POPULATION DYNAMICS WITH IMMIGRATION

by

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ABSTRACT

DAN HAN. Population Dynamics with Immigration. (Under the direction of DR. STANISLAV MOLCHANOV)

The paper contains the complete analysis of the Galton-Watson models with immigration, including the processes in the random environment, stationary or non-stationary ones. We also study the branching random walk on Z^d with immigration and prove the existence of the limits for the first two correlation functions. Additional results concerns the Lyapunov stability of the moments with respect to small perturbations of the parameters of the model such as mortality rate, birth rate and immigration rate.

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CHAPTER 1: INTRODUCTION

The subject of the population dynamics from the mathematical view point is the analysis of the evolution in space and time of some species (we call them particles) in the presence of such factors as birth and death processes, migration, immigration and etc. The simplest models of such kind exclude the interaction between particles and even the spatial distribution of the species.

The classical example is the Galton-Watson branching process with continuous time. One of the defects of this model is the absence of the statistical equilibrium. That is, $n(t)$, the number of the particles at time t, either goes to ∞ with positive probability or population degenerates P a.s. In the critical case when the mortality rate and birth rate are equal to each other, $P(n(t) = 0) \to 1$ and this was the central observation by Galton, the founder of the theory of branching processes.

The central problem in the population dynamics is the study of the models which demonstrates the convergence to statistical equilibrium. On the mathematical level, it is the theory of the infinite-dimensional Markov processes which phase space is the set of all possible configurations of the particles, either on Z^d (lattice models) or on R^d (continuous models).

Configurations are changing in the time (due to migration,, birth-death processes etc). Existence of statistical equilibrium is equivalent to the ergodicity of Markov process, mentioned above. The probability measure (distribution) $P(\cdot)$ is the space of configurations converges weakly to the limit if $t \to \infty$. The limiting measure $P(\infty)$ is the desirable stationary distribution (steady states).

The simplest population model with steady state is so called contact model.

In 2006, Y. Kondratiev and A. Skorokhod first proposed a continuous contact model [7] in R^d. A series of researches in continuous contact models have been done by Y. Kondratiev and his group, especially see the publication of Y. Kondratiev, O. Kutovyi and S. Pirogov [6] in 2008. Under certain general technical assumptions on the infection spreading characteristics, they constructed the non-equilibrium contact process as a a spatial birth-and-death Markov process on configuration space. The process describes a stochastic evolution in configurations, i.e, locally finite subsets $\gamma \subset \mathbb{R}^d$ as the phase space of the process in the language of papers [7],[6]. During the stochastic evolution, each particle independently generates a new particle according to a dispersion probability density $0 \leq a \in L^1(\mathbb{R}^d)$, which is an even function. The contact process generator is given on proper function $F(\gamma)$:

$$(LF)(\gamma) = \sum_{x\in\gamma} \mu(x,\gamma)[F(\gamma \setminus x) - F(\gamma)] + \int\limits_{R^d} \beta(x,\gamma)[F(\gamma \cup x) - F(\gamma)]dx$$

where $\mu(x,\gamma)$ describes the death rate of the particle x in the configuration γ. $\beta(x,\gamma)$ describes the rate at which, given the configuration γ, a new particle is born at $x \in R^d$. They considered a spatial branching process with killing. Assume the death rate $\mu = 1$ and the birth rate

$$\beta(x,\gamma) = \kappa \sum_{y\in\gamma} a(x-y)$$

$$0 \leq a \in L^1(R^d), \int_{R^d} a(z)dz = 1$$

And in the nearest future, we will set κ, the diffusivity as 1.

The above construction means each $y \in \gamma$ generates a new particle at $x \in R^d$ with the rate $\kappa a(x-y)dx$ independently. Additional technical assumption in [6] is the existence of the second moment: $\int_{R^d} a(z)z^2 < \infty$.

For the critical value $\kappa = 1$ and the dimension $d \geq 3$, they proved the existence of a continuous family of invariant measures parameterized by the density values. Starting with an admissible measure uniquely defined by the density of the initial state, the critical contact process converges to the equilibrium measure uniquely defined by the density of the initial state. But in the dimension $d = 2$, invariant measures for the model do not exist. Namely, when $d = 2$, correlations between population members are growing in time too fast and the second-order limiting correlation function will diverge to infinity.

From a biological perspective, Y. Kondratiev and his group's contact model is a "forest" model: there is no motion of parental particles "trees" in space, however, each parental "tree" can produce a new "seed", and the seeds can jump to other random positions around the parental "tree" and originate the new trees.

Denote $n(t,\Gamma)$ is the number of particles at moment t in the set $\Gamma \subset R^d$. Assume that the initial field of "trees" has a Poissonian structure with the density ρ_0, i.e. $\forall(\Gamma \subset \mathcal{B}(R^d), m(\Gamma) =\mid \Gamma \mid \leq \infty$, we have (for constant rates μ and β)

$$P\{n(0,\Gamma) = k\} = e^{-\lambda(\Gamma)} \frac{(\lambda(\Gamma))^k}{k!}, k \geq 0$$

$$\lambda(\Gamma) = |\Gamma|\rho_0$$

During a small period time dt, each particle located at $x \in R^d$can die with probability μdt or generates a new offspring with probability βdt. It means that the parental tree stays at the same point $x \in R^d$, but it generate the seed (new offspring) which jumps from x to $x + z$ with the distribution density $a(z)$.

The field $n(t, \Gamma)$ has multiplicity one and the correlation functions $k_t^{(n)}(x_1, x_2, ..., x_n)$ has the sense of the densities:

$k_t^{(n)}(x_1, x_2, \cdots, x_n)dx_1...dx_n = P\{$to find a particle at the moment $t \geq 0$

in the volumes $dx_1, \cdots, dx_n$ around n points $x_1, \cdots, x_n\}$

In [6],The following model was introduced by Kondratiev. Consider initial Poissonian field in R^d. Assume the death rate μ of the particles is equal to the birth rate β. Whenever there is a transformation, the probability of death or birth is equal to each other. That is, at moment t, a particle either dies with probability $\frac{1}{2}$ or produces a new seed, which will be randomly distributed according to $a(z), z \in R^d$. Assume this density function $a(z)$ is symmetric $a(z) = a(-z)$ and $\int_{R^d} a(z) = 1$. Differential equations of $k_t^{(n)}(x_1, \cdots, x_n)$ are derived. The first two moments have the following form:

$$\begin{aligned} &\frac{\partial k_t^{(1)}(x)}{\partial t} = -\beta k_t^1(x) + \beta \int_{R^d} a(x-z)k_t^{(1)}(z)dz \\ &k_0^{(1)}(x) = \rho_0 \end{aligned} \tag{1}$$

and

$$\frac{\partial k_t^{(2)}(x_1,x_2)}{\partial t} = -2\beta k_t^{(2)}(x_1,x_2) + \beta k_t^{(1)}(x_1)a(x_1-x_2) + \beta k_t^{(1)}(x_2)a(x_2-x_1) + \beta \int_{R^d} a(x_1-z)k_t^{(2)}(x_2,z)dz + \beta \int_{R^d} a(x_2-z)k_t^{(2)}(x_1,z)dz \quad (2)$$

$$k_0^{(2)}(x_1,x_2) = \rho_0^2$$

Under such a critical case $\beta = \mu$, the equation (1) has unique solution $k_t^{(1)} = \rho_0$. The density is invariant of dynamics. For $d \geq 3$,there exists a limiting distribution of $k_t^{(2)}$. For $d \leq 2$, $k_t^{(2)}$ has a limiting distribution only when continuous density $a(z)$ satisfies some specific conditions.

