ for small $|z|$ and by a constant times $|z|^{n-1}$ for large $|z|$. Thus, $\ln|f(z)| \le \Gamma_n |z|^n$, and so

$$\ln|\det_n(1+K)| \le \Gamma_n \sum_j |\lambda_j(K)|^n$$

By Weyl's inequality [28, Thm. 1.15], the sum on the right side does not exceed $\|K\|_{\mathfrak{S}_n}^n$. A simple computation shows that for $n=1$ and $n=2$ one can take $\Gamma_1 = 1$ and $\Gamma_2 = 1/2$, respectively (see [29]). $\square$

CHAPTER 3: DISCRETE SCHRÖDINGER OPERATOR ON $\mathbb{Z}_+$

In this chapter we look at the discrete Schrödinger operator with a complex potential. We obtain bounds on the total number of eigenvalues in the case where V decays exponentially at infinity.

3.1 Introduction and Main Results

Let $\mathfrak{H} = \ell^2(\mathbb{Z}_+)$ be the Hilbert space of square summable sequences on $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$. Let $V : \mathfrak{H} \mapsto \mathfrak{H}$ be the operator of multiplication by a bounded complex-valued function on $\mathbb{Z}_+$. We study the spectral properties of the Schrödinger operator H, defined in $\mathfrak{H}$ by

$$(H\,u)_j = \sum_{|l-j|=1} u_l + V_j u_j, \qquad \forall j \geq 2. \tag{3.1}$$

Additionally, we set

$$(Hu)_1 = u_2 + V_1 u_1.$$

Note that H is a bounded operator. The spectrum of the self-adjoint operator $H_0 = H - V$ coincides with the interval $[-2, 2]$ and is absolutely continuous. This is due to the fact that the Schrödinger operator, as we defined it here, is unitary equivalent to the operator on multiplication by the $2\cos(p)$ function on the $L^2([-\pi, \pi))$. That is $H_0 = F^{-1}[2\cos(p)]F$, where F is a unitary operator such that $F : \ell^2(\mathbb{Z}) \to L^2([-\pi, \pi))$. Consequently, the range of the function $2\cos(p)$ will coincide with the continuous spectrum of the free Schrödinger operator H_0.

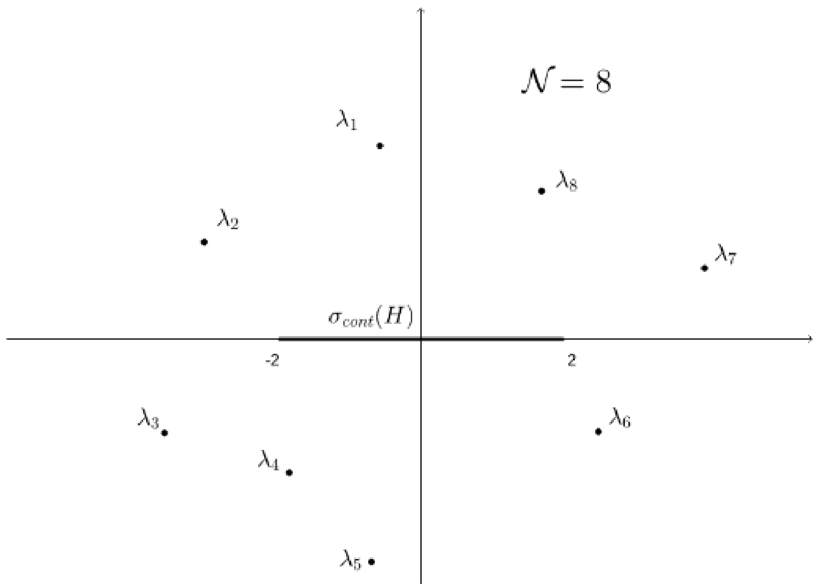

Figure 3.1: Example of spectrum of discrete Schrödinger operator

Let λ_j denote the eigenvalues of the operator (3.1). We are interested in an estimate of the total number $\mathcal{N}$ of eigenvalues λ_j in the case where the sequence V_j decays exponentially fast at infinity. Refer to figure 3.1.

More precisely, we shall prove the following two theorems:

Theorem 3.1. *The number $\mathcal{N}$ of eigenvalues of H in $\ell^2(\mathbb{Z}_+)$, counting algebraic multiplicities, satisfies*

$$\mathcal{N} \leq \frac{1}{2\ln\Lambda}\left(\frac{2\Lambda^2}{\Lambda^2-1}\sum_{n=1}^{\infty}\Lambda^{2n}|V_n|\right)^2,$$

for any $\Lambda > 1$.

A similar result for a continuous operator was proved in [15] by Frank, Laptev and Safronov.

We also establish a slightly different estimate:

Theorem 3.2. *The number $\mathcal{N}$ of eigenvalues of H in $\ell^2(\mathbb{Z}_+)$, counting algebraic* *multiplicities, satisfies*

$$\mathcal{N} \leq \frac{1}{\ln \Lambda} \frac{\Lambda^2}{(\Lambda^2 - 1)} \left(\sum_{n=1}^{\infty} \Lambda^n |V_n|^{1/2} \right)^2 ,$$

for any $\Lambda > 1$.

Note that the right hand sides of both estimates can be finite only in the case where V is an exponentially decaying potential. Moreover, we choose Λ in such a way that the summation converges. For example, if $V_n = q^{2n}$, $q \in (0, 1)$, then we choose Λ in such a way that $\Lambda|q| < 1$. In this case the summations become the geometric series and so our estimate in Theorem 3.1 will become:

$$\mathcal{N} \leq \frac{1}{2 \ln \Lambda} \left(\frac{2\Lambda^2}{\Lambda^2 - 1} \right)^2 \frac{(\Lambda q)^2}{1 - (\Lambda q)^2}.$$

Similarly our estimate in Theorem 3.2 will become:

$$\mathcal{N} \leq \frac{1}{\ln \Lambda} \frac{\Lambda^2}{(\Lambda^2 - 1)} \frac{\Lambda|q|}{1 - \Lambda|q|}.$$

Doing so will allow us to find Λ_{min} for each q, which will give us the smallest estimate for $\mathcal{N}$. For example, if we consider the case $q = 1/2$, then from the estimate in Theorem 3.2 we get $\Lambda_{min} \approx 1.3351$ and $\mathcal{N} \leq 32$. However, Theorem 3.1 gives us a better estimate: $\Lambda_{min} \approx 1.3765$ and $\mathcal{N} \leq 23$.

It turns out that $\mathcal{N}$ might be finite even in the case when the potential decays slower. For instance, as discussed in Section 2.1, the operator $-d^2/dx^2 + V(x)$ on the half-line $[0, \infty)$ has finitely many eigenvalues if $|V| \leq C \exp(-c\sqrt{x})$ for some C, $c > 0$.

3.2 Zeros of analytic functions

The following proposition gives a useful bound on the zeros of an analytic function in the compliment of the disc of radius $R > 0$.

Proposition 3.3. *Let $0 < R < 1$. Let $a(\cdot)$ be an analytic function in $\{k : |k| > R\}$. Assume that $a(\cdot)$ is continuous up to the boundary and satisfies*

$$a(k) = 1 + O(|k|^{-1}) \qquad \text{as } |k| \to \infty \text{ in } \{k : |k| > R\}. \tag{3.2}$$

Assume also that for some $A \geq 1$,

$$|a(k)| \leq A, \qquad \text{if } |k| = R\,. \tag{3.3}$$

Then the zeros k_j of $a(\cdot)$ in $\{k : |k| > R\}$, repeated according to their multiplicities, satisfy

$$\prod_j \left(\frac{|k_j|}{R} \right) \leq A. \tag{3.4}$$

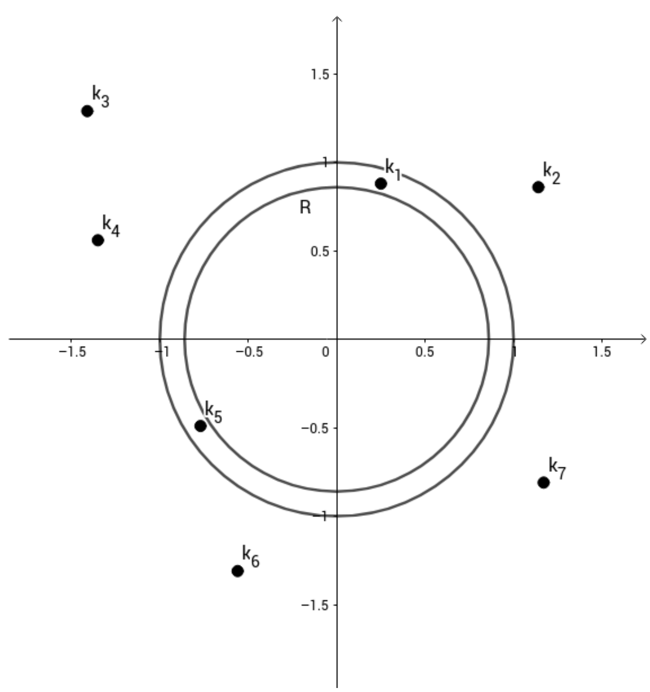

Figure 3.2: Zeros of analytic function outside of circle of radius R

Proof. We introduce the Blaschke product

$$B(k) = \prod_j \frac{k - k_j}{R - R^{-1}\overline{k_j}k}\,.$$

Clearly, $a(k)/B(k)$ is an analytic and non-zero in $\{k:\ |k| > R\}$. Consequently, $\log(a(k)/B(k))$ exists and is analytic in $\{k:\ |k| > R\}$. Let C_R denote the circle $\{k \in \mathbb{C}:\ |k| = R\}$, traversed counterclockwise.

Then, according to the residue calculus,

$$\int_{C_R} \log\frac{a(k)}{B(k)}\,\frac{dk}{k} = 2\pi i \lim_{k\to\infty} \log\frac{a(k)}{B(k)} = 2\pi i \sum_j \log\frac{\bar{k}_j}{-R}\,,$$

and therefore

$$\int_{-\pi}^{\pi} \log\frac{a(Re^{i\varphi})}{B(Re^{i\varphi})}\,d\varphi = 2\pi \sum_j \log\frac{\bar{k}_j}{-R}\,. \tag{3.5}$$

We note that $|B(Re^{i\varphi})| = 1$ if $\varphi \in \mathbb{R}$ and, therefore,

$$\mathrm{Re}\int_{-\pi}^{\pi} \log\frac{a(Re^{i\varphi})}{B(Re^{i\varphi})}\,d\varphi = \int_{-\pi}^{\pi} \ln\Big|\frac{a(Re^{i\varphi})}{B(Re^{i\varphi})}\Big|\,d\varphi = \int_{-\pi}^{\pi} \ln|a(Re^{i\varphi})|\,d\varphi. \tag{3.6}$$

On the other hand,

$$\mathrm{Re}\sum_j \log\frac{\bar{k}_j}{-R} = \sum_j \ln\frac{|k_j|}{R}. \tag{3.7}$$

We conclude from (3.5), (3.6) and (3.7) that

$$\int_{-\pi}^{\pi} \ln|a(Re^{i\varphi})|\,d\varphi = 2\pi \sum_j \ln\frac{|k_j|}{R}. \tag{3.8}$$

Finally, by (3.3),

$$\int_{-\pi}^{\pi} \ln|a(Re^{i\varphi})|\,d\varphi \le 2\pi \ln A,\,. \tag{3.9}$$

Inequality (3.4) now follows from (3.8) and (3.9). □

Corollary 3.4. *Let* $0 < R < 1$*. Let* $a(\cdot)$ *be an analytic function in* $\{k: \ |k| > R\}$ *satisfying* (3.2)*. Assume that, for any* $R' > R$ *sufficiently close to* R*, condition* (3.3) *holds with* R *replaced by* R'*. Then the number*

$$\mathcal{N} := \#\{j : |k_j| \geq 1\}$$

of zeros k_j *of* $a(\cdot)$ *in* $\{k : |k| \geq 1\}$*, repeated according to their multiplicities, satisfies*

$$\mathcal{N} \leq \frac{\ln A}{\ln 1/R} \,.$$

Proof. We apply Proposition 3.3 for every $R' > R$ sufficiently close to R and obtain

$$\sum_j \left(\ln|k_j| - \ln R'\right)_+ \leq \ln A$$

Clearly, we have

$$\sum_j \left(\ln|k_j| - \ln R'\right)_+ \geq |\ln R'| \cdot \# \left\{j : \ |k_j| \geq 1\right\} \,.$$

Consequently,

$$|\ln R'| \cdot \mathcal{N} \leq \ln A.$$

The corollary follows by passing to the limit $R' \to R$. □

3.3 Resolvent bounds

In this section we collect trace ideal bounds for the Birman–Schwinger operator

$$K(k) = \sqrt{V}(H_0 - z)^{-1}\sqrt{|V|}, \qquad z = k + k^{-1}, \quad |k| \geq 1 \,. \tag{3.10}$$

We use the notation $\sqrt{V(x)} = V(x)/\sqrt{|V(x)|}$ if $V(x) \neq 0$ and $\sqrt{V(x)} = 0$ if $V(x) = 0$.

We remind the reader that $\mathfrak{H} = \ell^2(\mathbb{Z}_+)$, and H_0 in (4.9) denotes the free Jacobi operator on $\mathbb{Z}_+$. From the explicit expression of its matrix it is easy to see that, if V has a compact support, then $K(k)$ admits an analytic continuation to $\mathbb{C} \setminus \{0\}$. The following proposition gives a bound on the Hilbert–Schmidt norm.

Proposition 3.5. *For any $k \in \mathbb{C} \setminus \{0\}$ with $|k| < 1$,*

$$\|K(k)\|_{\mathfrak{S}_2} \leq \frac{2}{1 - |k|^2} \sum_{n=1}^{\infty} |k|^{-2n} |V_n| \,,$$

Proof. The matrix of $(H_0 - z)^{-1}$ is given by

$$g_k(n,m) = \frac{k}{k^2 - 1} \left(k^{-|n-m|} - k^{-(n+m)} \right) \,,$$

which satisfies

$$|g_k(n,m)| \leq \frac{2}{1 - |k|^2} |k|^{-(n+m)} \,.$$

Combining this bound with the identity

$$\|K(k)\|^2_{\mathfrak{S}_2} = \sum_{1}^{\infty} \sum_{1}^{\infty} |V_n| |g_k(n,m)|^2 |V_m|$$

we obtain the claimed bound. □

Proposition 3.6. *For any $k \in \mathbb{C} \setminus \{0\}$ with $|k| < 1$,*

$$\|K(k)\|_{\mathfrak{S}_1} \leq \frac{2}{1 - |k|^2} \Big(\sum_{n=1}^{\infty} |k|^{-n} |V_n|^{1/2} \Big)^2 \,,$$

Proof. The matrix of $(H_0 - z)^{-1}$ is defined by

$$g_k(n,m) = \frac{k}{k^2-1}\left(k^{-|n-m|} - k^{-(n+m)}\right) ,$$

which satisfies

$$|g_k(n,m)| \leq \frac{2}{1-|k|^2}|k|^{-(n+m)} .$$

Combining this bound with the identity

$$\|K(k)\|_{\mathfrak{S}_1} \leq \sum_1^\infty \sum_1^\infty |V_n|^{1/2} |g_k(n,m)| |V_m|^{1/2}$$

we obtain the claimed bound. □

3.4 Proof of Theorem 3.1

In this section we prove Theorem 3.1. Let us assume that V has compact support. The bound in this case implies the bound in the general case by a simple continuity argument.

As discussed in Section 4.2.1, the Birman–Schwinger operators $K(k)$ from (4.9) extends analytically to $\mathbb{C} \setminus \{0\}$. The same proof shows that they are not only analytic with respect to the norm of bounded operators, but even with respect to the norm in $\mathfrak{S}_2$.

We will apply Corollary 4.4 to the function

$$a(k) := \det_2(1 + K(k))$$

with $\Lambda = 1/R$. Since $K(k)$ is analytic with values in $\mathfrak{S}_2$, the function a is analytic. It is easy to see that assumption (3.2) is valid. Moreover, combining them with Lemma

2.2, we see that assumption (3.3) holds with

$$\ln A = \frac{1}{2}\left(\frac{2\Lambda^2}{\Lambda^2-1}\sum_{n=1}^{\infty}\Lambda^{2n}|V_n|\right)^2 .$$

Thus, Corollary 4.4 implies that

$$\#\{j:\ \operatorname{Im} k_j \geq 0\} \leq \frac{1}{2\ln\Lambda}\left(\frac{2\Lambda^2}{\Lambda^2-1}\sum_{n=1}^{\infty}\Lambda^{2n}|V_n|\right)^2 .$$

It remains to use Lemma 2.1, which says that the $k_j + k_j^{-1}$, with $|k_j| > 1$, coincide with the eigenvalues of H, counting algebraic multiplicities. This proves Theorem 3.1.

3.5 Proof of Theorem 3.2

In this section we prove Theorem 3.2. Let us assume again that V has compact support.

As discussed in Section 4.2.1, the Birman–Schwinger operators $K(k)$ from (4.9) extend analytically to $\mathbb{C}\setminus\{0\}$. The same proof shows that they are not only analytic with respect to the norm of bounded operators, but even with respect to the norm in $\mathfrak{S}_1$.