The papers [6] and [7] covered only the continuous situation: particle fields in R^d, $d \geq 1$. The methods mentioned in [7] and [6] are not working in the lattice case. In the papers [7],[6], the assumption that two particles cannot appear in the same site (i.e the particle field has the multiplicity one is the central point of analysis. For the lattice models, this restriction will lead to highly complicated and highly non-linear equations.

In 2010, Y. Feng, S. Molchanov and J. Whitmeyer [3] started the study of the lattice contact models. They derived the equations for first correlation functions: $m_1(t,x) = E[n(t,x)]$, $m_2(t,x_1,x_2) = E[n(t,x_1)n(t,x_2)]$, $m_3(t,x_1,x_2,x_3) = E[n(t,x_1)n(t,x_2)n(t,x_3]$. And under assumptions about criticality ($\beta = \mu$) and transitivity of the the underlying random walk with the generator

$$\mathcal{L}_a\psi(x) = \kappa \sum_{z \neq 0} [\psi(x+z) - \psi(x)]a(z)$$

, they proved the existence of the limits

$$\lim_{t\to\infty} m_i(t,\cdot) = m_i(t,\cdot), i = 1,2,3$$

For the first two moments,they presented the exact formulas for the limiting density and correlation function.

Let us stress that in [3], the dimension d can be arbitrary and there are no any moment conditions. As the result, in dimension $d = 1,2$ under regularity assumption on the heavy tailed density $a(z)$, the limiting moments exist .

Now the existence of the steady state for lattice contact process or branching random walk on Z^d is proven under the full generality, that is underlying random walk is transient and $\beta = \mu$ (criticality), see S. Molchanov, J. Whitmeyer [9]

In 2017, S. Molchanov and J. Whitmeyer [9] proposed the new method to study the problem of the steady states for the critical contact process. The model in [9] and [3] is different from the model in [7] and [6]. It includes the spatial motion of the particles. Particularly, at the moment of the birth of a new particle (offspring), it can stay at the same site as the parental particle does.

Let us give the detailed description of the contact model (lattice case),see details in [9].

We now consider our process with birth, death and migration on a countable space, specifically the lattice $\mathbb{Z}^d$. We denote $N(t,y)$, $t \leq 0$, $y \in Z^d$ as the global population at time t in the position $y \in Z^d$ and denote $n(t,y,x)$ as the subpopulation at site $y \in Z^d$ generated by a single initial particles in the site $x \in Z^d$. Those subpopulations are independent, thus

$$N(t,y) = \sum_{x\in Z^d} n(t,y,x), N(0,y) = 1$$

Each particle follows a random walk with generator $\kappa\mathcal{L}_a\psi(x) = \kappa \sum_{y\in Z^d} [\psi(x+y) - \psi(x)]a(y)$ with a symmetrical $a(y) = a(-y)$ and normalization $\sum_y a(y) = 1$. Death occurs at rate μ and birth occurs at rate β. That means,during a period of time $(t, t+dt)$, the probability that one particle will split into two particles is βdt and the probability that one particle will die is μdt. In splitting, parental particle stays at the same site and the new offspring jumps from x to $x+y$ with distribution $b(y)$ and $\sum_{y\in Z^d} b(y) = 1$ so that b is a probability distribution and b is symmetric so that the probabilities to jump to opposite directions are equal: $b(y) = b(-y)$. Using the jump distribution $b(\cdot)$, on can introduce the new linear operator

$$\mathcal{L}_b\psi(x) = \beta \sum_{z\neq 0} [\psi(x+z) - \psi(x)]b(z)$$

For the study of subpopulation $n(t,y,x)$, $x, y \in Z^d$, the generating function $u_z(t,x,y) = E_x z^{n(t,y,x)} = \sum_{j=0}^{\infty} P\{n(t,y,x) = j\}z^j$. $u_z(t,x,y)$ satisfies the Kolmogorov-Petrovski-Piskunov type equation:

$$\frac{\partial u_z}{\partial t} = \kappa\mathcal{L}_a u_z - (\beta+\mu)u_z + \beta u_z \sum_{y\in Z^d} u_z(t, x+v, y)b(v) + \mu$$
$$u(0,x,y) = \begin{cases} z, & x = y \\ 1, & x \neq y \end{cases} \tag{3}$$

Then the factorial moments $m_l(t,x,y) = E(n(n-1)\cdots(n-l+1))$ can obtain their differential equations by differentiating 3 over z and substituting $z = 1$. For example,

$m_1(t,x,y) = E_x n(t,y,x)$ is given by

$$\frac{\partial m_1}{\partial t} = (\kappa\mathcal{L}_a + \beta\mathcal{L}_b)m_1 + (\beta-\mu)m_1$$
$$m_1(0,x,y) = \delta(y-x) \tag{4}$$

In the paper of S. Molchanov and J. Whitmeyer [9],this result uses backward equation technique instead of forward Kolmogorov equations, a strategy not feasible in the continuous space. When birth and mortality rates are equal to each other and the underlying random walk generated by $\mathcal{L}_a$ is transient, then $m_l(t) = EN(t,x)(N(t,x) - 1)\cdots(N(t,x) - l + 1) \xrightarrow{t\to\infty} m_l(\infty)$ and therefore $N(t,x) \xrightarrow[law]{t\to\infty} N(\infty,x)$, where $N(\infty,x)$ is a steady state, that is a random variable with a finite distribution. Thus the whole population reaches a stationary distribution (steady state).

The second fundamental question of the population dynamics is the stability (or instability) of the steady state with respect to small perturbation of the parameters of the model. How to measure of this "smallness"? There are two possibilities:

(a) use L^∞ norm. Instead of constant rates β, μ, one can consider functions

$$\beta(x) = \beta_0 + \epsilon\xi(x)$$

$$\mu(x) = \mu_0 + \epsilon\eta(x)$$

where $|\xi(x)| \le 1, x \in Z^d$, $|\eta(x)| \le 1$ and ϵ is a small parameter.

The function $\xi(x)$ and $\eta(x)$ can be random or deterministic. It is so-called Lyapunov stability .

(b) use local perturbations, i.e. $\beta(x) = \beta_0$, $\mu(x) = \mu_0$ when $x \in Z^d - \Gamma$ but on the finite set Γ, the difference $\beta(x) - \mu(x) = V(x)$ is not zero.

It is clear that for the contact processes, we cannot expect the Lyapunov stability. For instance, if $\beta_0 = \mu_0$ and the underlying random walk is transient, then the steady state exists but for $\xi(x) \equiv 1$,$\eta(x) \equiv 0$, the process with $\beta(x) = \beta_0 + \epsilon$, $\mu(x) = \mu_0 = \beta_0$ is supercritical. For the first moment, we have the obvious formula

$$m_1(t,x) = m_0 e^{\epsilon t} \to \infty$$

as $t \to \infty$.

For $\xi(x) \equiv -1$, the result is opposite,

$$m_1(t,x) = m_0 e^{-\epsilon t} \to 0$$

as $t \to \infty$.

Even assumption that $\xi(x)$, $\eta(x)$ are independent random symmetrically distributed fields, i.e. $V(x) = \epsilon(\xi(x) - \eta(x))$ is symmetrically distributed potential cannot lead to stabilization.

The local perturbations for the contact model were studied in several papers by E.Yarovaya [1],[13],[12]. Roughly speaking, the result is the following one: For fixed set Γ where $\beta(x) - \mu(x) > 0$, the steady states exists if $\underset{x \in \Gamma}{max}(\beta - \mu) \leq \delta$, δ is sufficiently small. If $\beta - \mu$ is large enough at least in one point, then $m_1(t,x) = E[n(t,x)] \to \infty$.

CHAPTER 2: SPATIAL GALTON-WATSON PROCESS WITH IMMIGRATION.NO MIGRATION AND NO RANDOM ENVIRONMENT.

2.1 Moments

In this section, we will study the branching process with immigration but without migration. Assume that at each site for each particle we have birth of one new particle with rate β and death of the particle with rate μ. Also assume that regardless of the number of particles at the site we have immigration of one new particle with rate k (this is a simplified version of the process in [10]). Assume that $\beta < \mu$, for otherwise the population will grow exponentially. Assume we start with one particle at each site. In continuous time, for a given site x, $x \in Z^d$, we can obtain all moments recursively by means of the Laplace transform with respect to $n(t,x)$, where $n(t,x)$ is the population size at time t at x

$$\varphi(t,\lambda) = E\, e^{-\lambda n(t,x)} = \sum_{j=0}^{\infty} P\{n(t,x) = j\} e^{-\lambda j}.$$

Specifically, for the jth moment, m_j

$$m_j(t,x) = (-1)^j \frac{\partial^j \varphi}{\partial \lambda^j}|_{\lambda=0}. \tag{5}$$

A partial differential equation for $\varphi(t,\lambda)$ can be derived using the forward Kolmogorov equations

$$n(t+dt,x) = n(t,x) + \xi_{dt}(t,x) \tag{6}$$

where the r.v. ξ is defined

$$\xi_{dt}(t,x) = \begin{cases} +1 & \beta n(t,x)dt + kdt \\ -1 & \mu n(t,x)dt \\ 0 & 1 - ((\beta+\mu)n(t,x) + k)dt \end{cases} \tag{7}$$

In other words, our site (x) in a small time interval $(t,t+dt)$ can gain a new particle with probability βdt for every particle at the site or through immigration with probability kdt; it can lose a particle with probability μdt for every particle at the site; or no change at all can happen.

For each site $x \in Z^d$, $n(t,x)$ is a branching process. All these branching process $n(t,x)$ for different site x are independent with each other and there is no interactions among those branching processes for different site x since there is no migration among those sites. Because of this fact, it is sufficient to study $n(t,x)$ for one particular site x. Thus we write $n(t)$ for $n(t,x)$ of this particular site x.

Now $\varphi(t+dt,\lambda) = E[e^{-\lambda n(t+dt)}] = E[E[e^{-\lambda n(t+dt}|n(t)]]$.

And $E[e^{-\lambda n(t+dt)}|n(t)] = e^{-\lambda n(t)}(1-(\beta+\mu)n(t)dt - kdt) + (\beta n(t)+k)dte^{-\lambda n(t)-\lambda} + \mu n(t)dte^{-\lambda n(t)+\lambda}$.

One can apply the total expectation at both sides and use the formula $-\dfrac{\partial\varphi}{\partial\lambda} = E[e^{-\lambda n(t)}n(t)]$. This leads to the general differential equation

$$\frac{\partial\varphi(t,\lambda)}{\partial t} = k(e^{-\lambda}-1)\varphi(t,\lambda) + \beta\frac{\partial\varphi(t,\lambda)}{\partial\lambda}(1-e^{-\lambda}) + \mu\frac{\partial\varphi(t,\lambda)}{\partial\lambda}(1-e^{\lambda}) \tag{8}$$

$$\varphi_0(\lambda) = e^{-\lambda} \tag{9}$$

from which we can calculate the recursive set of differential equations and we denote

$\frac{\varphi(t,\lambda)}{\partial t}$ as φ_t

$$\frac{\partial^j \varphi_t}{\partial \lambda^j} = k\sum_{i=0}^{j}\binom{j}{i}(e^{-\lambda}-1)^{(j-i)}\frac{\partial^i \varphi}{\partial \lambda^i} + \beta\sum_{i=0}^{j}\binom{j}{i}(1-e^{-\lambda})^{(j-i)}\frac{\partial^{i+1}\varphi}{\partial \lambda^{i+1}}$$
$$+\mu\sum_{i=0}^{j}\binom{j}{i}(1-e^{\lambda})^{(j-i)}\frac{\partial^{i+1}\varphi}{\partial \lambda^{i+1}}$$
$$\frac{\partial^j \varphi_0(\lambda)}{\partial \lambda^j} = (-1)^j e^{-\lambda}$$

Applying Eq. 5 we obtain a set of recursive differential equations for the moments

$$(-1)^j \frac{dm_j(t)}{dt} = \sum_{i=0}^{j-1}\binom{j}{i}[k(-1)^j m_i + \beta(-1)^j m_{i+1} + (-1)^i \mu m_{i+1}] \tag{10}$$
$$m_j(0) = 1$$

where we define $m_0 = 1$. For example, the differential equations for the first and second moments are

$$\frac{dm_1(t)}{dt} = (\beta-\mu)m_1(t) + k$$
$$m_1(0) = 1$$

and

$$\frac{dm_2(t)}{dt} = 2(\beta-\mu)m_2(t) + (\beta+\mu+2k)m_1(t) + k$$
$$m_2(0) = 1$$

These have the solutions:

$$m_1(t) = \frac{k}{\mu-\beta} + (1-\frac{k}{\mu-\beta})e^{-(\mu-\beta)t}$$

and

$$m_2(t) = \frac{k(k+\mu)}{(\mu-\beta)^2} + \frac{\mu^2 - 2k^2 - \beta^2 + k\mu - 3k\beta}{(\mu-\beta)^2} e^{-(\mu-\beta)t} + \\ + \frac{k^2 + 2\beta^2 + 3k\beta - 2\mu\beta - 2k\mu}{(\mu-\beta)^2} e^{-2(\mu-\beta)t}$$

Again, given that we have assumed that $\mu > \beta$, in other words, the birth rate is not high enough to maintain the population size, as $t \to \infty$

$$m_1(t) \xrightarrow[t\to\infty]{} \frac{k}{\mu-\beta}$$

$$m_2(t) \xrightarrow[t\to\infty]{} \frac{k(k+\mu)}{(\mu-\beta)^2}$$

and

$$\mathrm{Var}(n(t)) = m_2(t) - m_1^2(t) \xrightarrow[t\to\infty]{} \frac{\mu k}{(\mu-\beta)^2}.$$

Moreover, it is clear from Eq. 10 that all the moments are finite.

In other words, the population size will approach a finite limit, which can be regulated by controlling the immigration rate k, and this population size will be stable, as indicated by the fact that the limiting variance is finite. Without immigration, i.e., if $k = 0$, the population size will decay exponentially.

2.2 Local CLT

Setting $\lambda_n = n\beta + k$, $\mu_n = n\mu$, we see that the model given by Eqs. 6 and 7 is a particular case of the general random walk on $Z_+^1 = \{0, 1, 2, \cdots\}$ with generator

$$\mathcal{L}\psi(n) = \psi(n+1)\lambda_n - (\lambda_n + \mu_n)\psi(n) + \mu_n\psi(n-1), \quad n \geqslant 0 \tag{11}$$

$$\mathcal{L}\psi(0) = k\psi(1) - k\psi(0) \tag{12}$$

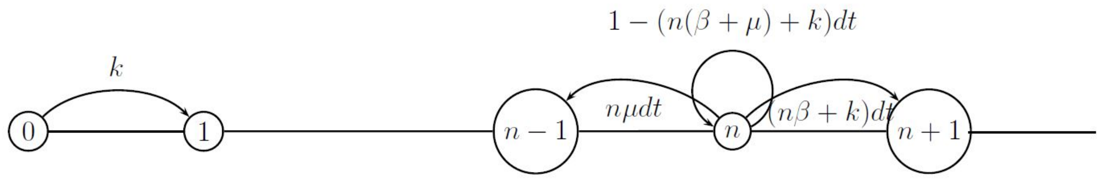

Figure 1: $n(t)$ is the random walk on Z^1_+

The theory of such chains has interesting connections to the theory of orthogonal polynomials, the moments problem, and related topics (see [4]). We recall several facts of this theory.