We apply Corollary 4.4 to the function

$$a(k) := \det{}_1(1+K(k)) = \det(1+K(k))$$

with $\Lambda = 1/R$. Since $K(k)$ is analytic with values in $\mathfrak{S}_1$, the function a is analytic. Assumption (3.3) holds with

$$\ln A = \frac{2\Lambda^2}{\Lambda^2-1}\left(\sum_{n=1}^{\infty}\Lambda^{n}|V_n|^{1/2}\right)^2 .$$

Thus, Corollary 4.4 implies that

$$\#\{j:\ \mathrm{Im}\ k_j \geq 0\} \leq \frac{1}{\ln\Lambda}\frac{2\Lambda^2}{\Lambda^2-1}\left(\sum_{n=1}^{\infty}\Lambda^n|V_n|^{1/2}\right)^2 .$$

It remains to use Lemma 2.1, which says that the $k_j + k_j^{-1}$, with $|k_j| > 1$, coincide with the eigenvalues of H, counting algebraic multiplicities. This proves Theorem 3.2. □

CHAPTER 4: DISCRETE SCHRÖDINGER OPERATOR ON $\mathbb{Z}$

In this chapter we estimate the number of eigenvalues for the discrete Schrödinger operator with complex potential on $\mathbb{Z}$. That is, we extend the result obtained in the previous chapter to the whole $\mathbb{Z}$. Many of the results we acquire in this section will be very similar to those in Chapter 2.

4.1 Introduction and Main Results

We consider the Hilbert space $\mathfrak{H} = \ell^2(\mathbb{Z}, \mathbb{C})$. Let the Schrödinger operator H be defined on $\mathfrak{H}$ as follows:

$$(H\,u)_j = \sum_{|l-j|=1} u_l + V_j u_j, \tag{4.1}$$

where $H = H_0 + V$. The spectrum of the self-adjoint operator $H_0 = H - V$ coincides with the interval $[-2, 2]$ and is absolutely continuous. Moreover, the sequence V_j decays exponentially fast. The following two theorems extend Theorems 3.1 and 3.2, obtained in last chapter, to the whole $\mathbb{Z}$.

Theorem 4.1. *The number $\mathcal{N}_S$ of eigenvalues of H in $\ell^2(\mathbb{Z})$, counting algebraic multiplicities, satisfies*

$$\mathcal{N}_S \leq \frac{1}{2\ln\Lambda}\left(\frac{\Lambda^2}{\Lambda^2-1}\sum_{n=-\infty}^{\infty}\Lambda^{2|n|}|V_n|\right)^2 + 2,$$

for any $\Lambda > 1$.

Theorem 4.2. *The number $\mathcal{N}_S$ of eigenvalues of H in $\ell^2(\mathbb{Z})$, counting algebraic* *multiplicities, satisfies*

$$\mathcal{N}_S \leq \frac{1}{\ln\Lambda}\frac{\Lambda^2}{(\Lambda^2-1)}\left(\sum_{n=-\infty}^{\infty}\Lambda^{|n|}|V_n|^{1/2}\right)^2+2,$$

for any $\Lambda > 1$.

4.2 Preliminary Results

Before we can prove theorems above, however, we will need to prove a few preliminary results. We consider space $\mathfrak{H} = \ell^2(\mathbb{Z})$, and let H_0 denote the free Jacobi operator on $\mathbb{Z}$. The results we obtain in this section will be similar to those obtained in Chapter 3 and will be used in both, this chapter and Chapter 5 in estimating Schrödinger and Dirac operators.

The first result will be very similar to Proposition 3.3 and next result will resemble very closely Corollary 3.4 in previous Chapter.

Proposition 4.3. *Let* $0 < R < 1$. *Let* $a(\cdot)$ *be a function in the circle* $\Omega := \{k : |k| > R\}$ *of the form* $a(k) = a_0(k)e^{\frac{ck}{k^2-1}}$, *where* $a_0(k)$ *is meromorphic and has poles at* $k = \pm 1$ *of order* n, *and* $c \in \mathbb{R}$. *Also assume that* $a(\cdot)$ *and satisfies*

$$a(k) = 1 + O(|k|^{-1}) \qquad \text{as } |k| \to \infty \text{ in } \Omega = \{k: \ |k| > R\}. \tag{4.2}$$

Assume also that for some $A \geq 1$,

$$|a(k)| \leq A, \qquad \text{if } |k| = R\,. \tag{4.3}$$

Then the zeros k_j *of* $a(\cdot)$ *in* Ω, *repeated according to their multiplicities, satisfy*

$$R^{2n}\prod_j\left(\frac{|k_j|}{R}\right) \leq A. \tag{4.4}$$

Proof. The idea of the proof of this Proposition will be similar to that of the proof of Proposition 3.3 in Chapter 3. We want to look at the function $\log(a(k)) = \log\left[a_0(k)e^{\frac{ck}{k^2-1}}\right] = \log[a_0(k)] + \frac{ck}{k^2-1}$, but $\log[a_0(k)]$ in not analytic in Ω. To make it analytic we need to get rid of all zeros and poles in $a_0(k)$. To do so, as discussed in Chapter 2, we introduce a Blaschke product as follows

$$B(k) = \left(\prod_j \frac{k-k_j}{R-R^{-1}\overline{k_j}k}\right)\frac{(R-R^{-1}k)^n}{(k-1)^n}\frac{(R+R^{-1}k)^n}{(k+1)^n}.$$

As a result, the function $\log\left[a_0(k)/B(k)\right]$ exists and is analytic in Ω, $|B(k)| = 1$ on $C_R := \partial\Omega = \{k : |k| = R\}$. Then

$$\log\left[\frac{a_0(k)}{B(k)}\right] = \alpha_0 + \frac{\alpha_1}{k} + \frac{\alpha_2}{k^2} + \cdots \Rightarrow \frac{1}{k}\log\left[\frac{a_0(k)}{B(k)}\right] = \frac{\alpha_0}{k} + \frac{\alpha_1}{k^2} + \frac{\alpha_2}{k^3} + \cdots .$$

By residue calculus we get the following:

$$\int_{C_R}\log\left[\frac{a(k)}{B(k)}\right]\frac{dk}{k} = \int_{C_R}\left(\log\left[\frac{a_0(k)}{B(k)}\right] + \frac{ck}{k^2-1}\right)\frac{dk}{k} = 2\pi i\alpha_0 + 0 = 2\pi i\alpha_0, \quad (4.5)$$

since $\int_{C_R}\frac{c}{k^2-1}dk = 0$ for $R < 1$.
So to calculate the integral we just compute α_0 as follows:

$$\begin{aligned}\alpha_0 = \lim_{k\to\infty}\log\left[\frac{a(k)}{B(k)}\right] &= \lim_{k\to\infty}\log\left[\frac{a_0(k)}{B(k)}\right] = \log\left[\lim_{k\to\infty}\frac{a_0(k)}{B(k)}\right] \\ &= \log\left[\left(\prod_j\frac{-\overline{k_j}}{R}\right)(-1)^n R^{2n}\right]\end{aligned} \quad (4.6)$$

As a result, from 4.5 and 4.6, we get:

$$\int_{C_R}\log\left[\frac{a(k)}{B(k)}\right]\frac{dk}{k} = 2\pi i\log\left[\left(\prod_j\frac{-\overline{k_j}}{R}\right)(-1)^n R^{2n}\right]$$

After the change of variables $k = R\,e^{i\varphi}$ the equation above becomes

$$\begin{aligned}\int_0^{2\pi} \ln\left|\frac{a\,(R\,e^{i\varphi})}{B\,(R\,e^{i\varphi})}\right| d\varphi &= 2\pi \ln\left|(-1)^n R^{2n}\left(\prod_j \frac{-\overline{k_j}}{R}\right)\right| \\ &= 2\pi \ln\left[R^{2n}\left(\prod_j \frac{|\overline{k_j}|}{R}\right)\right]\end{aligned} \tag{4.7}$$

On the other hand we have the following estimate

$$\int_0^{2\pi} \ln\left|\frac{a\,(R\,e^{i\varphi})}{B\,(R\,e^{i\varphi})}\right| d\varphi \leq 2\pi \ln A\,, \tag{4.8}$$

since $|B(k)| = 1$ and $|a(k)| \leq A$ on C_R.

Inequality 4.4 follows from 4.7 and 4.8. □

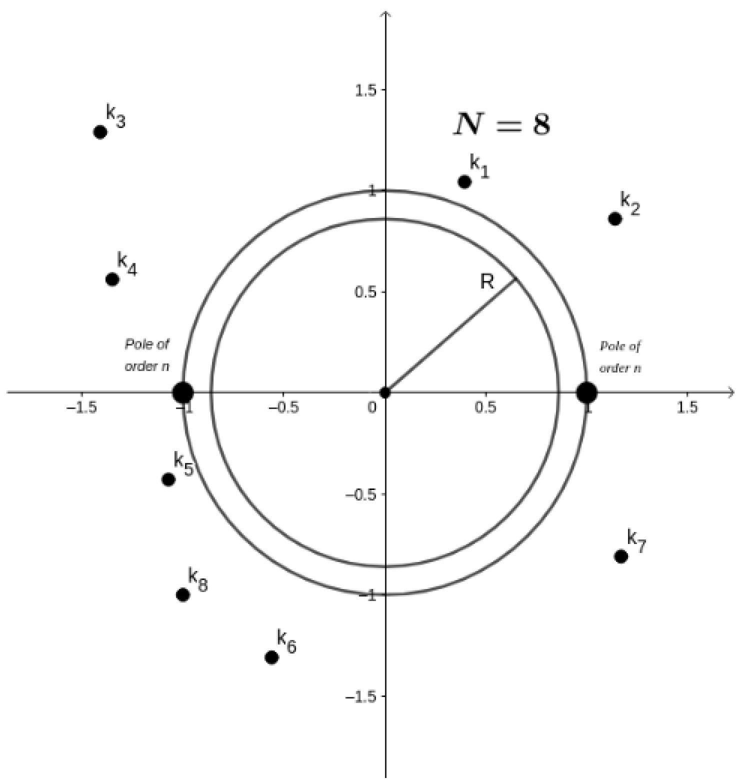

Figure 4.3: Zeros of $a(k)$ with poles of order n at ± 1

Corollary 4.4. *Let $0 < R < 1$. Let $a(\cdot)$ be a meromorphic function in $\{k : |k| > R\}$ with poles of order n at $k = \pm 1$ and satisfying (4.2). Assume that, for any $R' > R$ sufficiently close to R, condition (4.3) holds with R replaced by R'. Then the number*

$$\mathcal{N} := \#\{j : |k_j| \geq 1\}$$

of zeros k_j of $a(\cdot)$ in $\{k : |k| \geq 1\}$, repeated according to their multiplicities, satisfies

$$\mathcal{N} \leq \frac{\ln A}{\ln 1/R} + 2n\,.$$

Please refer to figure 4.3 for the help with visual presentation of this result.

Proof of this Corollary is analogous to the proof of Corollary 3.4 in Chapter 3.

4.2.1 Resolvent bounds

In this section we collect trace ideal bounds for the Birman–Schwinger operator

$$K(k) = \sqrt{V}(H_0 - z)^{-1}\sqrt{|V|}, \qquad z = k + k^{-1}, \quad |k| \geq 1\,. \tag{4.9}$$

Where $\sqrt{V}$ is defined as $\sqrt{V(x)} = \dfrac{V(x)}{\sqrt{|V(x)|}}$ if $V(x) \neq 0$ and $\sqrt{V(x)} = 0$ if $V(x) = 0$.

Please recall that the space is $\mathfrak{H} = \ell^2(\mathbb{Z})$, and H_0 in (4.9) denotes the free Jacobi operator on $\mathbb{Z}$. From the matrix representation of V it is easy to see that, if V is compactly supported, then $K(k)$ admits an analytic continuation to $\mathbb{C} \setminus \{0\}$. The following propositions give bounds on the Hilbert–Schmidt norm of $K(k)$.

Proposition 4.5. *For any $k \in \mathbb{C} \setminus \{0\}$ with $|k| < 1$,*

$$\|K(k)\|_{\mathfrak{S}_2} \leq \frac{1}{1 - |k|^2} \sum_{n=-\infty}^{\infty} |k|^{-2|n|} |V_n|\,,$$

Proof. The kernel of $(H_0 - z)^{-1}$ is given by

$$g_k(n,m) = \frac{-k}{k^2-1}\left(k^{-|n-m|}\right) .$$

The equation above can be estimated as follows:

$$|g_k(n,m)| \leq \frac{1}{1-|k|^2}|k|^{-(|n|+|m|)} .$$

After plugging the estimation above into the identity

$$\|K(k)\|_{\mathfrak{S}_2}^2 = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} |V_n||g_k(n,m)|^2|V_m|$$

we obtain the desired bound. □

Proposition 4.6. *For any $k \in \mathbb{C} \setminus \{0\}$ with $|k| < 1$,*

$$\|K(k)\|_{\mathfrak{S}_1} \leq \frac{1}{1-|k|^2}\Big(\sum_{n=-\infty}^{\infty} |k|^{-|n|}|V_n|^{1/2}\Big)^2 ,$$

Proof. The kernel of $(H_0 - z)^{-1}$ is defined by

$$g_k(n,m) = \frac{-k}{k^2-1}\left(k^{-|n-m|}\right) .$$

The equation above can be estimated as follows:

$$|g_k(n,m)| \leq \frac{1}{1-|k|^2}|k|^{-(|n|+|m|)} .$$

After plugging the estimation above into the identity

$$\|K(k)\|_{\mathfrak{S}_1} \leq \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} |V_n|^{1/2}|g_k(n,m)||V_m|^{1/2}$$

we obtain the desired bound. □

Note that in the case of $\mathbb{Z}_+$ in Chapter 3 the kernel does not have poles at ± 1. This means that the $\det_1(1+K)$ does not have poles, and, as a result, Theorems 3.1 and 3.2 do not have the "+2" term.

4.3 Proofs of Theorems

Before proving the theorem we would like to note that the class of potentials for which $k = \pm 1$ are poles of $\det_1(1+X)$ forms a dense subset of the space of potentials for which $\|V\|_{\infty,q} < \infty$. This fact allows us to apply Proposition 4.3 to the functions $a(k) := \det_1(1+K(k))$ and $a(k) := \det_2(1+K(k))$ in proofs of all the consequent theorems in this chapter and chapter 5.

Proof. Suppose V is compactly supported. The Birman–Schwinger operators $K(k)$ from (4.9) can be extended analytically to $\mathbb{C} \setminus \{0\}$, as discussed in Section 4.2.1. The same proof shows that the operators are not only meromorphic with respect to the infinity norm, but even with respect to the norm in $\mathfrak{S}_2$.

We will apply Corollary 4.4 to the function $a(k) := \det_2(1+K(k))$, then we get the following estimates:

$$
\begin{aligned}
\ln|a(K)| &\le \frac{1}{2}\|K\|_{\mathfrak{S}_2}^2 && \text{by Lemma 2.2} \\
&\le \frac{1}{2}\left(\frac{1}{1-|k|^2}\sum_{n=-\infty}^{\infty}|k|^{-2|n|}|V_n|\right) && \text{by Proposition 4.5} \\
&\le \frac{1}{2}\left(\frac{1}{1-R^2}\sum_{n=-\infty}^{\infty}R^{-2|n|}|V_n|\right) && \text{since } |k| > R
\end{aligned}
$$

So plugging the estimate above into result in Corollary 4.4 for $\ln(A)$ and setting

$\Lambda = 1/R$ we get the following result

$$\#\{j: \ \mathrm{Im}\ k_j \geq 0\} \leq \frac{1}{2\ln\Lambda}\left(\frac{\Lambda^2}{\Lambda^2-1}\sum_{n=-\infty}^{\infty}\Lambda^{2|n|}|V_n|\right)^2 + 2\,.$$

Now, by Lemma 2.1 for $|k_j| > 1$, the $k_j + k_j^{-1}$ coincide with the eigenvalues of H, counting algebraic multiplicities.

The established bound for compact V can be easily extended to the general case with the use of the continuity argument. Hence the result in Theorem 4.1 holds. □

The proof of Theorem 4.2 will be almost identical to the proof of Theorem 4.1 with the few slight differences. If we consider $a(k) := \det_1(1+K(k)) = \det(1+K(k))$ instead of $a(k) := \det_2(1+K(k))$, substitute $\|K\|_{\mathfrak{S}_1}^1$ for $\|K\|_{\mathfrak{S}_2}^2$, and apply Proposition 4.6 in place of Proposition 4.5, the result will follow.

CHAPTER 5: DISCRETE DIRAC OPERATOR ON $\mathbb{Z}$

In this chapter we establish similar bounds for the discrete Dirac operator. Such an operator is well known in the continuous case, however, in the discrete case it has not been defined yet (or at least we do not know if it has). In this paper, we define a discrete Dirac operator on the Hilber space $\mathfrak{H} = \ell^2(\mathbb{Z}, \mathbb{C}^2)$. We then show that the spectrum of the free Dirac operator coincides with $[-\sqrt{5}, -1] \cup [1, \sqrt{5}]$ and is absolutely continuous. Moreover, we use results from Chapter 4 to establish the upper bound for the number of eigenvalues $\mathcal{N}_D$ of the operator in the case where the potential V decays exponentially at infinity.

5.1 Introduction and Main Results

Let $\mathfrak{H} = \ell^2(\mathbb{Z}, \mathbb{C}^2)$ be the Hilbert space of square summable sequences of two-dimensional comples vectors on $\mathbb{Z}$. Let $V : \mathfrak{H} \mapsto \mathfrak{H}$ be the operator of multiplication by a bounded complex-valued function on $\mathbb{Z}$. We define the free Dirac operator on $\mathfrak{H}$ by

$$D_0 = \begin{bmatrix} 1 & S-1 \\ S^*-1 & -1 \end{bmatrix}.$$

Here the operator S is the shift operator in $\ell^2(\mathbb{Z}, \mathbb{C})$ and is defined as follows:

$$(Su)_n = u_{n-1}.$$

And finally we define the operator D_V to be $D_V = D_0 + V$.

In this chapter we prove the following two theorems regarding the Dirac operator on $\mathbb{Z}$:

Theorem 5.1. *The number $\mathcal{N}_D$ of eigenvalues of D_V in $\ell^2(\mathbb{Z},\mathbb{C}^2)$, counting algebraic multiplicities, satisfies*

$$\mathcal{N}_D \leq \frac{\left(\|V\|_{\infty,q}\left(4\Lambda^{1+\frac{\varepsilon}{2}}+2\sqrt{2}\right)+\|V\|^2_{\infty,q}\right)^2}{\ln\Lambda}\left(\frac{\Lambda^2}{\Lambda^2-1}\right)^2\left(\frac{\Lambda^\varepsilon+1}{\Lambda^\varepsilon-1}\right)^2+4,$$

where Λ is any constant greater than 1, $q=\dfrac{1}{\Lambda^{2+\varepsilon}}$, and $\|V\|_{\infty,q}=\sup\limits_{-\infty<n<\infty}|V_n q^{-|n|}|$.

Theorem 5.2. *The number $\mathcal{N}_D$ of eigenvalues of D_V in $\ell^2(\mathbb{Z},\mathbb{C}^2)$, counting algebraic multiplicities, satisfies*

$$\mathcal{N}_D \leq \frac{\|V\|_{\infty,q}\left(4\Lambda^{1+\frac{\varepsilon}{2}}+2\sqrt{2}+\|V\|_{\infty,q}\right)}{\ln\Lambda}\frac{\sqrt{2}\Lambda^2}{\Lambda^2-1}\left(\frac{\Lambda^{\varepsilon/2}+1}{\Lambda^{\varepsilon/2}-1}\right)^2+4,$$

where Λ is any constant greater than 1, $q=\dfrac{1}{\Lambda^{2+\varepsilon}}$, and $\|V\|_{\infty,q}=\sup\limits_{-\infty<n<\infty}|V_n q^{-|n|}|$.

Here we keep in mind that $\Lambda > 1$ in such a way that the infinity-q norm is finite. This means that in the case when the potential V is of the form $V_n = \dfrac{1}{k}$ then $\Lambda \in (1, (k)^{\frac{1}{2+\varepsilon}})$.

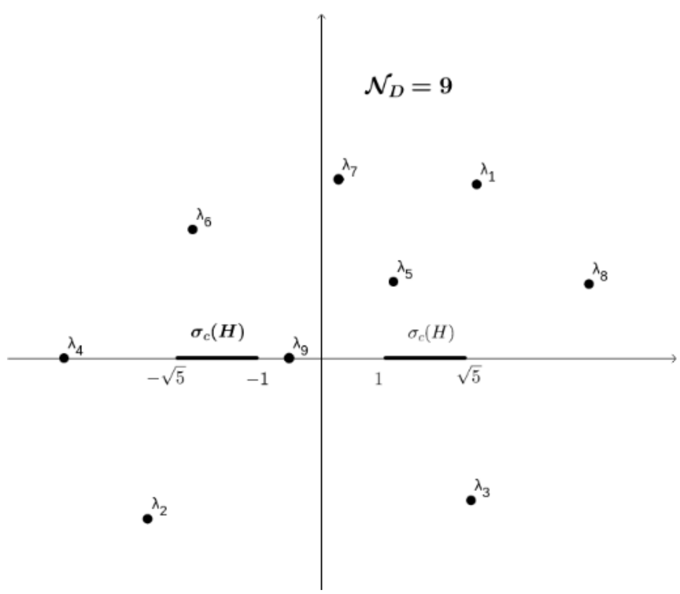

Figure 5.4: Example of spectrum of a discrete Dirac operator

Also note that the additional "+4" term is related to the number of edges of the spectrum of the free Dirac operator. Under small perturbations of D_0, the eigenvalues of D_V appear near the edges of the continuous spectrum (see Figure 5.4) .

5.1.1 Independence of two results

One may claim that Theorem 5.2 implies Theorem 5.1, since the results are very similar. At first glance we see that the numerator over the $\ln(\Lambda)$ in the Theorem 5.1 is squared, as opposed to it being linear in the Theorem 5.2. This may provide a false impression to draw such a conclusion. However, after some analysis, it can be seen that it is not the case. In other words, Theorem 5.1 does not imply Theorem 5.2, nor does Theorem 5.2 imply Theorem 5.1. And so the two results are independent of each other.

To demonstrate this we will look at the ratios of the two estimates. For the simplicity of notation let $C_* = \|V\|_{\infty,q}$, N_{D1} be the estimate in Theorem 5.1 and N_{D2} be the estimate from Theorem 5.2. Then

$$\begin{aligned}\frac{N_{D1}}{N_{D2}} &= \frac{\sqrt{2}\Lambda^2 C_*(4\Lambda^{1+\varepsilon/2}+2+C_*)(\Lambda^\varepsilon+1)^2}{(\Lambda^2-1)(\Lambda^{\varepsilon/2}+1)^4} \\ &\leq \frac{\sqrt{2}\Lambda^2 C_*(4\Lambda^{1+\varepsilon/2}+2\Lambda^{1+\varepsilon/2}+C_*\Lambda^{1+\varepsilon/2})(\Lambda^\varepsilon+1)^2}{(\Lambda^2-1)(\Lambda^\varepsilon+1)^2} \\ &= \frac{\sqrt{2}\Lambda^{3+\varepsilon/2}C_*(6+C_*)}{\Lambda^2-1}\end{aligned} \tag{5.1}$$

Note that if the last ratio is less than 1 then we get that the estimate in the Theorem 5.1 is better that the estimate from Theorem 5.2. So now we will find a condition for C_* which will guarantee this. In order for the ratio to be less than 1 we need to have the following:

$$C_*^2 + 6C_* < \frac{\Lambda^2-1}{\sqrt{2}\Lambda^{3+\varepsilon/2}}$$

$$\Rightarrow C_*^2 + 6C_* - \frac{\Lambda^2-1}{\sqrt{2}\Lambda^{3+\varepsilon/2}} < 0$$

Note that the left hand side of the inequality above is a quadratic function of C_*. This means that in order for the inequality to be satisfied we get the following condition:

$$0 < C_* < -3 + \left(9 + \frac{\Lambda^2 - 1}{\sqrt{2}\Lambda^{3+\varepsilon/2}}\right)^{1/2}$$

So whenever the above condition is satisfied then the estimate in the Theorem 5.1 is better that the estimate from Theorem 5.2.

Similarly we look at the ratio $\dfrac{N_{D2}}{N_{D1}}$:

$$\begin{aligned}
\frac{N_{D2}}{N_{D1}} &= \frac{(\Lambda^2-1)(\Lambda^{\varepsilon/2}+1)^4}{\sqrt{2}\Lambda^2 C_*(4\Lambda^{1+\varepsilon/2}+2+C_*)(\Lambda^{\varepsilon}+1)^2} \\
&\le \frac{\Lambda^2(\Lambda^{\varepsilon}+1)^4}{\sqrt{2}\Lambda^2 C_*(4\Lambda^{1+\varepsilon/2}+2+C_*)(\Lambda^{\varepsilon}+1)^2} \\
&= \frac{(\Lambda^{\varepsilon}+1)^4}{\sqrt{2}C_*(4\Lambda^{1+\varepsilon/2}+2+C_*)(\Lambda^{\varepsilon}+1)^2} \\
&\le \frac{(\Lambda^{\varepsilon}+1)^2}{\sqrt{2}C_*(4\Lambda^{1+\varepsilon/2}+2+C_)} \\
&\le \frac{4\Lambda^{2\varepsilon}}{\sqrt{2}C_*(6+C_*)} \le \frac{2\sqrt{2}\Lambda^{2\varepsilon}}{C_*(6+C_*)}
\end{aligned} \tag{5.2}$$

Now again we want to know when the ration is less than 1, as in that case the estimate obtained by the Theorem 5.2 is better than the one given by Theorem 5.1. And so again we will find a condition on C_* which will make this to be true:

$$C_*^2 + 6C_* > 2\sqrt{2}\Lambda^{2\varepsilon}$$

$$\Rightarrow C_*^2 + 6C_* - 2\sqrt{2}\Lambda^{2\varepsilon} > 0$$

And again we get the left hand side of the inequality to be a quadratic function of C_*. So in order for the inequality to hold we get the following condition on C_* :

$$C_* > \frac{-6 + (36 + 8\sqrt{2}\Lambda^{2\varepsilon})^{1/2}}{2}$$

So for all C_* which satisfy the inequality above we get that the estimate given by Theorem 5.2 is better than the estimate given by Theorem 5.1. (i.e. $N_{D2} < N_{D1}$)

5.1.2 Correlation between the Dirac Operator to Schrödinger Operator

After we square the free Dirac Operator D_0 we get the following:

$$D_0^2 = \begin{bmatrix} 1 & S-1 \\ S^*-1 & -1 \end{bmatrix} \begin{bmatrix} 1 & S-1 \\ S^*-1 & -1 \end{bmatrix} = \begin{bmatrix} 3-(S+S^*) & 0 \\ 0 & 3-(S+S^*) \end{bmatrix} = 3 - H_0,$$

where H_0 is the free discrete Schrödinger operator on $\mathbb{Z}$.

5.1.3 Continuous spectrum of Dirac operator

We know from Chapter 3 that the absolutely continuous spectrum of the H_0 is $\sigma_c(H_0) = \sigma_c(-H_0) = [-2, 2]$. Then the continuous spectrum for $3 - H_0$ is $\sigma_c(3 - H_0) = [1, 5]$. As a result, the continuous spectrum of D_0 then must correspond to $\sigma_c(D_0) = [-\sqrt{5}, -1] \cup [1, \sqrt{5}]$.

This can also be seen from the fact that the shift operator is unitary equivalent to the operator of multiplication by the function e^{-ip} on $L^2([-\pi, \pi))$. That is, $S = F^{-1}[e^{-ip}]F$, where F is a unitary operator mapping $\ell^2(\mathbb{Z})$ onto $L^2([-\pi, \pi))$. Then $S^* = F^{-1}[e^{ip}]F$. So we can express the free Dirac operator as follows:

$$\begin{bmatrix} I & S-I \\ S^*-I & -I \end{bmatrix} = F^{-1} \begin{bmatrix} 1 & e^{-ip}-1 \\ e^{ip}-1 & -1 \end{bmatrix} F$$

Then the characteristic equation of the multiplication operator is:

$$(1-\lambda)(-1-\lambda) - (e^{-ip}-1)(e^{ip}-1) = 0$$

$$\Rightarrow (1-\lambda^2) + (1 + 1 - 2\cos(p)) = 0$$

$$\Rightarrow \lambda^2 = 1 + (2 - 2\cos(p)) = 3 - 2\cos(p)$$

$\Rightarrow \lambda_1(p) = \sqrt{3 - 2\cos(p)},\ \lambda_2(p) = -\sqrt{3 - 2\cos(p)}$

So as p runs through $[-\pi, \pi)$ the continuous spectrum of the Dirac operator runs through $[-\sqrt{5}, -1] \cup [1, \sqrt{5}]$.

Next, if we look at the square of the operator D_V, we will get

$$D_V^2 = (D_0 + V)^2 = D_0^2 + D_0 V + V D_0 + V^2 = (3 - H_0) + Q\,,$$

where $Q = D_0 V + V D_0 + V^2$.

5.2 Estimation for the Dirac Operator

In this section we will prove our estimations for the Dirac operator D_V.

Let us first recall that $D_V^2 = (3 - H_0) + Q$, where $Q = D_0 V + V D_0 + V^2$. We then apply the Birman-Schwinger principle in the regular case to conclude that the number of eigenvalues, λ, of D_V s.t. $\lambda \in \rho(D_0)$ corresponds to the number of zeros of the function $d(k) = det_p(1 + Q(H_0 - \lambda)^{-1})$, for any $p \in \mathbb{N}$.

For our further estimations it will be usefull to define a new operator W as follows: $W_n = q^{\frac{|n|}{2}}$, for some $q \in (0, 1)$. Another useful result is that for any operators A and B the equality $\sigma_p(AB) = \sigma_p(BA)$ holds. As a result, we can rewrite the equation as follows:

$$\begin{aligned}
Q(H_0 - z)^{-1} &= \frac{1}{W} Q(H_0 - z)^{-1} W \\
&= \frac{1}{W}(D_0 V + V D_0 + V^2)(H_0 - z)^{-1} W \\
&= \frac{1}{W} D_0 V \frac{1}{W} W (H_0 - z)^{-1} W + \frac{1}{W} V D_0 \frac{1}{W} W (H_0 - z)^{-1} W \\
&\qquad + \frac{1}{W} V^2 (H_0 - z)^{-1} W
\end{aligned}$$

It is also known that for any two Schatten class operators S_1 and S_2, their Schatten norm can be estimated by $\|S_1 S_2\|_{\mathfrak{S}_p} \le \|S_1\|_\infty \|S_2\|_{\mathfrak{S}_p}$. Applying this property we get the following estimate:

$$
\begin{aligned}
\left\|Q(H_0-z)^{-1}\right\|_{\mathfrak{S}_p} &= \left\| \begin{matrix} \frac{1}{W}D_0V\frac{1}{W}W(H_0-z)^{-1}W + \frac{1}{W}VD_0\frac{1}{W}W(H_0-z)^{-1}W \\ +\frac{1}{W}V^2(H_0-z)^{-1}W \end{matrix} \right\|_{\mathfrak{S}_p} \\
&\leq \left\|\frac{1}{W}D_0V\frac{1}{W}\right\|_\infty \left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_p} \\
&\qquad + \left\|\frac{1}{W}VD_0\frac{1}{W}\right\|_\infty \left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_p} \\
&\qquad + \left\|\frac{1}{W}V^2\frac{1}{W}\right\|_\infty \left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_p} \\
&= \left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_p} \left(\begin{matrix} \left\|\frac{1}{W}D_0V\frac{1}{W}\right\|_\infty + \left\|\frac{1}{W}VD_0\frac{1}{W}\right\|_\infty \\ + \left\|\frac{1}{W}V^2\frac{1}{W}\right\|_\infty \end{matrix} \right) \qquad (\star)
\end{aligned}
$$

Now, applying Proposition 4.5 while substituting V in the proposition with W^2, we get the following:

$$
\left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_2} \leq \frac{1}{1-|k|^2}\sum_{n=-\infty}^{\infty}|k|^{-2|n|}q^{|n|} \tag{5.3}
$$

Similarly, applying Proposition 4.6 while substituting V with W^2, we get

$$
\left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_1} \leq \frac{1}{1-|k|^2}\left(\sum_{n=-\infty}^{\infty}|k|^{-|n|}q^{\frac{|n|}{2}}\right)^2 \tag{5.4}
$$

To estimate $\left\|\frac{1}{W}D_0V\frac{1}{W}\right\|_\infty$ we decompose the operator D_0 into the sum of two operators T_S and T_0:

$$
D_0 = \begin{bmatrix} 1 & S-1 \\ S^*-1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = T_S + T_0
$$

Then $\left\|\frac{1}{W}D_0V\frac{1}{W}\right\|_\infty \leq \left\|\frac{1}{W}T_SV\frac{1}{W}\right\|_\infty + \left\|\frac{1}{W}T_0V\frac{1}{W}\right\|_\infty$. Notice that the eigenvalues of T_0 are $\pm\sqrt{2}$ and T_0 is self-adjoint, so $\|T_0\|_\infty = \sqrt{2}$. As a result, since T_0 does not depend on W, we get the inequality,

$$\left\|\frac{1}{W}T_0V\frac{1}{W}\right\|_\infty \leq \|T_0\|_\infty \left\|\frac{V}{W^2}\right\|_\infty = \sqrt{2}\left\|\frac{V}{W^2}\right\|_\infty = \sqrt{2}\sup_n |V_n q^{-|n|}| := \sqrt{2}\,\|V\|_{\infty,q} \ .$$

The estimation of $\left\|\frac{1}{W}T_SV\frac{1}{W}\right\|_\infty$, however, will be a little more complicated:

We remind the reader that $W = \{W_n\}$ is acting on the Hilbert space $\mathfrak{H} = \ell^2(\mathbb{Z}, \mathbb{C}^2)$. Then for every $\xi = \begin{Bmatrix} \alpha_n \\ \beta_n \end{Bmatrix}_{n\in\mathbb{Z}} \in \mathfrak{H}\,,\ W\xi = \begin{Bmatrix} W_n\alpha_n \\ W_n\beta_n \end{Bmatrix}_n = \begin{Bmatrix} q^{\frac{|n|}{2}}\alpha_n \\ q^{\frac{|n|}{2}}\beta_n \end{Bmatrix}_n$.