a. Equation $\mathcal{L}\psi = 0, x \geqslant 1$, (i.e., the equation for harmonic functions) has two linearly independent solutions:

$$\begin{aligned}
\psi_1(n) &\equiv 1 \\
\psi_2(n) &= \begin{cases} 0 & n=0 \\ 1 & n=1 \\ 1+\frac{\mu_1}{\lambda_1}+\frac{\mu_1\mu_2}{\lambda_1\lambda_2}+\cdots+\frac{\mu_1\mu_2\cdots\mu_{n-1}}{\lambda_1\lambda_2\cdots\lambda_{n-1}} & n \geqslant 2 \end{cases}
\end{aligned} \tag{13}$$

b. Denoting the adjoint of $\mathcal{L}$ by $\mathcal{L}^*$, equation $\mathcal{L}^*\pi = 0$ (i.e., the equation for the stationary distribution, which can be infinite) has the positive solution

$$\pi(1) = \frac{\lambda_0}{\mu_1}\pi(0) \tag{14}$$

$$\pi(2) = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\pi(0) \tag{15}$$

$$\cdots \tag{16}$$

$$\pi(n) = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\pi(0) \tag{17}$$

This random walk is ergodic (i.e., $n(t)$ converges to a statistical equilibrium, a steady state) if and only if the series $1 + \frac{\lambda_0}{\mu_1} \cdots + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \cdots + \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}$ converges. In our case,

$$x_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \frac{k(k+\beta)\cdots(k+(\mu-1))\beta}{\mu(2\mu)\cdots(n\mu)}.$$

If $\beta > \mu$, then, for $n > n_0$, for some fixed $\varepsilon > 0$, $\frac{k+(n-1)\beta}{n\mu} > 1 + \varepsilon$, that is, $x_n \geq C^n$, for $C > 1$ and $n \geq n_1(\varepsilon)$, and so $\sum x_n = \infty$. In contrast, if $\beta < \mu$, then, for some $0 < \varepsilon < 1$, $\frac{k+(n-1)\beta}{n\mu} < 1 - \varepsilon$, and $x_n \leq q^n$, for $0 < q < 1$ and $n > n_1(\varepsilon)$; thus, $\sum x_n < \infty$. In this ergodic case, the invariant distribution of the random walk $n(t)$ is given by the formula

$$\pi(n) = \frac{1}{\tilde{S}} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n},$$

where

$$\tilde{S} = 1 + \frac{k}{\mu} + \frac{k(\beta+k)}{\mu(2\mu)} + \cdots + \frac{k(k+\beta)\cdots(\beta(n-1)+k)}{\mu(2\mu)\cdots(n\mu)} + \cdots.$$

Theorem 1 (Local Central Limit theorem). Let $\beta < \mu$. If $l = O(k^{2/3})$, then, for the invariant distribution $\pi(n)$

$$\pi(n_0 + l) \sim \frac{e^{-\frac{l^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \quad as \;\; k \to \infty \tag{18}$$

where $\sigma^2 = \frac{\mu k}{(\mu-\beta)^2}$, $n_0 \sim \frac{k}{\mu-\beta}$.

Proof.

$$\begin{aligned}\pi(n) &= \frac{1}{\tilde{S}} \frac{k(k+\beta)\cdots(k+\beta(n-1))}{\mu(2\mu)\cdots(n\mu)} \\ &= \frac{1}{\tilde{S}} \left(\frac{\beta}{\mu}\right)^n \frac{\frac{k}{\beta}(\frac{k}{\beta}+1)\cdots(\frac{k}{\beta}+n-1)}{n\,!} \\ &= \frac{1}{\tilde{S}} \left(\frac{\beta}{\mu}\right)^n \frac{\Gamma(\frac{k}{\beta}+n)}{\Gamma(\frac{k}{\beta})n\,!}.\end{aligned}$$

We see $\tilde{S}$ is a degenerate hypergeometric series, thus

$$\tilde{S} = \left(1 - \frac{\beta}{\mu}\right)^{-\frac{k}{\beta}}.$$

Set

$$a_n = \left(\frac{\beta}{\mu}\right)^n \frac{\Gamma(\frac{k}{\beta}+n)}{\Gamma(\frac{k}{\beta})n\,!}. \tag{19}$$

Then, $\pi(n) = \dfrac{a_n}{\tilde{S}}$. We have

$$\begin{aligned}a_{n+l} &= a_n \left(\frac{\beta}{\mu}\right)^l (1+\frac{\frac{k}{\beta}-1}{n+1})(1+\frac{\frac{k}{\beta}-1}{n+2})\cdots(1+\frac{\frac{k}{\beta}-1}{n+l}) \\ &= a_n \prod_{i=1}^{l} \frac{\beta}{\mu}(1+\frac{\frac{k}{\beta}-1}{n+i}) \\ &= a_n \prod_{i=1}^{l} \frac{1+\frac{\beta(i-1)(\mu-\beta)}{\mu k}}{1+\frac{i(\mu-\beta)}{k}} \\ &= a_n \prod_{i=1}^{l} \frac{\frac{\beta}{\mu}(n+i-1)+\frac{k}{\mu}}{n+i}\end{aligned}$$

and because $a_{n_0} \sim \frac{k}{\mu-\beta}$

$$\begin{aligned} a_{n_0+l} &\sim a_{n_0} \prod_{i=1}^{l} \frac{\frac{\beta}{\mu}(\frac{k}{\mu-\beta}+i-1)+\frac{k}{\mu}}{\frac{k}{\mu-\beta}+i} \\ &= a_{n_0} \prod_{i=1}^{l} \frac{\beta(i-1)(\mu-\beta)+k\mu}{i(\mu-\beta)\mu+k\mu} \\ &= a_{n_0} \prod_{i=1}^{l} \frac{1+\frac{\beta(i-1)(\mu-\beta)}{k\mu}}{1+\frac{i(\mu-\beta}{k}} \\ &= a_{n_0} e^{\sum\limits_{i=1}^{l}[\ln(1+\frac{\beta(i-1)(\mu-\beta)}{\mu k})-\ln(1+\frac{i(\mu-\beta)}{k})]} \end{aligned}$$

We consider $\sum\limits_{i=1}^{l} \ln(1+\frac{\beta(i-1)(\mu-\beta)}{\mu k}) = \int_1^l \ln(1+\frac{(x-1)(\mu-\beta)\beta}{\mu k})dx + O(\ln(1+\frac{(l-1)(\mu-\beta)\beta}{\mu}))$ and $\sum\limits_{i=1}^{l} \ln(1+\frac{i(\mu-\beta)}{k})dx = \int_1^l \ln(1+\frac{x(\mu-\beta)}{k})dx + O(\ln(1+\frac{l(\mu-\beta)}{k}))$

We integrate the series $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$, and take $l = O(k^{2/3})$

$$\begin{aligned} &\int_1^l \ln\left(1+\frac{(x-1)(\mu-\beta)\beta}{\mu k}\right)dx \\ &= \int_1^l \frac{(x-1)(\mu-\beta)\beta}{\mu k}dx - \frac{1}{2}\int_1^l \left(\frac{(x-1)(\mu-\beta)\beta}{\mu k}\right)^2 dx + \cdots \\ &= \frac{(\mu-\beta)\beta}{\mu k}\left(\frac{l^2}{2}-l\right) - \frac{(u-\beta)^2\beta^2}{6\mu^2 k^2}(l-1)^3 + \cdots \\ &= \frac{(\mu-\beta)\beta}{\mu k} l^2 + O(1) \end{aligned}$$

and

$$\begin{aligned} &\int_1^l \ln(1+\frac{x(\mu-\beta)}{k})dx \\ &= \int_1^l \frac{x(\mu-\beta)}{k}dx - \frac{1}{2}\int_1^l \left(\frac{x(\mu-\beta}{k}\right)^2 dx + \cdots \\ &= \frac{1}{2}\frac{\mu-\beta}{k} l^2 + O(1). \end{aligned}$$