Furthermore

$$T_SW\xi = \begin{Bmatrix} q^{\frac{|n-1|}{2}}\alpha_{n-1} \\ q^{\frac{|n+1|}{2}}\beta_{n+1} \end{Bmatrix}_n \quad \text{and} \quad \frac{1}{W}T_SW\xi = \begin{Bmatrix} \dfrac{q^{\frac{|n-1|}{2}}}{q^{\frac{|n|}{2}}}\alpha_{n-1} \\ \dfrac{q^{\frac{|n+1|}{2}}}{q^{\frac{|n|}{2}}}\beta_{n+1} \end{Bmatrix}_n .$$

As a result, we get the following estimate for $\left\|\frac{1}{W}T_SW\right\|_\infty$:

$$\left\|\frac{1}{W}T_SW\right\|_\infty \leq \sup_n \left|\frac{q^{\frac{|n-1|}{2}}}{q^{\frac{|n|}{2}}}\right| + \sup_n \left|\frac{q^{\frac{|n+1|}{2}}}{q^{\frac{|n|}{2}}}\right| = q^{\frac{-1}{2}} + q^{\frac{-1}{2}} = 2q^{\frac{-1}{2}} .$$

Similarly, we get the estimate for $\left\|WT_S\frac{1}{W}\right\|_\infty$ to be

$$\left\|WT_S\frac{1}{W}\right\|_\infty \leq \sup_n \left|\frac{q^{\frac{|n|}{2}}}{q^{\frac{|n-1|}{2}}}\right| + \sup_n \left|\frac{q^{\frac{|n|}{2}}}{q^{\frac{|n+1|}{2}}}\right| = q^{\frac{-1}{2}} + q^{\frac{-1}{2}} = 2q^{\frac{-1}{2}} .$$

Hence

$$\left\|\frac{1}{W}D_0V\frac{1}{W}\right\|_\infty = \left\|\frac{1}{W}VD_0\frac{1}{W}\right\|_\infty = \left\|\frac{V}{W^2}\right\|_\infty \left(\left\|\frac{1}{W}T_SW\right\|_\infty + \sqrt{2}\right) \tag{5.5}$$

$$= \|V\|_{\infty,q}\left(2q^{\frac{-1}{2}} + \sqrt{2}\right) \tag{5.6}$$

The only thing left for us to do is to note that

$$\left\|\frac{V^2}{W^2}\right\|_\infty \le \left\|\frac{V^2}{W^4}\right\|_\infty \le \left\|\frac{V}{W^2}\right\|_\infty^2 = \|V\|_{\infty,q}^2 \ .$$

As a result, from all the estimates above, we get the following estimate for the Schatten p-norm of the operator $Q(H_0-z)^{-1}$:

$$\begin{aligned}\left\|Q(H_0-z)^{-1}\right\|_{\mathfrak{S}_p} &\le \left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_p}\left(\begin{array}{c}\left\|\frac{1}{W}D_0V\frac{1}{W}\right\|_\infty + \left\|\frac{1}{W}VD_0\frac{1}{W}\right\|_\infty \\ +\left\|\frac{1}{W}V^2\frac{1}{W}\right\|_\infty\end{array}\right)\\ &\le \left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|_{\infty,q}^2\right)\left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_p}\end{aligned} \tag{5.7}$$

5.2.1 Proof of Theorems 5.1 and 5.2

We will apply similar technique to the one we applied in the proof of Theorem 4.1.

Proof of Theorem 5.1. Suppose V is compactly supported. The Birman–Schwinger operators $K(k)$ from (4.9) can be extended analytically to $\mathbb{C}\setminus\{0\}$, as discussed in Section 4.2.1. The same proof shows that the operators are not only meromorphic with respect to the infinity norm, but even with respect to the norm in $\mathfrak{S}_2$.

We will apply Corollary 4.4 with $n=2$ to the function $a(k) := \det_2(1+Q(H_0-z)^{-1})$ to get the following estimates:

$$\begin{aligned}\ln|a(K)| &\le \frac{1}{2}\|Q(H_0-z)^{-1}\|_{\mathfrak{S}_2}^2\\ &\qquad\text{by Lemma 2.2}\\ &\le \frac{1}{2}\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|_{\infty,q}^2\right)^2\left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_2}^2\\ &\qquad\text{by inequality 5.7}\\ &\le \frac{1}{2}\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|_{\infty,q}^2\right)^2\left(\frac{1}{1-|k|^2}\sum_{n=-\infty}^{\infty}|k|^{-2|n|}q^{|n|}\right)^2\end{aligned}$$

by (5.3)

$$\leq \frac{1}{2}\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|^2_{\infty,q}\right)^2\left(\frac{1}{1-R^2}\sum_{n=-\infty}^{\infty}R^{-2|n|}q^{|n|}\right)^2$$

since $|k| > R$

$$\leq \frac{1}{2}\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|^2_{\infty,q}\right)^2\left(\frac{1}{1-R^2}\left(\frac{2}{1-qR^{-2}}-1\right)\right)^2$$

Note that in order for the last inequality to make sense we must enforce the following restriction on $q:\ \ q = R^{2+\varepsilon}$, $\varepsilon > 0$. So plugging the estimate above into result in Corollary 4.4 for $\ln(A)$ and setting $\Lambda = 1/R$ yields the following inequality:

$$\#\{j:\ \mathrm{Im}\ k_j \geq 0\} \leq \frac{\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|^2_{\infty,q}\right)^2}{2\ln\Lambda}\left(\frac{\sqrt{2}\Lambda^2}{\Lambda^2-1}\right)^2\left(\frac{\Lambda^\varepsilon+1}{\Lambda^\varepsilon-1}\right)^2+4\,.$$

The extra $\sqrt{2}$ in front of the $\dfrac{\Lambda^2}{\Lambda^2-1}$ term is due to the fact that H_0, is an orthogonal sum of two operators (i.e. it is a matrix opeator), as can be seen in subsection 5.1.2. Now, by Lemma 2.1 for $|k_j| > 1$, the $k_j + k_j^{-1}$, coincide with the eigenvalues of H, counting algebraic multiplicities.

The established bound for compact V can be easily extended to the general case with the use of the continuity argument. Hence the result in Theorem 5.1 holds. □

The proof for Theorem 5.2 will be very similar to the proof of Theorem 5.1.

Proof of Theorem 5.2. Suppose V is compactly supported. The Birman–Schwinger operators $K(k)$ from (4.9) can be extended analytically to $\mathbb{C} \setminus \{0\}$, as discussed in Section 4.2.1. The same proof shows that the operators are not only meromorphic with respect to the infinity norm, but even with respect to the norm in $\mathfrak{S}_1$.

We will apply Corollary 4.4 with $n = 2$ to the function $a(k) := \det_1(1 + Q(H_0 -$

$z)^{-1})$, then we get the following estimates:

$$\begin{aligned}
\ln|a(K)| \leq\ & \|Q(H_0-z)^{-1}\|_{\mathfrak{S}_1}, \quad \text{by Lemma 2.2} \\
\leq\ & \left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|_{\infty,q}^2\right)\left\|W(H_0-z)^{-1}W\right\|_{\mathfrak{S}_1} \\
& \text{by inequality 5.7} \\
\leq\ & \left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|_{\infty,q}^2\right)\frac{1}{1-|k|^2}\left(\sum_{n=-\infty}^{\infty}|k|^{-|n|}q^{\frac{|n|}{2}}\right)^2 \\
& \text{by inequality 5.4} \\
\leq\ & \left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|_{\infty,q}^2\right)\frac{1}{1-R^2}\left(\sum_{n=-\infty}^{\infty}R^{-|n|}q^{\frac{|n|}{2}}\right)^2 \\
& \text{since } |k|>R \\
\leq\ & \frac{1}{2}\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|_{\infty,q}^2\right)^2\frac{1}{1-R^2}\left(\frac{2}{1-q^{\frac{1}{2}}R^{-1}}-1\right)^2
\end{aligned}$$

So after plugging the estimate above into result in Corollary 4.4 for $\ln(A)$ and setting $\Lambda = 1/R$ we get the following result

$$\#\{j:\ \mathrm{Im}\ k_j \geq 0\} \leq \frac{\left(2\|V\|_{\infty,q}\left(2q^{\frac{-1}{2}}+\sqrt{2}\right)+\|V\|_{\infty,q}^2\right)}{\ln\Lambda}\frac{\sqrt{2}\Lambda^2}{\Lambda^2-1}\left(\frac{\Lambda^{\varepsilon/2}+1}{\Lambda^{\varepsilon/2}-1}\right)^2+4\,.$$

Now, by Lemma 2.1 for $|k_j|>1$, the $k_j+k_j^{-1}$, coincide with the eigenvalues of H, counting algebraic multiplicities.

The established bound for compact V can be easily extended to the general case with the use of the continuity argument. Hence the result in Theorem 5.2 holds. □

CHAPTER 6: BIHARMONIC OPERATOR ON $\mathbb{R}^3$

In this chapter we study the biharmonic operator on $\mathbb{R}^3$ with a complex potential, which decays exponentially at infinity. We obtain bounds on the total number of eigenvalues of the said operator.

6.1 Introduction and Main Results

We consider the operator

$$H = (-\Delta)^2 + V(x), \qquad x \in \mathbb{R}^3$$

with a complex valued exponentially decaying potential V. We obtain an estimate for the total number $\mathcal{N}_B$ of eigenvalues of the operator H in the complex plane $\mathbb{C}$, minus the positive real line.

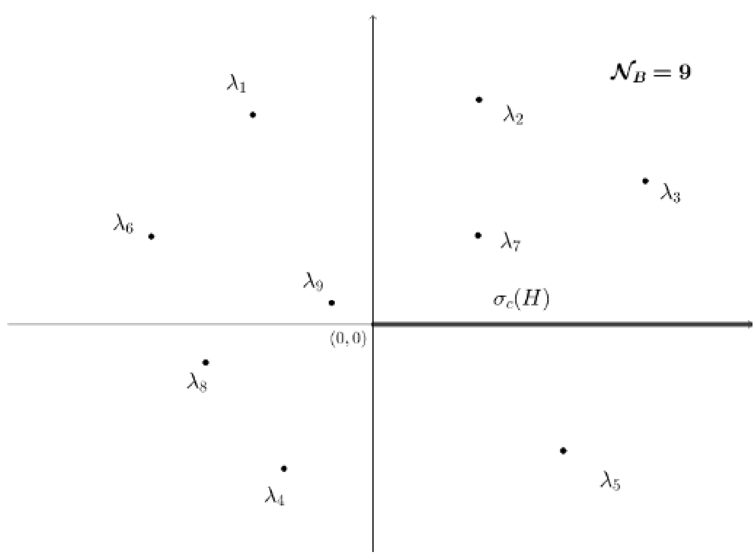

Figure 6.5: Spectrum of a biharmonic operator with a complex potential

Our work is motivated by a recent result of Frank, Laptev and Safronov [15] which gives a similar estimate for the number of eigenvalues of a Schrödinger operator. While Shrödinger operators are the most intensively studied operators in mathematical physics, polyharmonic operators of higher order have also been considered, as they too have some interesting applications.

We assume that V is a measurable function such that the integral

$$\|V\|_{1,\varepsilon} := \int_{\mathbb{R}^3} |V(x)| e^{\varepsilon|x|} dx < \infty \tag{6.1}$$

is finite. In this case, the operator H can be defined in the sense of quadratic forms as it was done in the paper of Laptev and Safronov [19]. We do not give the details of this definition, simply because they are quite standard. Note only, that the domain of the operator H is contained in the Sobolev space $\mathcal{H}^2(\mathbb{R}^3)$. However it depends on the potential V, which does not have to be bounded. It was also shown by Laptev and Safronov [19] that it is enough to establish the eigenvalue bounds for compactly supported smooth potentials V, since they could be extended to the general case by the limit procedure.

For the following theorem we will need to introduce a few things. First let $\mathfrak{S}_2$ denote the class of Hilbert-Schmidt operators and $\|\cdot\|_{\mathfrak{S}_2}$ denote the norm in this space, i.e.

$$\mathfrak{S}_2 = \{T : \quad \operatorname{tr} T^*T < \infty\}, \qquad \|T\|^2_{\mathfrak{S}_2} = \operatorname{tr} T^*T.$$

Theorem 6.1. *Let $\varepsilon > 0$ and let V satisfy (7.1). Then the number $\mathcal{N}_B$ of eigenvalues of H in $L^2(\mathbb{R}^3)$, counting algebraic multiplicities, located outside of the essential spectrum, satisfies*

$$\mathcal{N}_B \le \frac{1}{\varepsilon^3}\left(\frac{\varepsilon^2}{4\pi}\|V\|_{1,\varepsilon} + \frac{\varepsilon+2}{64\pi^2}\|V\|^2_{1,\varepsilon} + \frac{3\gamma}{64\pi^4}\|V\|^3_{1,\varepsilon}\right) + 1$$

where γ *is the best constant satisfying the inequality* $|\det_3(1+X)| \le e^{\gamma\|X\|^3_{\mathfrak{S}_2}}$.

Note that the potential V in our estimate must decay exponentially fast in the integral sense, so that $\|V\|_{1,\varepsilon}$ is finite. Recall that such exponential decay at infinity is not needed in order to guarantee that the number of eigenvalues of the corresponding Schrödinger operator to be finite. B. Pavlov used the notion of quasi-analyticity to obtain certain criteria for the number of eigenvalues to be finite (see [25], [26]). However, his methods do not lead to estimates for their total number $\mathcal{N}_p$. Also recall that Pavlov established that if the potential decays slower than $\gamma \exp(-\alpha|x|^{1/2})$ then the number of eigenvalues of the corresponding one-dimensional Schrödinger operator might be infinite. Another interesting result was recently established by Bögli [2]. In this paper, the author constructs a non-real potential $V \in L^p(\Omega) \cap L^\infty(\Omega)$, $p > d$ that decays at infinity so that the Schrödinger operator has infinitely many non-real eigenvalues accumulating at every point of the interval $[0, \infty)$. Also one may take a look at the paper [11], where the authors prove that the spectrum of Schrödinger operators in three dimensions is purely continuous and coincides with the non-negative semiaxis for all potentials satisfying a form-subordinate smallness condition.

Remark: In this chapter we use the fact that we have explicit form for the resolvent kernel of the Laplacian in the dimension $d = 3$. In the case when $d = 1$ the explicit form for the resolvent kernel is also known, but we will not consider this case. The reason being is that the case when $d = 1$ can be treated in the similar manner, but in this case the operator $W_1(-\Delta - k)^{-1}W_2$ is already a Hilbert Schmidt operator. One does not need additional cancellations of singularities, which makes it simpler.

6.2 Relation between operator of fourth order and the Schrödinger operator

In this section we show that it is possible to express the resolvent of the biharmonic operator through the difference of resolvents of two "regular" Schrödinger operators.

Let us first consider the case when $A = (-\Delta)^2$. Then, for every $z \in \rho(A)$, using the difference of two squares formula, we can express the operator $(A - z)^{-1}$ as follows:

$$(A - z)^{-1} = ((-\Delta)^2 - z)^{-1} = (-\Delta - k)^{-1}(-\Delta + k)^{-1}, \text{ where } k^2 = z.$$

Next, recall the Hilbert's identity, which states that for any two invertible operators T and S, if $T - S$ is bounded, the following holds: $T^{-1} - S^{-1} = T^{-1}(S - T)S^{-1}$. Applying this result we get the following identity:

$$(-\Delta - k)^{-1} - (-\Delta + k)^{-1} = (-\Delta - k)^{-1}(2k)(-\Delta + k)^{-1} = 2k(A - z)^{-1}.$$

Now solving for $(A - z)^{-1}$ yields the following difference of the two resolvent Schrödinger operators:

$$(A - z)^{-1} = \frac{1}{2k}\left[(-\Delta - k)^{-1} - (-\Delta + k)^{-1}\right], \tag{6.2}$$

where $k^2 = z$. Resolvents for the Schrödinger operators on $\mathbb{R}^3$ are already known. As a result we can use them to help us find the resolvent for the operator of our interest. In the next section we will prove some nontrivial results which will will be needed later.

6.3 Resolvent Bounds

In this section we obtain some results which will be very useful for the proof of the Theorem 6.1. We include the following figure to help understand the next Proposition.

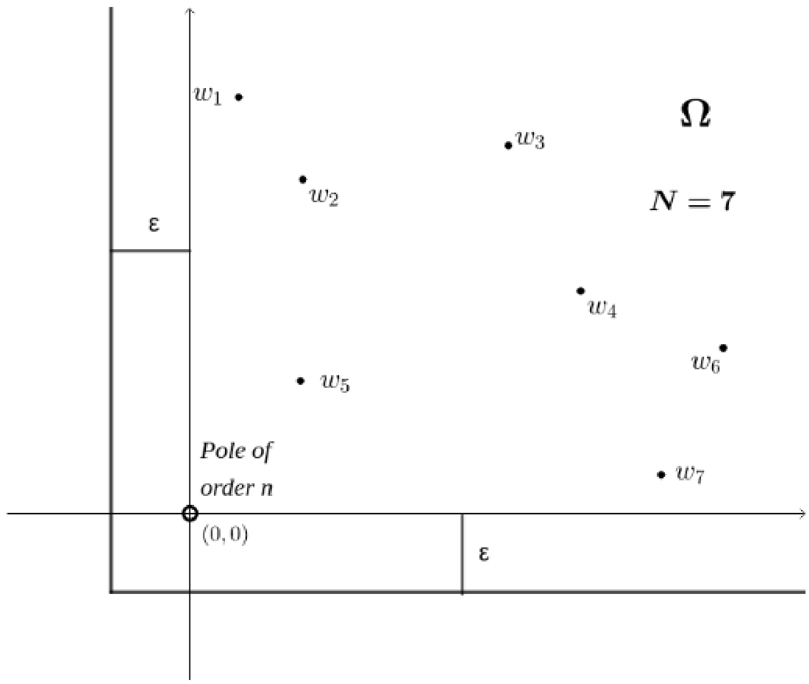

Figure 6.6: Zeros of $a(w)$ inside Ω with pole of order n at $(0,0)$ (Biharmonic)

Proposition 6.2. *Let $a(w)$ be a function in $\Omega := \{w \in \mathbb{C} : \operatorname{Im} w \geq -\varepsilon, \operatorname{Re} w \geq -\varepsilon\}$, such that $a(w) = 1 + O\left(\frac{1}{|w|^3}\right)$ as $|w| \to \infty$, and $\ln|a(w)| \leq \frac{D}{|w|^3}$ if w lies on the boundary of Ω. Moreover, assume that $a(w) = a_0(w)e^{f(w)}$, where $a_0(w)$ is meromorphic, having only one pole of order n at $w = 0$ and $f(w)$ is analytic everywhere except $w = 0$. Then the number of zeros N of $a(w)$ in the first quadrant satisfies*

$$N \leq \frac{\left|\int_{\partial\Omega_R} f(w)(w + (1+i)\varepsilon)dw\right|}{2\pi\varepsilon^2} + \frac{3D}{\pi\varepsilon^3} + n\,,$$

where $\Omega_R = \{w \in \Omega : |w| \leq R\}$, for any $R > 0$.

Proof. To get the desired estimate we would like to look at the function $\ln(a(w)) = \ln[a_0(w)e^{f(w)}] = \ln[a_0(w)] + f(w)$. However, $\ln[a_0(w)]$ is not analytic in Ω, due to a_0 having a pole at $w = 0$, as well as possibly having zeros in Ω. To make it analytic we need to get rid of the pole and all the zeros. To do so we introduce the following Blaschke product:

$$B(k) = \left(\frac{(w+(1+i)\varepsilon)^2 - ((1-i)\varepsilon)^2}{(w+(1+i)\varepsilon)^2 - ((1+i)\varepsilon)^2}\right)^n \prod_j \frac{(w+(1+i)\varepsilon)^2 - (w_j+(1+i)\varepsilon)^2}{(w+(1+i)\varepsilon)^2 - (\overline{w_j}+(1-i)\varepsilon)^2},$$

where w_j are zeros of $a_0(w)$. Let $\partial\Omega_R = I_R \cup J_R \cup C_R$, where $C_R = \{w \in \Omega : |w| = R\}$, I_R is the line $\{w \in \Omega : w = t - i\varepsilon \text{ } and \text{ } |w| \le R\}$, and J_R is the line $\{w \in \Omega : w = it - \varepsilon \text{ } and \text{ } |w| \le R\}$. Then the function $\ln[a_0(w)/B(w)]$ exists and is analytic in Ω, and $|B(w)| = 1$ on $\partial\Omega$. Now applying residue calculus yields the following equality

$$\begin{aligned}\operatorname{Re}\left[2\pi i Res_{w=0}\left(f(w)(w+(1+i)\varepsilon)\right)\right] &= \operatorname{Re}\int_{\partial\Omega_R}\left(\ln\left[\frac{a_0(w)}{B(w)}\right] + f(w)\right)(w+(1+i)\varepsilon)dw \\ = \operatorname{Re}\int_{C_R}\ln\frac{a(w)}{B(w)}&(w+(1+i)\varepsilon)dw + \operatorname{Re}\int_{I_R}\ln\frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw \\ &+ \operatorname{Re}\int_{J_R}\ln\frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw\,.\end{aligned} \tag{6.3}$$

As a result, from the equation above, after moving the last two integrals to the left side and using the triangle inequality, we get the following:

$$\begin{aligned}\left|\int_{\partial\Omega_R} f(w)(w+(1+i)\varepsilon)dw\right| &+ \operatorname{Re}\int_{I_R}\ln\frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw \\ + \operatorname{Re}\int_{J_R}\ln\frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw &\ge -\operatorname{Re}\int_{C_R}\ln\frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw\,.\end{aligned} \tag{6.4}$$

Now our goal is to estimate the right hand side from below and the left hand side from above. So first we will estimate the integral on the right.

We know that the integral of a function over a curve can be estimated by the maximum value of the function on that curve multiplied by the length of the curve. Namely,

$$\left|\int_{C_R}\psi(z)dz\right| \le \max_{z\in C_R}(\psi(z))|C_R|\,.$$

Here $|C_R|$ denotes the length of the curve C_R. Now, due to our two assumptions, since $\ln|a(w)| \leq \frac{D}{|w|^3}$ if $w \in \partial\Omega$ and $a(w) = 1 + O\left(\frac{1}{|w|^3}\right)$ as $|w| \to \infty$, we get the following estimate for the integral

$$\begin{aligned} \lim_{R\to\infty} \operatorname{Re} \int_{C_R} \ln\left[a(w)\right](w + (1+i)\varepsilon)dw &\leq \lim_{R\to\infty} \left\{ \frac{D}{|R|^3} |(R + (1+i)\varepsilon)||C_R| \right\} \\ &= \lim_{R\to\infty} \left\{ \frac{D\pi R}{2|R|^3} |(R + (1+i)\varepsilon)| \right\} = 0\,. \end{aligned} \tag{6.5}$$

And so we only need to estimate $\lim_{R\to\infty} \operatorname{Re} \int_{C_R} \ln B(w)(w + (1+i)\varepsilon)dw$ from below. After setting $\xi = w + (1+i)\varepsilon$ at first, then setting $\xi = Re^{i\varphi}$ and then computing the integral we arrive at the following inequality

$$\begin{aligned} \lim_{R\to\infty} \operatorname{Re} \int_{C_R} \ln B(w)(w + (1+i)\varepsilon)dw &= 4i\varepsilon^2 n \left(\frac{\pi i}{2}\right) + 2\left(\frac{\pi}{2}\right) \sum_j \operatorname{Im}(w_j + (1+i)\varepsilon)^2 \\ &\geq -2n\varepsilon^2\pi + \pi(2\varepsilon^2)N = 2\pi\varepsilon^2(N-n)\,, \end{aligned} \tag{6.6}$$

where N is the number of zeros w_j of the function $a_0(w)$ in the first quadrant. The last inequality is due to the fact that if we set $w = a + bi$ and compute $(w + (1+i)\varepsilon)^2$ we see that $\operatorname{Im}(w + (1+i)\varepsilon)^2 = 1(a+\varepsilon)(b+\varepsilon) \geq 2\varepsilon^2$. Hence, since we are taking the sum over all zeros of $a_0(w)$ which lie in the first quadrant, the desired inequality holds.

We now will estimate second and third terms on the left hand side of 7.3. Note that

$$\begin{aligned}
\operatorname{Re}\int_{I_R} \ln\frac{a(w)}{B(w)}(w+(1+i)\varepsilon)dw &= \int_{I_R} \ln\left|\frac{a(w)}{B(w)}\right|(w+(1+i)\varepsilon)dw, \\
&\qquad \text{since } (w+(1+i)\varepsilon)dw \text{ is real} \\
&= \int_{I_R} \ln|a(w)|\,(w+(1+i)\varepsilon)dw, \\
&\qquad \text{since } |B(w)|=1 \text{ on the boundary} \\
&\leq \int_{I_R} \frac{D}{|w|^3}(w+(1+i)\varepsilon)dw\,.
\end{aligned}$$

Analogous arguments show that the integral over the boundary J_R will yield the same estimate as above. So for $w \in I_R$ (i.e. $w = t - i\varepsilon$), or $w \in J_R$ (i.e. $w = it - \varepsilon$), after appropriate change of variables and integration we get

$$\begin{aligned}
\int_{J_R\,(\text{or } I_R)} \ln\left|\frac{a(w)}{B(w)}\right|(w+(1+i)\varepsilon)dw &\leq \frac{D}{\varepsilon}\int_{-1}^{\infty}\frac{s+1}{(s^2+1)^{3/2}}ds \\
\text{setting } u = s^2+1: \qquad &= \frac{D}{\varepsilon}\int_2^{\infty}\left(\frac{1}{2u^{3/2}\sqrt{u-1}}+\frac{1}{2u^{3/2}}\right)du+ \\
&\quad + \frac{2D}{\varepsilon}\int_1^{2}\left(\frac{1}{2u^{3/2}\sqrt{u-1}}+\frac{1}{2u^{3/2}}\right)du \\
&= \frac{D}{\varepsilon}\left(\sqrt{1-\frac{1}{u}}-\frac{1}{u^{1/2}}\right)\Bigg|_2^{\infty} + \frac{2D}{\varepsilon}\left(\sqrt{1-\frac{1}{u}}-\frac{1}{u^{1/2}}\right)\Bigg|_1^{2} \\
&= \frac{3D}{\varepsilon}\,.
\end{aligned} \tag{6.7}$$

Now combining inequalities 7.5, 7.6 and 7.3 we get a bound for the number of zeros N of $a(w)$ as follows

$$2\pi\varepsilon^2(N-n) \leq \left|\int_{\partial\Omega_R} f(w)(w+(1+i)\varepsilon)dw\right| + \frac{6D}{\varepsilon}\,.$$

The conclusion of the proposition follows immediately. □

6.3.1 Classes of compact operators and determinants

Recall the Birman-Schwinger principle for the special case, where H_0 is a self-adjoint operator and V is bounded. According to the Birman–Schwinger Principle from Chapter 2 and Lemma 2.1 we conclude that the algebraic multiplicities of eigenvalues of H can also be characterized by zeros of the of the determinant of the Birman-Schwinger operator mentioned above.

It is well known that the integral kernel for the operator $(-\Delta - z)^{-1}$ in the case when dimension $d = 3$ is given by $\rho_0(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$, where $k^2 = z$, $\operatorname{Im} k > 0$. So from this result and equation 6.2 we can derive the integral kernel for $(A - z)^{-1}$ to be the following:

$$\rho(x, y) = \frac{1}{2k}\left[\frac{e^{i\sqrt{k}|x-y|}}{4\pi|x-y|} - \frac{e^{i\sqrt{-k}|x-y|}}{4\pi|x-y|}\right] = \frac{1}{2k}\left[\frac{e^{i\sqrt{k}|x-y|} - 1 - (e^{i\sqrt{-k}|x-y|} - 1)}{4\pi|x-y|}\right].$$

Note that for any $z \in \mathbb{C}\backslash\mathbb{R}_+$, $w = \sqrt[4]{z}$ is always located in the first quadrant. However, as we look at the proposition 6.2, we consider all values of $w \in \Omega$, keeping in mind that $w^4 = z$ if w is in the first quadrant. Also using the inequality

$$\begin{aligned} \left|e^{i(a+bi)\tau} - 1\right| &\le \left|\int_0^\tau i(a+bi)e^{i(a+bi)s}ds\right| \le \int_0^\tau |i||a+bi|\left|e^{i(a+bi)s}ds\right| \\ &\le \tau|a+bi|e^{-b\tau} \le \tau|a+bi|e^{\varepsilon_1\tau} \end{aligned} \tag{6.8}$$

we arrive at the following estimate for $\rho(x, y)$:

$$|\rho(x, y)| \le \frac{1}{2|k|}\frac{|\sqrt{k}|\left|e^{\varepsilon_1(|x|+|y|)}\right||x-y| + |\sqrt{-k}|\left|e^{\varepsilon_1(|x|+|y|)}\right||x-y|}{4\pi|x-y|} = \frac{e^{\varepsilon_1(|x|+|y|)}}{4\pi\sqrt{|k|}}. \tag{6.9}$$

Next, we use this result to find a bound for the Hilbert-Schmidt norm of the operator $X = W_1(H_0 - z)^{-1}W_2$, where $W_1 = V|V|^{-1/2}$ and $W_2 = |V|^{1/2}$. Simple calculations show us that the Hilbert-Schmidt norm of X can be estimated as so:

$$
\begin{aligned}
\|X\|_{\mathfrak{S}_2}^3 &= \left[\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |W_1(x)\rho(x,y)W_2(y)|^2 dxdy\right]^{3/2} \\
&\le \left[\int_{\mathbb{R}^3} |W_2(y)|^2 \int_{\mathbb{R}^3} |W_1(x)|^2 |\rho(x,y)|^2 dxdy\right]^{3/2} \\
&\qquad\qquad (\text{since } \rho(x,y)=\rho(y,x)) \\
&\le \left[\left(\frac{1}{4\pi\sqrt{|k|}}\right)^2 \int_{\mathbb{R}^3} |W_2(y)|^2 \left|e^{2\varepsilon_1|y|}\right| \int_{\mathbb{R}^3} |W_1(x)|^2 \left|e^{2\varepsilon_1|x|}\right| dxdy\right]^{3/2} \qquad (6.10) \\
&\qquad\qquad (\text{by inequality 6.9}) \\
&= \left[\frac{1}{16\pi^2|k|}\int_{\mathbb{R}^3} |V(y)| \left|e^{2\varepsilon_1|y|}\right| dy \int_{\mathbb{R}^3} |V(x)| \left|e^{2\varepsilon_1|x|}\right| dx\right]^{3/2} \\
&= \frac{1}{64\pi^3|k|^{3/2}}\left(\int_{\mathbb{R}^3} \left|e^{2\varepsilon_1|x|}\right| |V(x)|dx\right)^3 .
\end{aligned}
$$

6.4 Proof of Theorem 6.1

As discussed in the section above we will be looking at $\det_3(1+W_1(H_0-\varsigma)^{-1}W_2)$. We know that

$$
\begin{aligned}
\det_3(1+W_1(H_0-\varsigma)^{-1}W_2) &= \det_1(1+W_1(H_0-\varsigma)^{-1}W_2)e^{-\operatorname{Tr}(X)+\frac{\operatorname{Tr}(X^2)}{2}} \\
&= \left[\prod_j (1+k_j)\right] e^{-\operatorname{Tr}(X)+\frac{\operatorname{Tr}(X^2)}{2}} .
\end{aligned}
$$

Note that the product has a pole of order 1 at $k=0$. Moreover, the function in the exponent will then have a pole of order 2 at $k=0$. Let us note that $|\det_3(1+X)| \le e^{\gamma\|X\|_{\mathfrak{S}_3}^3} \le e^{\gamma\|X\|_{\mathfrak{S}_2}^3}$, for some $\gamma > 0$. The proof of this statement can be found in Lemma 2.2 in Chapter 2; it is essentially due to Weyl's inequality [28, Thm. 1.15]. Now if we set $w^2 = k$ and apply Proposition 6.2 to the function $a(w) = \det_3(1+X(w))$ we will get an estimate for the number of zeros of the function $\det_3(1+X(w))$. However, to get an explicit bound we still need to compute

$\int_{\partial\Omega_R}\left(-\operatorname{Tr}(X)+\frac{\operatorname{Tr}(X^2)}{2}\right)(w+(1+i)\varepsilon)dw$. This is exactly what we will be doing in the next two subsections.

6.4.1 Estimating $\left|\int_{\partial\Omega_R} -\operatorname{Tr}(X)(w+(1+i)\varepsilon)dw\right|$

After a simple Taylor expansion and integration we get that $\operatorname{Tr}(X)=\left(\frac{i+1}{8w\pi}\right)\int_{\mathbb{R}^3}V(x)dx$. Now using residue calculus we get

$$\begin{aligned}\int_{\partial\Omega_R}\operatorname{Tr}(X)(w+(1+i)\varepsilon)dw &= \int_{\partial\Omega_R}\left(\frac{i+1}{8w\pi}\right)(w+(1+i)\varepsilon)\int_{\mathbb{R}^3}V(x)dxdw\\ &= \frac{i+1}{8\pi}\int_{\mathbb{R}^3}V(x)dx\int_{\partial\Omega_R}\left(1+\frac{(1+i)\varepsilon}{w}\right)dw\\ &= \frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)dx\,.\end{aligned}$$

This yields the following estimate from above:

$$\left|\int_{\partial\Omega_R}-\operatorname{Tr}(X)(w+(1+i)\varepsilon)dw\right| = \left|\frac{\varepsilon}{2}\int_{\mathbb{R}^3}V(x)dx\right| \le \frac{\varepsilon}{2}\int_{\mathbb{R}^3}|V(x)|e^{\varepsilon|x|}dx\,. \qquad (6.11)$$

6.4.2 Estimating $\left|\int_{\partial\Omega_R}\frac{\operatorname{Tr}(X^2)}{2}(w+(1+i)\varepsilon)dw\right|$

Note that

$$\begin{aligned}\operatorname{Tr}(X^2) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}V(y)\rho^2(x,y)V(x)dydx\\ &= \frac{i}{32w^2\pi^2}\left(\int_{\mathbb{R}^3}V(x)dx\right)^2-\frac{i+1}{32w\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}V(x)|x-y|V(y)dydx+\psi(w),\end{aligned} \qquad (6.12)$$

where $\psi(w)$ is analytic at $w=0$. Now when we use residue calculus to integrate $\operatorname{Tr}(X^2)$ over $\partial\Omega_R$ we get the following bound

$$\begin{aligned}\left|\int_{\partial\Omega_R}\frac{\operatorname{Tr}(X^2)}{2}(w+(1+i)\varepsilon)dw\right| &= \left|\int_{\partial\Omega_R}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}V(y)\rho^2(x,y)V(x)(w+(1+i)\varepsilon)dydxdw\right|\\ &= \left|\frac{\varepsilon}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}V(x)|x-y|V(y)dydx-\frac{1}{32\pi}\left(\int_{\mathbb{R}^3}V(x)dx\right)^2\right|\\ &\le \frac{\varepsilon}{16\pi}\left(\int_{\mathbb{R}^3}|V(x)(1+|x|)dx\right)^2+\frac{1}{32\pi}\left(\int_{\mathbb{R}^3}V(x)dx\right)^2\\ &\le \frac{1}{16\pi\varepsilon}\left(\int_{\mathbb{R}^3}|V(x)|e^{\varepsilon|x|}dx\right)^2+\frac{1}{32\pi}\left(\int_{\mathbb{R}^3}|V(x)|e^{\varepsilon|x|}dx\right)^2\\ &= \frac{\varepsilon+2}{32\pi\varepsilon}\left(\int_{\mathbb{R}^3}|V(x)|e^{\varepsilon|x|}dx\right)^2 .\end{aligned} \tag{6.13}$$

Note that the last inequality is valid, since, due to our assumption, the potential decays exponentially at infinity, forcing the integrals to be finite.