Hence

$$a_{n_0+l} \sim a_{n_0} e^{\frac{(\mu-\beta)\beta}{2\mu k} l^2 - \frac{1}{2}\frac{\mu-\beta}{k} l^2} = a_{n_0} e^{-\frac{l^2}{\frac{2k\mu}{(\mu-\beta)^2}}},$$

or, setting $\sigma^2 = \frac{k\mu}{(\mu-\beta)^2}$

$$a_{n_0+l} \sim a_{n_0} e^{-\frac{l^2}{2\sigma^2}}.$$

From Eq.19

$$a_{n_0} = \left(\frac{\beta}{\mu}\right)^{n_0} \frac{\Gamma(\frac{k}{\beta}) + n_0}{\Gamma(\frac{k}{\beta}) n_0\,!}$$

and, using Stirling's formula and the fact that $n_0 \sim \frac{k}{\mu-\beta}$

$$\begin{aligned}
a_{n_0} &= \left(\frac{\beta}{\mu}\right)^{n_0} \frac{\sqrt{\frac{2\pi}{\frac{k}{\beta}+n_0}}}{\sqrt{\frac{2\pi}{\frac{k}{\beta}}}} \frac{(\frac{\frac{k}{\beta}+n_0}{e})^{\frac{k}{\beta}+n_0}}{(\frac{\frac{k}{\beta}}{e})^{\frac{k}{\beta}} \sqrt{2\pi n_0} (n_0/e)^{n_0}} \\
&\sim \left(\frac{\beta}{\mu}\right)^{\frac{k}{\mu-\beta}} \frac{\sqrt{\frac{2\pi}{\frac{k}{\beta}+\frac{k}{\mu-\beta}}}}{\sqrt{\frac{2\pi}{\frac{k}{\beta}}}} \frac{(\frac{\frac{k}{\beta}+\frac{k}{\mu-\beta}}{e})^{\frac{k}{\beta}+\frac{k}{\mu-\beta}}}{\left(\frac{\frac{k}{\beta}}{e}\right)^{\frac{k}{\beta}} \sqrt{2\pi \frac{k}{\mu-\beta}} (\frac{k}{\mu-\beta}/e)^{\frac{k}{\mu-\beta}}} \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left(1 + \frac{\beta}{\mu-\beta}\right)^{\frac{k}{\beta}}
\end{aligned}$$

where $\sigma = \sqrt{\frac{\mu k}{(\mu-\beta)^2}}$. Thus

$$\begin{aligned}
\frac{a_{n_0}}{\tilde{S}} &\sim \frac{\frac{1}{\sqrt{2\pi}\sigma}(1 + \frac{\beta}{\mu-\beta})^{\frac{k}{\beta}}}{(1-\frac{\beta}{\mu})^{1-\frac{k}{\beta}}} \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left((1 + \frac{\beta}{\mu-\beta})(1 - \frac{\beta}{\mu})\right)^{\frac{k}{\beta}} \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left(\frac{\mu}{\mu-\beta} \frac{\mu-\beta}{\mu}\right)^{\frac{k}{\beta}} \\
&= \frac{1}{\sqrt{2\pi}\sigma}
\end{aligned}$$

and so

$$\pi(n_0 + l) = \frac{a_{n_0+l}}{\tilde{S}} \sim \frac{a_{n_0}}{\tilde{S}} e^{-\frac{l^2}{2\sigma^2}} \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{l^2}{2\sigma^2}} \quad \text{as} \quad n_0 \to \infty.$$

$\square$

CHAPTER 3: BRANCHING PROCESS WITH MIGRATION AND IMMIGRATION FOR BINARY SPLITTING

We now consider our process with birth, death, migration, and immigration on a countable space, specifically the lattice $\mathbb{Z}^d$. As in the other models, we have $\beta > 0$, the rate of duplication at $x \in Z^d$; $\mu > 0$, the rate of death; and $k > 0$, the rate of immigration. Here, we add migration of the particles with rate $\kappa > 0$ and probability kernel $a(z)$, $z \in \mathbb{Z}^d$, $z \neq 0$, $a(z) = a(-z)$, $\sum\limits_{z \neq 0} a(z) = 1$. That is, a particle jumps from site x to $x + z$ with probability $\kappa a(z)dt$. Here we put $\kappa = 1$ to simplify the notation.

For $n(t, x)$ the number of particles at x at time t, the forward equation for this process is given by $n(t + dt, x) = n(t, x) + \xi(dt, x)$, where

$$\xi(dt, x) = \begin{cases} 1 & \text{w. pr. } n(t,x)\beta dt + kdt + \sum\limits_{z \neq 0} a(z)n(t, x+z)dt \\ -1 & \text{w. pr. } n(t,x)(\mu+1)dt \\ 0 & \text{w. pr. } 1 - (\beta+\mu+1)n(t,x)dt - \sum\limits_{z \neq 0} a(z)n(t,x+z)dt - kdt \end{cases} \tag{20}$$

Note that $\xi(dt, x)$ is independent on $\mathcal{F}_{\leqslant t}$ (the σ-algebra of events before or including t) and

a) $E[\xi(dt, x)|\mathcal{F}_{\leqslant t}] = n(t, x)(\beta - \mu - 1)dt + kdt + \sum\limits_{z \neq 0} a(z)n(t, x + z)dt.$

b) $E[\xi^2(dt, x)|\mathcal{F}_{\leqslant t}] = n(t, x)(\beta + \mu + 1)dt + kdt + \sum\limits_{z \neq 0} a(z)n(t, x + z)dt.$

c) $E[\xi(dt, x)\xi(dt, y)|\mathcal{F}_{\leqslant t}] = a(x - y)n(t, x)dt + a(y - x)n(t, y)dt.$

A single particle jumps from x to y or from y to x. Other possibilities have probability $O((dt)^2) \approx 0$. Here, of course, $x \neq y$.

d) If $x \neq y$, $y \neq z$, and $x \neq z$, then $E[\xi(dt, x)\xi(dt, y)\xi(dt, z)] = 0$.

We will not use property d) in this paper but it is crucial for the analysis of moments of order greater or equal to 3.

From here on, we concentrate on the first two moments.