6.4.3 Proof of the Theorem

Combining estimates 6.11 and 6.13 we get the following inequality:

$$\left|\int_{\partial\Omega_R}\left(-\operatorname{Tr}(X)+\frac{\operatorname{Tr}(X^2)}{2}\right)(w+(1+i)\varepsilon)dw\right| \le \frac{\varepsilon}{2}\|V\|_{1,\varepsilon}+\frac{\varepsilon+2}{32\pi\varepsilon}\|V\|_{1,\varepsilon}^2, \tag{6.14}$$

where $\|V\|_{1,\varepsilon}=\int_{\mathbb{R}^3}|V(x)|e^{\varepsilon|x|}dx$. Then we apply Proposition 6.2 to the function $a(w)=\det_3(1+X)$ with $a_0(w)=\det_1(1+X)$ and $f(w)=-\operatorname{Tr}(X)+\dfrac{\operatorname{Tr}(X^2)}{2}$, as discussed in the beginning of this section. After this we use the estimate 6.14. Moreover, we set $D=\dfrac{\gamma}{64\pi^3}\left(\int_{\mathbb{R}^3}e^{2\varepsilon_1|x|}|V(x)|dx\right)^3$ due to 6.10. Choosing $\varepsilon_1=\varepsilon/2$ we conclude that the total number $\mathcal{N}$ of zeros of the function $\det_3(1+X)$ satisfies

$$\mathcal{N}\le\frac{1}{\varepsilon^3}\left(\frac{\varepsilon^2}{4\pi}\|V\|_{1,\varepsilon}+\frac{\varepsilon+2}{64\pi^2}\|V\|_{1,\varepsilon}^2+\frac{3\gamma}{64\pi^4}\|V\|_{1,\varepsilon}^3\right)+1\,,$$

where $\|V\|_{1,\varepsilon}=\int_{\mathbb{R}^3}|V(x)|e^{\varepsilon|x|}dx$. Then by the argument in the second paragraph of the subsection 7.3.1, the same estimate gives us the bound for the number of eigenvalues of the operator H. □

CHAPTER 7: POLYHARMONIC OPERATOR OF ORDER $2l$ ON $\mathbb{R}^d$, WITH $d > 2l$

In this chapter we study the polyharmonic operator of order $2l,\ l \in \mathbb{N}$ on $\mathbb{R}^d$ with a complex potential, which decays exponentially at infinity. We obtain bounds on the total number of eigenvalues of the said operator. Result in this Chapter is essentially a generalization of the result in Chapter 6.

7.1 Introduction and Main Results

We consider the operator

$$H = (-\Delta)^l + V(x), \qquad x \in \mathbb{R}^d$$

with a complex valued exponentially decaying potential V. We obtain an estimate for the total number $\mathcal{N}$ of eigenvalues of the operator H in the complex plane $\mathbb{C}$, minus the positive real line. This work is essentially an extension of the result obtained in Chapter 6 to more general order $2l$ and dimension d.

Just like in Chapter 6 we assume that V is a measurable function such that the integral

$$\|V\|_{\frac{d+1}{2},\varepsilon} := \int_{\mathbb{R}^d} |V(x)|^{\frac{d+1}{2}} e^{\varepsilon|x|} dx < \infty \tag{7.1}$$

is finite. In this case, once again, the operator H can be defined in the sense of quadratic forms as it was done in the paper of Laptev and Safronov [19]. Note only, that the domain of the operator H is contained in the Sobolev space $\mathcal{H}^l(\mathbb{R}^d)$. However it depends on the potential V, which does not have to be bounded. It was also shown by Laptev and Safronov [19] that it is enough to establish the eigenvalue bounds for compactly supported smooth potentials V, since they could be extended to the general case by the limit procedure.

Note that in the case when $d > 2l$ the resolvent will have singularity at the the line $x = y$. We note that even Cwikel estimate for the polyharmonic operator considers this case separately. In this case, for the real potential V, Cwikel proved the following inequality:

$$\mathcal{N} \leq C_{l,d} \int_{\mathbb{R}^d} |V|^{\frac{d}{2l}} dx \ ,$$

where $C_{l,d}$ is a constant which depends only on the dimension d and the power of the Laplacian l [4].

For the following theorem we will need to introduce a few things. First let $\mathfrak{S}_p$ denote the pth Schatten-class operator and $\|\cdot\|_{\mathfrak{S}_p}$ denote the norm in this space, i.e.

$$\mathfrak{S}_p = \left\{ T : \left(\sum_k s_k^p(T) \right)^{\frac{1}{p}} < \infty \right\}, \qquad \|T\|_{\mathfrak{S}_p}^p = \sum_k s_k^p(T).$$

Theorem 7.1. *Let $\varepsilon > 0$, $l > 1$, $d -$ odd and satisfy $d > 2l$. Also let V satisfy (7.1). Then the number $\mathcal{N}_P$ of eigenvalues of H in $L^2(\mathbb{R}^d)$, counting algebraic multiplicities, located outside of the essential spectrum, satisfies*

$$\mathcal{N}_P \leq \frac{C_{d,l}}{\varepsilon^{2(ld-d+l)}} \left(\int_{\mathbb{R}^d} e^{\varepsilon|x|} |V(x)|^{\frac{d+1}{2}} dx \right)^2$$

where $C_{d,l}$ is a constant which depends only on the dimension d and power of the Laplacian l.

Note that the potential V in our estimate must decay exponentially fast in the integral sense, so that $\|V\|_{\frac{d+1}{2},\varepsilon}$ is finite.

7.2 Relation between operator of oredr $2l$ and the Schrödinger operator

In this section we will to express the resolvent of the polyharmonic operator of order $2l$ through the sum of resolvents of "regular" Schrödinger operators.

Consider $A = (-\Delta)^l$. We can express $(A-z)^{-1} = \dfrac{1}{(-\Delta)^l - z}$ as follows:

$$\frac{1}{(-\Delta)^l - z} = \sum_{j=1}^{l} \frac{c_j}{-\Delta - k_j}, \quad c_j = \prod_{\substack{n=1 \\ n\neq j-1}}^{l} \frac{1}{z_1\left(e^{\frac{(j-1)2i\pi}{l}} - e^{\frac{2in\pi}{l}}\right)}$$

where $k_j = z_1 e^{\frac{(j-1)2\pi i}{l}}$ and $z_1^l = z$.

Claim 7.2. *For the c_j and k_j defined above the following holds:*

$$\sum_{j=1}^{l} c_j = 0 \quad \textit{and} \quad \sum_{j=1}^{l} c_j \cdot k_j = 0\,.$$

Proof. To prove the first inequality consider the function $\dfrac{1}{p^l - |z|}$. Then the following inequality holds:

$$\left|\int_{C_R} \frac{dp}{p^l - |z|}\right| \leq 2\pi R \max \left|\frac{1}{R^l - |z|}\right| \to 0, \text{ as } R \to \infty$$

So the sum of residues has to equal to zero. But the residues are the C_j's. Hence the first equality holds.

The proof of the second equality is analogous to the prove of the first one. □

7.3 Resolvent Bounds

In this section we will prove some nontrivial results which will be very useful for the proof of the Theorem 7.1.

Proposition 7.3. *Let $a(w)$ be a function defined on*

$$\Omega := \left\{w \in \mathbb{C} : \; 0 \le \arg\left(w + \varepsilon e^{i\pi/2l}\right) < \pi/l\right\}$$

where $w_1 = |w_1|e^{i\varphi}, \quad \varphi \in \left[0, \frac{\pi}{l}\right)\Big\}$, *such that* $a(w) = 1 + O\left(\frac{1}{|w|^{l+1}}\right)$ *as* $|w| \to \infty$, *and* $\ln|a(w)| \le \frac{D}{|w|^{l+1}}$ *if* w *lies on the boundary of* Ω*. Moreover, assume that* $a(w) = a_0(w)e^{f(w)}$*, where* $a_0(w)$ *is meromorphic, having only one pole of order* n *at* $w = 0$ *and* $f(w)$ *is analytic everywhere except* $w = 0$*. Then the number of zeros* N *of* $a(w)$ *in* $\Omega + \varepsilon e^{\frac{i\pi}{2l}} = \left\{w \in \mathbb{C} : w = |w|e^{i\varphi}, \quad \varphi \in \left[0, \frac{\pi}{l}\right)\right\}$ *satisfies*

$$N \le \frac{l\left|\int_{\partial\Omega_R} f(w)(w + \varepsilon e^{\frac{i\pi}{2l}})^{l-1}dw\right|}{2\pi\varepsilon^l} + \frac{lDC_l}{\pi\varepsilon^{l+1}} + n\,,$$

where $\Omega_R = \{w \in \Omega : |w| \le R\}$*, for any* $R > 0$*.*

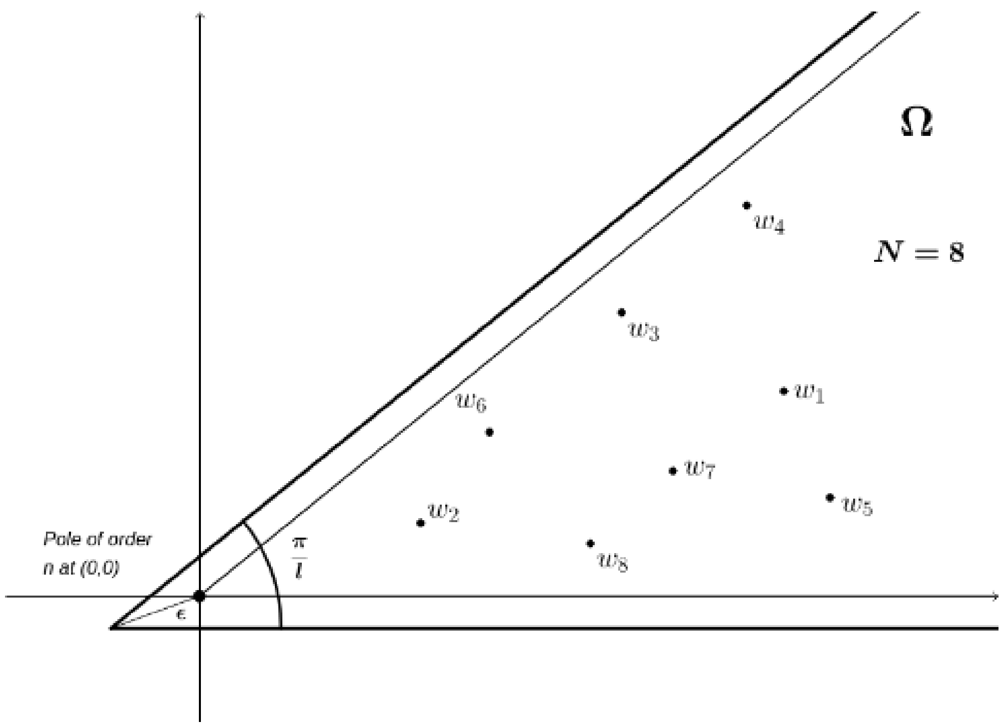

Figure 7.7: Zeros of $a(w)$ inside Ω with pole of order n at $(0, 0)$ (Polyharmonic)

The proof of this Proposition will be similar to the proof of the Proposition 6.2.

Proof. To get the desired estimate we would like to look at the function $\ln(a(w)) = \ln[a_0(w)e^{f(w)}] = \ln[a_0(w)] + f(w)$. However, $\ln[a_0(w)]$ is not analytic in Ω, due to a_0 having a pole at $w = 0$, as well as possibly having zeros in Ω. To make it analytic we need to get rid of the pole and all the zeros. To do so we introduce the following Blaschke product:

$$B(k) = \left(\frac{(w + \varepsilon e^{\frac{i\pi}{2l}})^l - (\varepsilon e^{\frac{-i\pi}{2l}})^l}{(w + \varepsilon e^{\frac{i\pi}{2l}})^l - (\varepsilon e^{\frac{i\pi}{2l}})^l} \right)^n \prod_j \frac{(w + \varepsilon e^{\frac{i\pi}{2l}})^l - (w_j + \varepsilon e^{\frac{i\pi}{2l}})^l}{(w + \varepsilon e^{\frac{i\pi}{2l}})^l - (\overline{w_j} + \varepsilon e^{\frac{-i\pi}{2l}})^l},$$

where w_j are zeros of $a_0(w)$. Then the function $\ln[a_0(w)/B(w)]$ exists and is analytic in Ω, and $|B(w)| = 1$ on $\partial\Omega$.

Let $\partial\Omega_R = I_R \cup J_R \cup C_R$, where $C_R = \{w \in \Omega : |w| = R\}$, I_R is the line $\{w \in \Omega : w = t - \varepsilon e^{\frac{i\pi}{2l}} \text{ } and \text{ } |w| \leq R\}$, and J_R is the line $\{w \in \Omega : w = te^{i\pi/l} - \varepsilon e^{\frac{i\pi}{2l}} \text{ } and \text{ } |w| \leq R\}$ (refer to Figure 7.8).

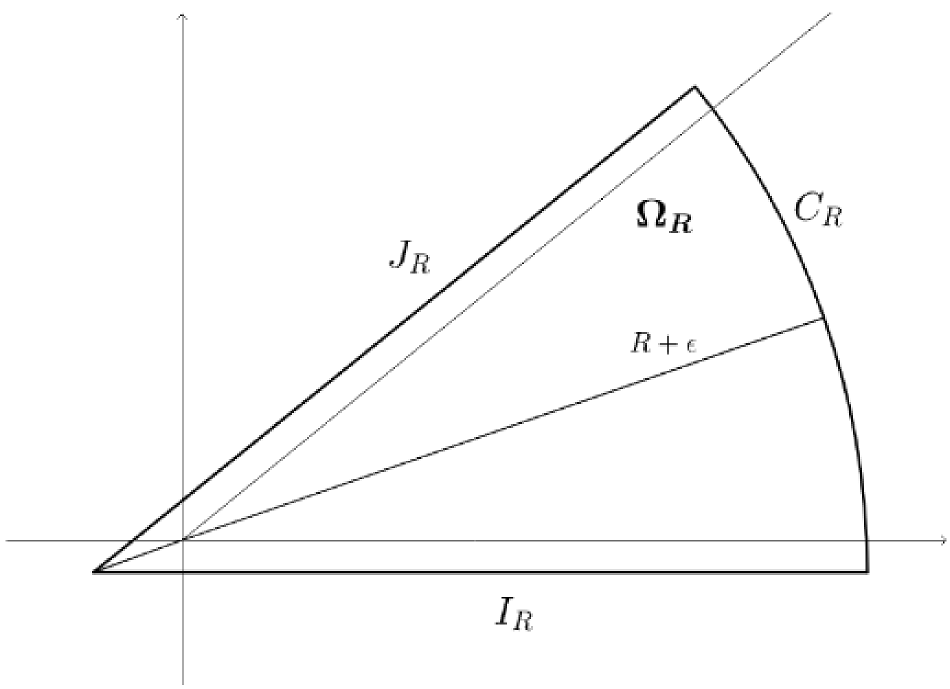

Figure 7.8: Ω_R for Polyharmonic Operator

Now applying residue calculus yields the following equality

$$\begin{aligned}
\operatorname{Re}\left[2\pi i \operatorname{Res}_{w=0}\left(f(w)(w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1}\right)\right] &= \\
&= \operatorname{Re}\int_{\partial\Omega_R}\left(\ln\left[\frac{a_0(w)}{B(w)}\right]+f(w)\right)(w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1}dw \\
&= \operatorname{Re}\int_{C_R}\ln\frac{a(w)}{B(w)}(w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1}dw \\
&\quad + \operatorname{Re}\int_{I_R}\ln\frac{a(w)}{B(w)}(w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1}dw \\
&\quad + \operatorname{Re}\int_{J_R}\ln\frac{a(w)}{B(w)}(w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1}dw\,.
\end{aligned} \tag{7.2}$$

As a result, from the equation above, after moving the last two integrals to the left side and using the triangle inequality, we get the following

$$\begin{aligned}
&\left|\int_{\partial\Omega_R} f(w)(w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1}dw\right| + \operatorname{Re}\int_{I_R}\ln\frac{a(w)}{B(w)}(w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1}dw \\
&\quad + \operatorname{Re}\int_{J_R}\ln\frac{a(w)}{B(w)}(w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1}dw \geq -\operatorname{Re}\int_{C_R}\ln\frac{a(w)}{B(w)}(w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1}dw\,.
\end{aligned} \tag{7.3}$$

Now our goal is to estimate the right hand side from below and the left hand side from above. So first we will estimate the integral on the right.

We know that the integral of a function over a curve can be estimated by the maximum value of the function on that curve multiplied by the length of the curve. Namely,

$$\left|\int_{C_R}\psi(z)dz\right| \leq \max_{z\in C_R}|\psi(z)||C_R|\,.$$

Here $|C_R|$ denotes the length of the curve C_R. Now, due to our two assumptions, since $\ln|a(w)| \leq \dfrac{D}{|w|^{l+1}}$ if $w \in \partial\Omega$ and $a(w) = 1 + O\left(\dfrac{1}{|w|^{l+1}}\right)$ as $|w| \to \infty$, we get the following estimate for the integral:

$$\lim_{R\to\infty} \operatorname{Re} \int_{C_R} \ln\left[a(w)\right](w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1}dw \le \lim_{R\to\infty} \left\{ \frac{D}{|R|^{l+1}} \left| R+\varepsilon e^{\frac{i\pi}{2l}} \right|^{l-1} |C_R| \right\}$$
$$= \lim_{R\to\infty} \left\{ \frac{D\pi R \left| R+\varepsilon e^{\frac{i\pi}{2l}}) \right|^{l-1}}{l|R|^{l+1}} \right\} = 0\,, \tag{7.4}$$

since $|C_R| = \dfrac{\pi R}{l}$.