3.0.1 First moment

Due to the fact that $\beta < \mu$, the system has a short memory, and we can calculate all the moments under the condition that $n(0, x)$, $x \in \mathbb{Z}^d$, is a system of independent and identically distributed random variables with expectation $\frac{k}{\mu-\beta}$. We will select Poissonian random variables with parameter $\lambda = \frac{k}{\mu-\beta}$. Setting $m_1(t, x) = E[n(t, x)]$, we have

$$\begin{aligned} m_1(t+dt, x) &= E[E[n(t+dt, x)|\mathcal{F}_t]] = E[E[n(t, x) + \xi(t, x)|\mathcal{F}_t]] \\ &= m_1(t, x) + (\beta-\mu)m_1(t, x)dt + kdt + \sum_{z\neq 0} a(z)[m_1(t, x+z) - m_1(t, x)]dt \end{aligned} \tag{21}$$

Defining the operator $\mathcal{L}_a(f(t, x)) = \sum_{z\neq 0} a(z)[f(t, x+z) - f(t, x)]$, then, from Eq. 27 we get the differential equation

$$\begin{cases} \dfrac{\partial m_1(t, x)}{\partial t} &= (\beta-\mu)m_1(t, x) + k + \mathcal{L}_a m_1(t, x) \\ m_1(0, x) &= 0 \end{cases}$$

Because of spatial homogeneity, $\mathcal{L}_a m_1(t,x) = 0$, giving

$$\begin{cases} \dfrac{\partial m_1(t,x)}{\partial t} & = (\beta-\mu)m_1(t,x) + k \\ m_1(0,x) & = 0 \end{cases}$$

which has the solution

$$m_1(t,x) = \frac{k}{\beta-\mu}(e^{(\beta-\mu)t} - 1).$$

Thus, if $\beta \geq \mu$, $m_1(t,x) \to \infty$, and if $\mu > \beta$,

$$\lim_{t\to\infty} m_1(t,x) = \frac{k}{\mu-\beta}.$$

3.0.2 Second moment

We derive differential equations for the second correlation function $m_2(t,x,y)$ for $x = y$ and $x \neq y$ separately, then combine them and use a Fourier transform to prove a useful result concerning the covariance.

I. $x = y$

$$\begin{aligned} m_2(t+dt,x,x) &= E[E[(n(t,x)+\xi(dt,x))^2|\mathcal{F}_{\leqslant t}]] \\ &= m_2(t,x,x) + 2E[n(t,x)[n(t,x)(\beta-\mu-1)dt + kdt + \sum_{z\neq 0} a(z)n(t,x+z)]dt] \\ &\quad + E[n(t,x)(\beta+\mu+1)dt + kdt + \sum_{z\neq 0} a(z)n(t,x+z)dt] \end{aligned}$$

Denote $\mathcal{L}_{ax} m_2(t,x,y) = \sum_{z\neq 0} a(z)(m_2(t,x+z,y) - m_2(t,x,y))$.

From this follows the differential equation

$$\begin{cases} \dfrac{\partial m_2(t,x,x)}{\partial t} & = 2(\beta-\mu)m_2(t,x,x) + 2\mathcal{L}_{ax} m_2(t,x,x) + \frac{2k^2}{\mu-\beta} + \frac{2k(\mu+1)}{\mu-\beta} \\ m_2(0,x,x) & = 0 \end{cases}$$

II. $x \neq y$

Because only one event can happen during dt

$$P\{\xi(dt,x)=1,\xi(dt,y)=1\} = P\{\xi(dt,x)=-1,\xi(dt,y)=-1\} = 0,$$

while the probability that one particle jumps from y to x is

$$P\{\xi(dt,x)=1,\xi(dt,y)=-1\} = a(x-y)n(t,y)dt,$$

and the probability that one particle jumps from x to y is

$$P\{\xi(dt,x)=-1,\xi(dt,y)=1\} = a(y-x)n(t,x)dt.$$

Then, similar to above

$$m_2(t+dt,x,y) = E[E[(n(t,x)+\xi(t,x))(n(t,y)+\xi(t,y))|\mathcal{F}_{\leqslant t}]]$$

$$= m_2(t,x,y) + (\beta-\mu)m_2(t,x,y)dt + km_1(t,y)dt + \sum_{z\neq 0} a(z)(m_2(t,x+z,y) - m_2(t,x,y))dt$$

$$+ (\beta-\mu)m_2(t,x,y)dt + km_1(t,x)dt + \sum_{z\neq 0} a(z)(m_2(t,x,y+z) - m_2(t,x,y))dt$$

$$+ a(x-y)m_1(t,y)dt + a(y-x)m_1(t,x)dt$$

$$= m_2(t,x,y) + 2(\beta-\mu)m_2(t,x,y)dt + k(m_1(t,y)+m_1(t,x))dt + (\mathcal{L}_{ax}+\mathcal{L}_{ay})m_2(t,x,y)dt$$

$$+ a(x-y)(m_1(t,x)+m_1(t,y))dt$$

The resulting differential equation is

$$\frac{\partial m_2(t,x,y)}{\partial t} = 2(\beta-\mu)m_2(t,x,y) + (\mathcal{L}_{ax}+\mathcal{L}_{ay})m_2(t,x,y) + k(m_1(t,x)+m_1(t,y))$$
$$+ a(x-y)[m_1(t,x)+m_1(t,y)] \tag{22}$$

That is

$$\frac{\partial m_2(t,x,y)}{\partial t} = 2(\beta-\mu)m_2(t,x,y) + (\mathcal{L}_{ax}+\mathcal{L}_{ay})m_2(t,x,y) + \frac{2k^2}{\mu-\beta} + 2a(x-y)\frac{k}{\mu-\beta}$$

Because, for fixed t, $n(t,x)$ is homogeneous in space, we can write $m_2(t,x,y) = m_2(t,x-y) = m_2(t,u)$. Then, we can condense the two cases into a single differential equation

$$\begin{cases} \dfrac{\partial m_2(t,u)}{\partial t} & = 2(\beta-\mu)m_2(t,u) + 2\mathcal{L}_{au}m_2(t,u) + \frac{2k^2}{\mu-\beta} + 2a(u)\frac{k}{\mu-\beta} + \delta_0(u)\frac{2k(\mu+1)}{\mu-\beta} \\ m_2(0,u) & = En^2(0,x) \end{cases}$$

Here $u = x-y \neq 0$ and $a(0) = 0$.

We can partition $m_2(t,u)$ into $m_2(t,u) = m_{21} + m_{22}$, where the solution for m_{21} depends on time but not position and the solution for m_{22} depends on position but not time. Thus, $\mathcal{L}_{au}m_{21} = 0$ and m_{21} corresponds to the source $\frac{2k^2}{\mu-\beta}$, which gives

$$\frac{\partial m_{21}(t,u)}{\partial t} = 2(\beta-\mu)m_{21}(t,u) + \frac{2k^2}{\mu-\beta}$$

As $t \to \infty$, $m_{21} \to \bar{M_2} = m_1^2(t,x) = \frac{k^2}{(\mu-\beta)^2}$.

For the second part, m_{22} , $\dfrac{\partial m_{22}}{\partial t} = 0$, i.e.

$$\frac{\partial m_{22}(t,u)}{\partial t} = 2(\beta-\mu)m_{22}(t,u) + 2\mathcal{L}_{au}m_{22}(t,u) + 2a(u)\frac{k}{\mu-\beta} + \delta_0(u)\frac{2k(\mu+1)}{\mu-\beta} = 0$$

As $t \to \infty$, $m_{22} \to \tilde{M_2}$. $\tilde{M_2}$ is the limiting correlation function for the particle field $n(t,x)$, $t \to \infty$. It is the solution of the "elliptic" problem

$$2\mathcal{L}_{au}\tilde{M_2}(u) - 2(\mu-\beta)\tilde{M_2}(u) + \delta_0(u)\frac{2k(\mu+1)}{\mu-\beta} + 2a(u)\frac{k}{\mu-\beta} = 0$$

Applying the Fourier transform $\widehat{\tilde{M}_2}(\theta) = \sum_{u \in Z^d} \tilde{M}_2(u) e^{i(\theta,u)}$, $\theta \in T^d = [-\pi, \pi]^d$, we obtain

$$\widehat{\tilde{M}_2}(\theta) = \frac{\dfrac{k}{\mu - \beta} + \dfrac{k\hat{a}(\theta)}{\mu - \beta}}{(\mu - \beta) + (1 - \hat{a}(\theta)}.$$

We have proved the following result.