And so we only need to estimate $\lim_{R\to\infty} \operatorname{Re} \int_{C_R} \ln B(w) \left(w+\varepsilon e^{\frac{i\pi}{2l}}\right)^{l-1} dw =$ $\lim_{R\to\infty} \int_{C_R} \ln|B(w)| \left(w+\varepsilon e^{\frac{i\pi}{2l}}\right)^{l-1} dw$ from below. After setting

$$\xi = w+\varepsilon e^{\frac{i\pi}{2l}},\ \xi_0 = \varepsilon e^{\frac{i\pi}{2l}} \quad \text{and} \quad \xi_j = w_j + \varepsilon e^{\frac{i\pi}{2l}}$$

the $B(w)$ becomes

$$B(w) = \left(\frac{\xi^l - \overline{\xi_0}^l}{\xi^l - \xi_0^l} \right)^n \prod_j \left(\frac{\xi^l - \xi_j^l}{\xi^l - \overline{\xi_j}^l} \right) .$$

Now using the fact that $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \cdots$ we get the following expression for $\ln B(w)$:

$$\ln B(w) = \frac{2ni \operatorname{Im}\, \xi_0^l}{\xi^l} - \sum_j \frac{2i \operatorname{Im}\, \xi_j^l}{\xi^l} + O\left(\frac{1}{\xi^4}\right) ,$$

where we are summing over the zeros of the function $a(w)$ in Ω.

Recall that $\xi_0 = \varepsilon e^{\frac{i\pi}{2l}}$, hence $\operatorname{Im}\, \xi_0^l = \varepsilon^l$. Also recall that $w_j = |w_j|e^{i\varphi_j}$, $\varphi_j \in \left[0, \dfrac{\pi}{l}\right)$. Then

$$\xi_j^l = \left(w_j + \varepsilon e^{\frac{i\pi}{2l}}\right)^l = \left(|w_j|e^{i\varphi_j} + \varepsilon e^{\frac{i\pi}{2l}}\right)^l = \sum_{k=0}^{l} \alpha_k |w_j|^k e^{ik\varphi} e^{i\pi\left(\frac{l-k}{2l}\right)} \varepsilon^{l-k} ,$$

where $\alpha_k > 0$ are the binomial coefficients.

Now we have

$$\text{Im } \xi_j^l = \sum_{k=0}^{l} \alpha_k |w_j|^k \sin\left(k\varphi + \frac{l-k}{2l}\pi\right) \varepsilon^{l-k}$$

and

$$\sin\left(k\varphi + \frac{l-k}{2l}\pi\right) > 0 \ .$$

Hence, since $\varepsilon < 1$, we get

$$\text{Im } \xi_j^l \geq \sum_{k=0}^{l} \alpha_k |w_j|^k \sin\left(k\varphi + \frac{l-k}{2l}\pi\right) \varepsilon^l = \varepsilon^l \sum_{k=0}^{l} \alpha_k |w_j|^k \sin\left(k\varphi + \frac{l-k}{2l}\pi\right) \ .$$

Clearly $\sum_{k=0}^{l} \alpha_k |w_j|^k \sin\left(k\varphi + \frac{l-k}{2l}\pi\right) \geq 1$, since for $k = 0$ we have $\alpha_k \cdot 1 \cdot 1 \geq 1$. This yields that $\text{Im } \xi_j^l \geq \varepsilon^l$. This means that the following estimate holds:

$$\begin{aligned} \lim_{R\to\infty} \text{Re} \int_{C_R} \ln B(w)(w + \varepsilon e^{\frac{i\pi}{2l}})^{l-1} dw &= \left(2ni \,\text{Im } \xi_0^l - \sum_j 2i \,\text{Im } \xi_j^l\right) \lim_{R\to\infty} \int_{C_R} \frac{d\xi}{\xi} \\ &\geq \left(-2n\varepsilon^l + \sum_j 2\varepsilon^l\right) \frac{\pi}{l} \\ &\geq 2\varepsilon^l (N-n) \frac{\pi}{l} \ , \end{aligned} \tag{7.5}$$

where N is the number of zeros w_j of the function $a_0(w)$ in $\Omega + \varepsilon e^{\frac{i\pi}{2l}}$. Since we are taking the sum over all zeros of $a_0(w)$ which lie in Ω, the last inequality holds.

We now will estimate second and third terms on the left hand side of 7.3 from above. Note that

$$\begin{aligned}
\mathrm{Re} \int_{I_R} \ln \frac{a(w)}{B(w)} (w + \varepsilon e^{\frac{i\pi}{2l}})^{l-1} dw &= \int_{I_R} \ln \left| \frac{a(w)}{B(w)} \right| (w + \varepsilon e^{\frac{i\pi}{2l}})^{l-1} dw, \\
&\qquad \text{since } (w + \varepsilon e^{\frac{i\pi}{2l}})^{l-1} dw \text{ is real} \\
&= \int_{I_R} \ln |a(w)| \, (w + \varepsilon e^{\frac{i\pi}{2l}})^{l-1} dw, \\
&\qquad \text{since } |B(w)| = 1 \text{ on the boundary of } \Omega \\
&\leq \int_{I_R} \frac{D}{|w|^{l+1}} (w + \varepsilon e^{\frac{i\pi}{2l}})^{l-1} dw, \\
&\qquad \text{since } \ln |a(w)| \leq \frac{D}{|w|^{l+1}} \text{ on the boundary of } \Omega\,.
\end{aligned}$$

Now if $w \in I_R$ then $w = t - \varepsilon e^{\frac{i\pi}{2l}}$, $t \in \left[0, R + \varepsilon \sin\left(\frac{\pi}{2l}\right)\right)$ and $dw = dt$. Also $\left(w + \varepsilon e^{\frac{i\pi}{2l}}\right)^{l-1} dw = t^{l-1} dt$. Moreover, $|w| = \left|t - \varepsilon e^{\frac{i\pi}{2l}}\right|$. Then

$$\begin{aligned}
\lim_{R\to\infty} \int_{I_R} \ln \left| \frac{a(w)}{B(w)} \right| (w + \varepsilon e^{\frac{i\pi}{2l}})^{l-1} dw &\leq \lim_{R\to\infty} D \int_0^{R+\varepsilon \sin\left(\frac{\pi}{2l}\right)} \frac{t^{l-1} dt}{\left|t - \varepsilon e^{\frac{i\pi}{2l}}\right|^{l+1}} \\
\text{setting } t = \varepsilon s: \qquad &= D \lim_{R\to\infty} \int_0^{\frac{R}{\varepsilon}+\sin\left(\frac{\pi}{2l}\right)} \frac{\varepsilon^{l-1} s^{l-1} \varepsilon ds}{\varepsilon^{l+1} \left|s - e^{\frac{i\pi}{2l}}\right|^{l+1}} \qquad (7.6) \\
&= \frac{DC_l}{\varepsilon}\,,
\end{aligned}$$

where $C_l = \displaystyle\int_0^{\infty} \frac{s^{l-1} ds}{\left|s - e^{\frac{i\pi}{2l}}\right|^{l+1}}$.

Now if $w \in J_R$ then $w = te^{\frac{i\pi}{l}} - \varepsilon e^{\frac{i\pi}{2l}}$, where $t \in \left[0, R + \varepsilon \sin \frac{\pi}{2l}\right)$ and $dw = e^{\frac{i\pi}{l}} dt$. Moreover,

$$\left(w + \varepsilon e^{\frac{i\pi}{2l}}\right)^{l-1} dw = -t^{l-1} dt$$

and

$$|w| = \left|e^{\frac{i\pi}{l}} t - \varepsilon e^{\frac{i\pi}{2l}}\right| = \left|e^{\frac{i\pi}{l}}\right| \left|t - \varepsilon e^{\frac{-i\pi}{2l}}\right| = \left|t - \varepsilon e^{\frac{i\pi}{2l}}\right| \ .$$

The last equality holds since the terms are complex conjugates.

Then

$$\lim_{R\to\infty}\int_{J_R}\ln\left|\frac{a(w)}{B(w)}\right|(w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1}dw \le \lim_{R\to\infty} D\int_{R+\varepsilon\sin\left(\frac{\pi}{2l}\right)}^{0}\frac{-t^{l-1}dt}{\left|t-\varepsilon e^{\frac{i\pi}{2l}}\right|}\le\frac{DC_l}{\varepsilon}\,, \quad (7.7)$$

which is same as the integral over I_R.

Now combining inequalities 7.7, 7.6, 7.5 and 7.3 we get a bound for the number of zeros N of $a(w)$ as follows

$$2\varepsilon^l(N-n)\frac{\pi}{l}\le\left|\int_{\partial\Omega_R}f(w)(w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1}dw\right|+\frac{2DC_l}{\varepsilon}\,.$$

The conclusion of the proposition follows immediately.

□

7.3.1 Classes of compact operators and determinants

Recall the Birman-Schwinger principle for the special case, where $H_0=\Delta^l$ is the polyharmonic operator of order $2l$ and $V\in C_0^\infty(\mathbb{R}^d)$. In this case the Birman-Schwinger principle states that for $z\in\rho(H_0)$, z is the eigenvalue of the operator H_0+V if and only if -1 is an eigenvalue of the operator $X:=W_1(H_0-z)^{-1}W_2$, where $W_1=V|V|^{-1/2}$ and $W_2=|V|^{1/2}$. Moreover, the corresponding geometric multiplicities coincide.

Let us also recall well known results that are valid for $H_0=(-\Delta)^l$ and $V\in C_0^\infty(\mathbb{R}^d)$. Firstly, the function $\varsigma\mapsto \det_n(1+W_1(H_0-\varsigma)^{-1}W_2)$ is analytic on the whole $\rho(H_0)$ and for all n such that $2ln>d$. Also a point $z\in\rho(H_0)$ is an eigenvalue of H_0+V if and only if $\det_n(1+W_1(H_0-z)^{-1}W_2)=0$, and the order of the zero coincides with the algebraic multiplicity of the corresponding eigenvalue of H (see [13, 20, 29]). This means that the algebraic multiplicities of eigenvalues of H can also be characterized by multiplicities of zeros of the perturbation determinant mentioned above.

Now we have the following expansion for the Birman-Schwinger operator:

$$K = W_1(A-z)^{-1}W_2 = W_1\left(\sum_{j=1}^{l}\frac{c_j}{-\Delta-k_j}\right)W_2 = \sum_{j=1}^{l}W_1\frac{c_j}{-\Delta-k_j}W_2\,,$$

where $k_j = w_j^2$. Now we will use a result obtained by Frank, Laptev and Safronov to get an upper estimate for $\|K\|_{\mathfrak{S}_{d+1}}$.

$$\begin{aligned}\|K\|_{\mathfrak{S}_{d+1}} &= \left\|\sum_{j=1}^{l}W_1\frac{c_j}{-\Delta-k_j}W_2\right\|_{\mathfrak{S}_{d+1}}\\ &\leq \sum_{j=1}^{l}\left\|W_1\frac{c_j}{-\Delta-k_j}W_2\right\|_{\mathfrak{S}_{d+1}}\\ &= \sum_{j=1}^{l}|c_j|\left\|W_1\frac{1}{-\Delta-k_j}W_2\right\|_{\mathfrak{S}_{d+1}}\\ &\leq \sum_{j=1}^{l}|c_j|C_d\left(\frac{1}{|w_j|}\int_{\mathbb{R}^d}e^{\beta_d|x|(\mathrm{Im}\ w_j)_-}\,|V(x)|^{\frac{d+1}{2}}\,dx\right)^{\frac{2}{d+1}},\end{aligned} \tag{7.8}$$

where $\beta_d = \dfrac{2\left(e^{\frac{d+1}{2}}-1\right)}{e-1}$. The last inequality is due to the result obtained by Frank, Laptev and Safronov [15, Prop. 4.2]. Note that $|c_j| = \left(\frac{1}{|z_1|}\right)^{l-1} C_1(l,j)$, and $|k_j| = |w_j|^2$, which yield the following inequality:

$$\frac{|c_j|}{|w_j|^{\frac{2}{d+1}}} = \frac{C_1(l,j)}{|z_1|^{\frac{1}{d+1}+l-1}} \leq \frac{1}{z_1^{\frac{1}{d+1}+l-1}}\max_j C_1(l,j) =: \frac{1}{|z_1|^{\frac{1}{d+1}+l-1}}C_1(l)\,.$$

Hence our estimate becomes

$$\|K\|_{\mathfrak{S}_{d+1}} \leq \frac{lC_dC_1(l)}{|z|_1^{\frac{1}{d+1}+l-1}}\max_j\left(\int_{\mathbb{R}^d}e^{\beta_d|x|(\mathrm{Im}\ w_j)_-}\,|V(x)|^{\frac{d+1}{2}}\,dx\right)^{\frac{2}{d+1}}.$$

Recall that our area of interest is Ω, which means that all of our w_j lie above the line

$t - i\varepsilon \sin\left(\frac{\pi}{2l}\right)$, which implies that $(\text{Im } w_j)_- \leq \varepsilon \sin\left(\frac{\pi}{2l}\right)$. And so we get the "final" estimate estimate to be

$$\|K\|_{\mathfrak{S}_{d+1}} \leq \frac{C_d C_2(l)}{|z_1|^{\frac{1}{d+1}+l-1}} \left(\int_{\mathbb{R}^d} e^{\varepsilon' \beta_d |x|} \, |V(x)|^{\frac{d+1}{2}} \, dx \right)^{\frac{2}{d+1}} , \tag{7.9}$$

where $\varepsilon' = \varepsilon \sin\left(\frac{\pi}{2l}\right)$.

7.4 Proof of Theorem 7.1

As discussed in the section above we will be looking at $\det_{d+1}(1 + W_1(H_0 - \varsigma)^{-1} W_2)$. Note that this is the case when $d > 2l$. This means that $(d+1)$th determinant does not have a pole. Let us note that

$$|\det_n(1 + X)| \leq e^{\gamma_n \|X\|^n_{\mathfrak{S}_n}} ,$$

for some $\gamma_n > 0$. The proof of this statement can be found in Lemma 2.1; it is essentially due to Weyl's inequality [28, Thm. 1.15]. Now if we set $w^2 = k$ and apply Proposition 7.3 to the function $a(w) = \det_{d+1}(1 + K(w))$ we will get an estimate for the number of zeros of the function $\det_{d+1}(1 + K(w))$. Consequently we get the following estimate for $\ln |a(w)| = \ln |\det_{d+1}(1 + K)|$:

$$\begin{aligned}
\ln |\det_{d+1}(1 + K)| \leq \gamma_{d+1} \, \|K\|^{d+1}_{\mathfrak{S}_{d+1}} &\leq \frac{\gamma_{d+1} C_{d,l}}{|w_1|^{2(ld+l-d)}} \left(\int_{\mathbb{R}^d} e^{\varepsilon' \beta_d |x|} \, |V(x)|^{\frac{d+1}{2}} \, dx \right)^2 \\
&= \frac{1}{|w_1|^{l+1}} \frac{\gamma_{d+1} C_{d,l}}{|w_1|^{2ld-2d+l-1}} \left(\int_{\mathbb{R}^d} e^{\varepsilon' \beta_d |x|} \, |V(x)|^{\frac{d+1}{2}} \, dx \right)^2 \\
&\leq \frac{1}{|w_1|^{l+1}} \frac{\gamma_{d+1} C_{d,l}}{(\varepsilon')^{2ld-2d+l-1}} \left(\int_{\mathbb{R}^d} e^{\varepsilon' \beta_d |x|} |V(x)|^{\frac{d+1}{2}} dx \right)^2 ,
\end{aligned} \tag{7.10}$$

where the last inequality holds due to the fact that we will be considering $w \in \partial\Omega$ which lie at least ε' away from zero. So we will apply Proposition 7.3 to the

$\det_{d+1}(1+K)$ with

$$D = \frac{\gamma_{d+1} C_{d,l}}{(\varepsilon')^{2ld-2d+l-1}} \left(\int_{\mathbb{R}^d} e^{\varepsilon' \beta_d |x|} |V(x)|^{\frac{d+1}{2}} dx \right)^2 .$$

Note that $\left| \int_{\partial\Omega_R} f(w)(w+\varepsilon e^{\frac{i\pi}{2l}})^{l-1} dw \right| = 0$ and $n = 0$, since $a(w)$ is analytic. Hence the total number of zeros of $a(w)$ in $\Omega + \varepsilon e^{\frac{i\pi}{2l}} = \left\{ w \in \mathbb{C} : w = |w| e^{i\varphi}, \quad \varphi \in \left[0, \frac{\pi}{l}\right) \right\}$ can be estimated as follows:

$$N \leq \frac{lDC_l}{\pi(\varepsilon')^{l+1}} = \frac{\gamma_{d+1} C_{d,l}}{\pi(\varepsilon')^{2(ld-d+l)}} \left(\int_{\mathbb{R}^d} e^{\varepsilon' \beta_d |x|} |V(x)|^{\frac{d+1}{2}} dx \right)^2 .$$

And finally, after changing the notation by setting $\varepsilon = \varepsilon'$ and then, once again, taking $\beta\varepsilon$ as a new ε and adjusting $C_{d,l}$, we arrive at the following bound:

$$N \leq \frac{C_{d,l}}{\varepsilon^{2(ld-d+l)}} \left(\int_{\mathbb{R}^d} e^{\varepsilon |x|} |V(x)|^{\frac{d+1}{2}} dx \right)^2 .$$

But now, by the argument in the second paragraph of the subsection 7.3.1, the same estimate gives us the bound for the number of eigenvalues of the operator H. □

Most of the papers listed in the references section contain results on the eigenvalues of non-self-adjoint operators. More specifically, those are the articles [1]-[19], [21]-[27] and [30]-[31]. The remaining references were needed for technical reasons.

CHAPTER 8: FUTURE WORK

Spectral Analysis is a very vast and important field of mathematics. It is very widely used in other sciences, such as physics, chemistry, quantum mechanics, and many others. There are many ways that the results in this dissertation may be extended or generalized. For once, the results about the discrete Schrödinger or Dirac operators can be generalized to an arbitrary dimension d. Moreover, for the Polyharmonic operator one could also get an estimate for the case when $2l < d$, or maybe even get a similar estimate for the case when $d - even$. Such results may not be very easy to do. However, as some of the best mathematicians put it, "quality is more important than quantity".

Another interesting extension one may consider is to study the operator $(-\Delta)^l + V$, where $0 < l < 1$, which was suggested by Dr. Molchanov. Many of the qualities of such operator are known, so it would be nice to also estimate the total number of eigenvalues.

There are also some questions that arise from physics. The following were suggested by Dr. D. Jacobs:

1) Construct specific $V(x)$ that could be interesting. Have it such that $V(x) = 0$ outside some finite range (i.e. $V(x)$ is compactly supported), say $x > L$, but on $0 \leq x \leq L$, let $V(x)$ be an interesting function. Actually, an interesting family of functions, where you can control the shape of $V(x)$. Perhaps the degree of "roughness". I will call this family of functions $V(x|shape)$. Then, using your formulas, you can find the lowest possible bound of number of eigenvalues for a particular

$V(x|shape)$. In this way, you might be able to gain insight into the type of shapes that gives more/less eigenvalues. How does the variations in $V(x)$ effect the number of eigenvalues?

2) Normally in physics we like self-adjoint V, but if you do allow for non-self-adjoint potential V, how can you control the number of eigenvalues that have an imaginary component, (+ or -). Can you design a $V(x|shape)$ that might correspond to a loss of particles in a system or gain of particles?

I believe this would correspond to cases where there are eigenvalues below or above the real line. Complex eigenvalues with positive imaginary parts would respond physically different than negative imaginary parts. Could you provide bounds for numbers of eigenvalues with positive imaginary components and also for negative imaginary components. My intuition tells me that the positive half plane would correspond to loss of particles and the negative half plane would correspond to gain of particles (or vice versa depending on how the original operator is defined).

3) Finally, if we push further, could there be bounds for ranges of eigenvalues within concentric circles. Could you reduce this radius, and count number of eigenvalues within a smaller radius? If you could, you would be able to give estimates of the number of eigenvalues with magnitude within a ring. $R_1 < |\lambda| < R_2$. I realize all this is probably too difficult to calculate, but these sort of questions would be very interesting.

Often times mathematicians obtain results without knowing if it is useful in the "real world applications". And quite often, many years later, it comes out that those results do indeed have some very important applications. In case of Spectral Theory it might not always be clear right away when a result will be very useful, but with time some interesting applications might come to light [22].

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APPENDIX 1: EXISTENCE OF POLES OF THE $det_2(1+K(k))$ in CHAPTER 5

Since we would like to apply the Proposition 4.3 to the function $a(k) = det_2(I+K(k))$ we need to do some analysis to demonstrate that the second determinant contains poles at $k = \pm 1$.

First notice that $\det{}_2(I + K(k)) = \det{}_1(I + K(k)) \cdot e^{-Tr[K(k)]}$.

Let us recall that the kernel of $K(k)$ is $W(n)\dfrac{-k}{k^2-1}W(m)Sign(V(m))$. We can rewrite it as follows:

$$\begin{aligned} S(n)W(n)\frac{-k}{k^2-1}W(m) = &\frac{-k}{k^2-1}W(n)S(n)\left(k^{-|n-m|}-1\right)W(m) \\ &+\frac{-k}{k^2-1}W(n)S(n)W(m) \end{aligned}$$

where $S(n)$ is the sign function of V.

As a result, we can split the operator $K(k)$ into the sum of two operators, as so

$$K(k) = X_0(k) + \widetilde{X}(k), \tag{A.1}$$

where $X_0(k) = \dfrac{-k}{k^2-1}\Gamma = \dfrac{-k}{k^2-1}WSW$, and $\widetilde{X}(k)$ is analytic at $k = \pm 1$. Note that Γ is a rank 1 operator, since Γ can be expressed as

$$\Gamma u = SW < u, W > . \tag{A.2}$$

From this we can also see that $u = SW$ is an eigenvector of Γ. Consequently, we get

$$\lambda =< SW, W >= \sum_n S(n)W(n)W(n) = \sum_n V(n)$$

to be the eigenvalue of Γ.

Now from equation A.1, after some algebra, we also get the following

$$I + K(k) = 1 + \widetilde{X}(k) + X_0(k) = (I + X_0(k))[I + (I + X_0(k))^{-1}\widetilde{X}(k)]$$

which, in turn, implies that

$$\det\left(I + K(k)\right) = \det\left(I + X_0(k)\right)\det\left(I + (I + X_0(k))^{-1}\widetilde{X}(k)\right) \tag{A.3}$$

Note that the eigenvalue of the operator $X_0(k) = \frac{-k}{k^2-1}\Gamma$ is equal to $\frac{-k}{k^2-1}$ times the eigenvalue of Γ. Which, in turn, equals to $\frac{-k}{k^2-1}\sum_{n=-\infty}^{\infty} V(n)$. Hence the determinant of the operator $I + X_0(k)$ is

$$\det(I + X_0(k)) = \prod_j (1 + \lambda_j) = 1 - \frac{k}{k^2-1}\sum_{n=-\infty}^{\infty} V(n)\,. \tag{A.4}$$

Note that from equation A.2 we see that there is only one nonzero eigenvalue, namely

$$\lambda = -\frac{k}{k^2-1}\sum_{n=-\infty}^{\infty} V(n)\ .$$

Hence A.4 holds.

Now, since SW is an eigenvector of Γ and $< SW, W >$ is its corresponding eigenvalue, we can decompose the space $\mathfrak{H}$ into orthogonal subspaces as follows:

$$\mathfrak{H} = \ell^2(\mathbb{Z}, \mathbb{C}^2) = \mathfrak{H}_0 \oplus \mathfrak{H}_0^{\perp},$$

where $\mathfrak{H}_0 = \alpha SW + \beta W$. Then the following two properties hold:

1) $\Gamma : \mathfrak{H}_0 \to \mathfrak{H}_0$ 2) $\Gamma : \mathfrak{H}_0^{\perp} \to 0.$

This means that $X_0\mathfrak{H}_0 \to \mathfrak{H}_0$ and $X_0\mathfrak{H}_0^{\perp} \to 0$, since X_0 is proportional to Γ. As a result, we can write the identity as $I = I_{\mathfrak{H}_0} \oplus I_{\mathfrak{H}_0^{\perp}}$. From this we get

$I + X_0(k) = (I_{\mathfrak{H}_0} + X_0(k)) + I_{\mathfrak{H}_0^\perp}$.

Note that $\{SW, W\}$ makes up a basis for $\mathfrak{H}_0$. Moreover, $\Gamma(SW) = \lambda SW$ (since SW is an eigenvector of Γ) and

$$\Gamma(W) = \mu SW \ ,$$

where

$$\lambda = \sum_{n=-\infty}^{\infty} V(n), \quad \mu = \sum_{n=-\infty}^{\infty} |V(n)| \ .$$

Hence, we can express the operators Γ and X_0 as so

$$\Gamma = \begin{bmatrix} \lambda & \mu \\ 0 & 0 \end{bmatrix} \text{ and } X_0(k) = \frac{-k}{k^2-1} \begin{bmatrix} \lambda & \mu \\ 0 & 0 \end{bmatrix} \Rightarrow \ I + X_0(k) = \begin{bmatrix} 1 + \dfrac{-k\lambda}{k^2-1} & \dfrac{-k\mu}{k^2-1} \\ 0 & 1 \end{bmatrix} .$$

Now let us find $(I + X_0)^{-1}$:

$$(I + X_0)^{-1} = \frac{1}{1 + \frac{-k\lambda}{k^2-1}} \begin{bmatrix} 1 & \dfrac{k\mu}{k^2-1} \\ 0 & 1 + \dfrac{-k\lambda}{k^2-1} \end{bmatrix} = \frac{k^2-1}{k^2-k\lambda-1} \begin{bmatrix} 1 & \dfrac{k}{k^2-1}\mu \\ 0 & 1 + \dfrac{-k}{k^2-1}\lambda \end{bmatrix}$$

Notice that when $\lambda \neq 0$ we have

$$(I + X_0)^{-1} = \frac{1}{k^2-k\lambda-1} \begin{bmatrix} k^2-1 & k\mu \\ 0 & k^2-k\lambda-1 \end{bmatrix}, \qquad \text{and } k^2-k\lambda-1 \neq 0$$

Hence in this case $(I + X_0)^{-1}$ exists and is analytic at $k = \pm 1$ when $\lambda \neq 0$. As a result, from equations A.3 and A.4, we see that

$$\det(I + K(k)) = \left(1 - \frac{k}{k^2-1}\lambda\right) \det\left(I + (I + X_0(k))^{-1}\widetilde{X}(k)\right)$$

where $\lambda = \sum_{n=-\infty}^{\infty} V(n)$ and $\det\Big(I + (I + X_0(k))^{-1}\widetilde{X}(k)\Big)$ is analytic at $k = \pm 1$.

Now let us take a look at what happens when $\lambda = \sum_{n=-\infty}^{\infty} V(n) = 0$. So let us suppose that $\lambda = \sum_{n=-\infty}^{\infty} V(n) = 0$. Consider a new operator $V_{\varepsilon,i} = V + \varepsilon(\cdot, e_i)e_i$, for some $i \in \mathbb{Z}$, where $\{e_i\}$ is a basis for $\mathfrak{H}$. Then clearly $\|V - V_{\varepsilon,i}\| \leq \varepsilon$ and $\sum_{n=-\infty}^{\infty} V_{\varepsilon,i}(n) \neq 0$. Then, according to the argument above, the operator $H_{\varepsilon,i} := H_0 + V_{\varepsilon,i}$ has poles at ± 1. As a result we can get an estimate on the number of eigenvalues for the operator $H_{\varepsilon,i}$. After this we can apply the result obtained by Laptev and Safronov in Proposition 3 of [19]. From this proposition we conclude that the number of eigenvalues of operators H and $H_{\varepsilon,i}$ will be the same. As a result, the estimate for $H_{\varepsilon,i}$ will also hold for H. Hence, without loss of generality we can assume that $\lambda = \sum_{n=-\infty}^{\infty} V(n) \neq 0$, as the class of potentials for which $k = \pm 1$ are poles of $\det{}_1(1+X)$ forms a dense subset of potentials for which $\|V\|_{\infty,q} < \infty$. Where $\|V\|_{\infty,q}$ is defined as follows:

$$\|V\|_{\infty,q} = \sup_{-\infty<n<\infty} |V_n q^{-|n|}|$$

where Λ is any constant greater than 1 and $q = \frac{1}{\Lambda^{2+\varepsilon}}$, for any $\varepsilon > 0$.

APPENDIX 2: EXISTENCE OF POLES OF THE $det_3(1 + X(w))$ IN CHAPTER 6

In order for us to be able to use Proposition 6.2 we need to show that the function $a(k) = det_3(I + X(w))$ contains a pole of some order n at $w = 0$. In this subsection we will use similar agrumentation as in Appendix 1 to show that the function $a(k) = det_3(I + X(w))$ contains the pole of order 1 at $\sqrt{k} = w = 0$. To do so we need to decompose operator $X(w)$ into the sum of two operators, one analytic with the respect to w and one having a pole of order 1 at $w = 0$.

Recal that the integral kernel for $(A - z)^{-1}$, where $A = (-\Delta)^2$, is

$$\rho(x, y) = \frac{1}{2k}\left(\frac{e^{i\sqrt{k}|x-y|} - e^{-\sqrt{k}|x-y|}}{2\pi|x - y|}\right) ,$$

where $k^2 = z$. So after the Taylor expansion we get

$$\rho(x, y) = \frac{i + 1}{8\pi\sqrt{k}} + \alpha(k, x, y) ,$$

where α is an analytic function of k. Hence we can express the operator $X(w)$ as follows

$$X(w) = WSign(V)\frac{i + 1}{8\pi w}W + \widetilde{X}(w) ,$$

where $\widetilde{X}(w)$ is analytic at $w = 0$. The we can rewrite this as

$$X(w) = X_0(w) + \widetilde{X}(w) = \frac{i + 1}{8\pi w}\Gamma + \widetilde{X}(w) \tag{A.5}$$

where $\Gamma = WSW$, $S = Sign(V)$. Note that Γ is a rank 1 operator: $\Gamma u = SW < u, W >$. Then $u = SW$ is an eigenvector of Γ. Hence

$$< SW, W >= \int_{\mathbb{R}^3} V(x)dx$$

is an eigenvalue of the operator Γ.

Now from A.5, using same argumentation as in Appendix 1, we get the equality A.3, that is

$$\det\left(I + X(w)\right) = \det\left(I + X_0(w)\right)\det\left(I + (I + X_0(w))^{-1}\widetilde{X}(w)\right) \tag{A.6}$$

Note that the eigenvalue of $X_0(w)$ is

$$\frac{i+1}{8\pi w}\int_{\mathbb{R}^3} V(x)dx\ .$$

Using this fact we can calculate the first determinant on the right side of the equation above to get

$$\det(1 + X_0(w)) = \prod_j (1 + \lambda_j) \tag{A.7}$$

where λ_j are eigenvalues of the operator Γ. However, since Γ has only one eigenvalue we get the equality

$$\det(1 + X_0(w)) = 1 + \frac{i+1}{8\pi w}\int_{\mathbb{R}^3} V(x)dx \tag{A.8}$$

Now we will use same argumentation of Appendix 1 almost word for word, except at a few places, where the operators differ. First, will decompose our Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^3, \mathbb{C})$ into two orthogonal subspaces $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_0^{\perp}$, where $\mathfrak{H}_0 = \alpha SW + \beta W$. Then note that the following two properties hold:

$$1)\ \Gamma : \mathfrak{H}_0 \to \mathfrak{H}_0 \qquad\qquad 2)\ \Gamma : \mathfrak{H}_0^{\perp} \to 0.$$

This means that $X_0\mathfrak{H}_0 \to \mathfrak{H}_0$ and $X_0\mathfrak{H}_0^{\perp} \to 0$, since X_0 is proportional to Γ. Then $I + X_0(k) = (I_{\mathfrak{H}_0} + X_0(k)) + I_{\mathfrak{H}_0^{\perp}}$, where $I = I_{\mathfrak{H}_0} \oplus I_{\mathfrak{H}_0^{\perp}}$. Note that $\{SW, W\}$ makes up a basis for $\mathfrak{H}_0$. Moreover, $\Gamma(SW) = \lambda SW$ (since SW is an eigenvector of Γ) and

$\Gamma(W) = \mu SW$, where

$$\lambda = \int_{\mathbb{R}^3} V(x)dx \quad \text{and} \quad \mu = \int_{\mathbb{R}^3} |V(x)|dx \ .$$

Hence, we can express the operators Γ and X_0 as so

$$\Gamma = \begin{bmatrix} \lambda & \mu \\ 0 & 0 \end{bmatrix} \Rightarrow \ X_0(k) = \frac{i+1}{8\pi w} \begin{bmatrix} \lambda & \mu \\ 0 & 0 \end{bmatrix} \Rightarrow \ I + X_0(k) = \begin{bmatrix} 1 + \dfrac{(i+1)\lambda}{8\pi w} & \dfrac{(i+1)\mu}{8\pi w} \\ 0 & 1 \end{bmatrix}.$$

From this we get that $(I + X_0)^{-1}$ is

$$(I + X_0)^{-1} = \frac{1}{1 + \frac{(1+i)\lambda}{8\pi w}} \begin{bmatrix} 1 & \dfrac{-\mu(1+i)}{8\pi w} \\ 0 & 1 + \dfrac{(1+i)\lambda}{8\pi w} \end{bmatrix} = \frac{8\pi w}{8\pi w + (1+i)\lambda} \begin{bmatrix} 1 & \dfrac{-\mu(1+i)}{8\pi w} \\ 0 & 1 + \dfrac{(1+i)\lambda}{8\pi w} \end{bmatrix}$$

Notice that when $\lambda \neq 0$ we have

$$(I + X_0)^{-1} = \frac{1}{8\pi w + (1+i)\lambda} \begin{bmatrix} 8\pi w & -\mu(1+i) \\ 0 & 8\pi w + (1+i)\lambda \end{bmatrix}, \qquad \text{and } 8\pi w + (1+i)\lambda \neq 0$$

Hence in this case $(I + X_0)^{-1}$ exists and is analytic at $w = 0$ when $\lambda \neq 0$. As a result, from equations A.6 and A.8, we see that

$$\det(I + X(w)) = \left(1 + \frac{i+1}{8\pi w} \int_{\mathbb{R}^3} V(x)dx\right) \det\left(I + (I + X_0(w))^{-1}\widetilde{X}(w)\right)$$

where $\lambda = \displaystyle\int_{\mathbb{R}^3} V(x)dx$ and $\det\left(I + (I + X_0(k))^{-1}\widetilde{X}(k)\right)$ is analytic at $k = \pm 1$.

Now if $\lambda = \displaystyle\int_{\mathbb{R}^3} V(x)dx = 0$ the we use the exact argument as at the end of Appendix 1. To do so we replace all summations over $\mathbb{Z}$ by integrals over $\mathbb{R}^3$, all n replaced by x. Hence, without loss of generality we can assume that $\lambda = \displaystyle\int_{\mathbb{R}^3} V(x)dx \neq 0$, as the class of potentials for which $w = 0$ is a pole of $\det{}_1(1+X)$ forms a dense subset of all potentials for which $\|V\|_{1,\varepsilon} < \infty$. Here $\|V\|_{1,\varepsilon}$ is defined as follows:

$$\|V\|_{1,\varepsilon} = \int_{\mathbb{R}^3} |V(x)| e^{\varepsilon|x|} dx \,, \qquad \forall \varepsilon > 0 \,.$$