Theorem 2. If $t \to \infty$, then $\mathrm{Cov}(n(t,x), n(t,y)) = E[n(t,x)n(t,y)] - E[n(t,x)]E[n(t,y)]$ $= m_2(t,x,y) - m_1(t,x)m_1(t,y)$, tends to $\tilde{M}_2(x-y) = \tilde{M}_2(u) \in L^2(Z^d)$

The Fourier transform of $\tilde{M}_2(\cdot)$ is equal to

$$\widehat{\tilde{M}_2}(\theta) = \frac{c_1 + c_2\hat{a}(\theta)}{c_3 + (1 - \hat{a}(\theta))} \in C(T^d)$$

where $c_1 = \dfrac{k}{\mu - \beta}$, $c_2 = \dfrac{k}{\mu - \beta}$, $c_3 = \mu - \beta$

Let's compare our results with the corresponding results for the critical contact model [3] (where $k = 0$, $\mu = \beta$). In the last case, the limiting distribution for the field $n(t,x)$, $t \geqslant 0$, $x \in \mathbb{Z}^d$, exists if and only if the underlying random walk with generator $\mathcal{L}_a$ is transient. In the recurrent case, we have the phenomenon of clusterization. The limiting correlation function is always slowly decreasing (like the Green kernel of $\mathcal{L}_a$).

In the presence of immigration, the situation is much better: the limiting correlation function always exists and we believe that the same is true for all moments. The decay of $\tilde{M}_2(u)$ depends on the smoothness of $\hat{a}(\theta)$. Under minimal regularity conditions, correlations have the same order of decay as $a(z)$, $z \to \infty$. For instance, if $a(z)$ is finitely supported or exponentially decreasing, the correlation also has an exponential decay. If $a(z)$ has power decay, then the same is true for correlation $\tilde{M}_2(u)$, $u \to \infty$.

CHAPTER 4: PROCESSES IN A RANDOM ENVIRONMENT

The final four models involve a random environment. Two are Galton-Watson models with immigration and lack a spatial component. In the first, the parameters are random functions of the population size; in the second, they are random functions of a Markov chain on a finite space. The last two models are spatial and feature immigration, migration, and, most importantly, a random environment in space, still stationary in time for the third but not stationary in time for the fourth.

4.1 Galton-Watson processes with immigration in random environments

4.1.1 Galton-Watson process with immigration in random environment based on population size

Assume that rates of mortality $\mu(\cdot)$, duplication $\beta(\cdot)$, and immigration $k(\cdot)$ are random functions of the volume of the population $x \geq 0$. Namely, the random vectors $(\mu, \beta, k)(x, \omega)$ are i.i.d on the underlying probability space $(\Omega_e, \mathcal{F}_e, P_e)$ (e: environment).

The Galton-Watson Process is ergodic (P_e-a.s) if and only if the random series

$$S = \sum_{n=1}^{\infty} \frac{k(0)(\beta(1)+k(1))(2\beta(2)+k(2))\cdots((n-1)\beta(n-1)+k(n-1))}{\mu(1)(2\mu(2))\cdots(n\mu(n))} < \infty,\ P_e\text{-a.s.}$$

Theorem 3. Assume that the random variables $\beta(x,\omega)$, $\mu(x,\omega)$, $k(x,\omega)$ are bounded from above and below by the positive constants $C^{\pm}$: $0 < C^- \leq \beta(x,\omega) \leq C^+ < \infty$.

Then, the process $n(t, \omega_e)$ is ergodic P_e-a.s. if and only if $\langle \ln \frac{\beta(x,\omega)}{\mu(x,\omega)} \rangle = \langle \ln \beta(\cdot) \rangle - \langle \ln(\mu(\cdot)) \rangle < 0$

Proof. It is sufficient to note that

$$\frac{k(n-1,\omega) + (n-1)\beta(n-1,\omega)}{n\mu(x,\omega)} = \frac{\frac{k(n-1,\omega) - \beta(n-1,\omega))}{n} + \beta(n-1,\omega)}{\mu(n,\omega)} = e^{\ln \beta(n-1) - \ln \mu(n) + o(\frac{1}{n})}.$$

$\square$

It follows from the strong LLN that the series diverges exponentially fast for $\langle \ln \beta(\cdot) \rangle - \langle \ln \mu(\cdot) \rangle > 0$; it converges like a decreasing geometric progression for $\langle \ln \beta(\cdot) \rangle - \langle \ln \mu(\cdot) \rangle < 0$; and it is divergent if $\langle \ln \beta(\cdot) \rangle = \langle \ln \mu(\cdot) \rangle$. It diverges even when $\beta(x, \omega_e) = \mu(x, \omega_e)$ due to the presence of $k^- \geq C^- > 0$.

Note that $ES < \infty$ if and only if $\langle \frac{\lambda(x-1)}{\mu(x)} \rangle = \langle \lambda \rangle \langle \frac{1}{\mu} \rangle < \infty$, i.e., the fluctuations of S, even in the case of convergence, can be very high.

4.1.2 Random non-stationary(time dependent) environment

Assume that $k(t)$ and $\Delta = (\mu - \beta)(t)$ are stationary random processes on (Ω_m, P_m) and that $k(t)$ is independent of Δ. For a fixed environment, i.e., fixed $k(\cdot)$ and $\Delta(\cdot)$, the equation for the first moment takes the form

$$\frac{dm_1(t, \omega_m)}{dt} = -\Delta(t, \omega_m) m_1 + k(t, \omega_m)$$

$$m_1(0, \omega_m) = m_1(0)$$

Then

$$m_1(t, \omega_m) = m_1(0) e^{-\int_0^t \Delta(u,\omega_m) du} + \int_0^t k(s, \omega_m) e^{-\int_s^t \Delta(u,\omega_m) du} ds$$

Assume that $\frac{1}{\delta} \geqslant \Delta(\cdot) \geqslant \delta > 0$, $\frac{1}{\delta} \geqslant k(\cdot) \geqslant \delta > 0$. Then

$$m_1(t, \omega_m) = \int_{-\infty}^{t} k(s, \omega_m) e^{-\int_s^t \Delta(u, \omega_m) du} ds + O(e^{-\delta t}).$$

Thus, for large t, the process $m_1(t, \omega_m)$ is exponentially close to the stationary process

$$\tilde{m}_1(t, \omega) = \int_{\infty}^{t} k(s, \omega_m) e^{-\int_s^t \Delta(u, \omega_m) du} ds$$

Assume now that $k(t)$ and $\Delta(s)$ are independent stationary processes and $-\Delta(t) = V(x(t))$, where $x(t)$, $t \geqslant 0$, is a Markov Chain with continuous time and symmetric geometry on the finite set X. (One can also consider $x(t)$, $t \geqslant 0$, as a diffusion process on a compact Riemannian manifold with Laplace-Beltrami generator Δ.) Let

$$\begin{aligned} u(t, x) &= E_x e^{\int_0^t V(x_s) dx} f(x_t) \\ &= E_x e^{\int_0^t -\Delta(x_s) dx} f(x_t) \end{aligned}$$

Then

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{L} u + V u = H u \\ \\ u(0, x) = f(x) \end{cases} \tag{23}$$

The operator $\mathcal{L}$ is symmetric in $L^2(x)$ with dot product $(f, g) = \sum_{x \in X} f(x) g(\bar{x})$. Thus, $H = \mathcal{L} + V$ is also symmetric and has real spectrum $0 > -\delta \geqslant \lambda_0 > \lambda_1 \geqslant \cdots$ with orthonormal eigenfunctions $\psi_0(x) > 0, \psi_1(x) > 0, \cdots$ Iinequality $\lambda_0 \leqslant \delta < 0$ follows from our assumption on $\Delta(\cdot)$.

The solution of equation 23 is given by

$$u(t, x) = \sum_{n=1}^{N} e^{\lambda_k t} \psi_k(x) (t, \psi_k).$$

Now, we can calculate $< \tilde{m}_1(t, x, \omega_m) >$.

$$< \tilde{m} >= \int_{-\infty}^{t} < k(\cdot) >< E_\pi e^{\int_s^t V(x_u)du} > ds \tag{24}$$

Here, $\pi(x) = \frac{1}{N} = \frac{\mathbb{1}(x)}{N}$ is the invariant distribution of x_s. Then

$$\begin{aligned}
< \tilde{m} > &= \int_{-\infty}^{t} < k > \sum_{k=0}^{k=N} e^{\lambda_k(t-s)}(\psi_k \pi)(\mathbb{1}\psi_k)ds \\
&= - < k > \sum_{k=0}^{k=N} \frac{1}{\lambda_k}(\psi_k \mathbb{1})^2 \frac{1}{N} \\
&= -\frac{< k >}{N} \sum_{k=0}^{N} \frac{(\psi_k \mathbb{1})^2}{\lambda_k}
\end{aligned}$$

4.1.3 Galton-Watson process with immigration in random environment given by Markov chain

Let $x(t)$ be an ergodic MCh on the finite space X and let $\beta(x), \mu(x), k(x)$, the rates of duplication, annihilation, and immigration, be functions from X to R^+, and, therefore, functions of t and ω_e. The process $(n(t), x(t))$ is a Markov chain on $\mathbb{Z}_+^1 \times X$.

Let $a(x,y)$, $x, y \in X$, $a(x,y) \geq 0$, $\sum_{y \in X} a(x,y) = 1$ for all $x \in X$, be the transition function for $x(t)$. Consider $E_{(n,x)} f(n(t), x(t)) = u(t, (n,x))$. Then

$$\begin{aligned}
u(t+dt, (n,x)) = (1 - (n\beta(x) + n\mu(x) + k(x) - a(x,x))dt)u(t,x) + n\beta(x)u(t,(n+1,x))dt \\
+ k(x)u(t,(n+1,x))dt + n\mu(x)u(t,(n-1,x))dt + \sum_{y:y \neq x} a(x,y)u(t,(n,y))dt
\end{aligned}$$

We obtain the backward Kolmogorov equation

$$\frac{\partial u}{\partial t} = \sum_{y:y\neq z} a(t,y)(u(t,(n,y)) - u(t,(n,x))) + (n\beta(x)+k(x))(u(t,(n+1,x)) - u(t,(n,x)))$$

$$+ n\mu(x)(u(t,(n-1,x)) - u(t,(n,x)))$$

$u(0,(n,x)) = 0$

Example. Two-state random environment.

Here, $x(t)$ indicates which one of two possible states, $\{1,2\}$ the process is in at time t. The birth, mortality, and immigration rates are different for each state: β_1 and β_2, μ_1 and μ_2, and k_1 and k_2. For a process in state 1, at any time the rate of switching to state 2 is α_1, with α_2 the rate of the reverse switch. This creates the two-state random environment. Let G be the generator for the process, as diagrammed in Figure 3.

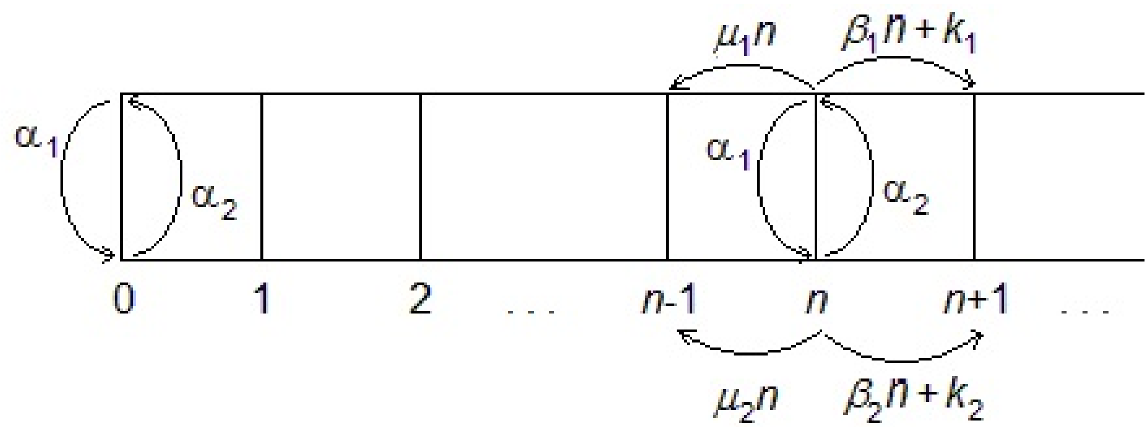

Figure 1: GW process with immigration with random environment as two states

The following theorem gives sufficient conditions for the ergodicity of the process $(n(t), x(t))$.

Theorem 4. Assume that for some constants $\delta > 0$ and $A > 0$

$$\mu_i - \beta_i \geq \delta,\ k_i \leq A, \quad i = 1, 2$$

Then, the process $(n(t), x(t))$ is an ergodic Markov chain and the invariant measure of this process has exponential moments, i.e., $E\, e^{\lambda n(t)} \leq c_0 < \infty$ if $\lambda \leq \lambda_0$ for appropriate (small) $\lambda_0 > 0$.

Proof. We take as a Lyapunov function $f(n, x) = n$.

Then, $Gf(n(t), x(t)) = (\beta_x - \mu_x)n + k_x$. So for sufficiently large n, specifically $n > \frac{A}{\delta}$, we have $Gf \leq 0$. $\square$

4.2 Models with immigration and migration in a random environment

For this most general case, we have migration and a non-stationary environment in space and time. The rates of duplication, mortality, and immigration at time t and position $x \in \mathbb{Z}^d$ are given by $\beta(t, x)$, $\mu(t, x)$, and $k(t, x)$. As in the above models, immigration is uninfluenced by the presence of other particles; also set $\delta_1 \leq k(t, x) \leq \delta_2$, $0 < \delta_1 < \delta_2 < \infty$. The rate of migration is given by κ, with the process governed by the probability kernel $a(z)$, the rate of transition from x to $x + z$, $z \in Z^d$.

If $n(t, x)$ is the number of particles at $x \in Z^d$ at time t, $n(t+dt, x) = n(t, x) + \xi(t, x)$, where

$$\xi(t,x) = \begin{cases} 1 & \text{w. pr. } n(t,x)\beta(t,x)dt + k(t,x)dt + \sum_{z\neq 0} a(-z)n(t,x+z)dt \\ -1 & \text{w. pr. } n(t,x)\mu(t,x)dt + \sum_{z\neq 0} a(z)n(t,x)dt \\ 0 & \text{w. pr. } 1-(\beta(t,x)+\mu(t,x))n(t,x)dt - \sum_{z\neq 0} a(z)n(t,x+z)dt \\ & \qquad - \sum_{z\neq 0} a(z)n(t,x)dt - k(t,x)dt \end{cases}$$

For the first moment, $m_1(t,x) = E[n(t,x)]$, we can write

$$\begin{aligned} m_1(t+dt,x) &= E[E[n(t+dt,x)|\mathcal{F}_t]] = E[E[n(t,x)+\epsilon(t,x)|\mathcal{F}_t]] \\ &= m_1(t,x) + (\beta(t,x)-\mu(t,x))m_1(t,x)dt + k(t,x)dt \\ &\quad + \sum_{z\neq 0} a(z)[m_1(t,x+z)-m_1(t,x)]dt \end{aligned}$$

and so, defining, as above, $\mathcal{L}_a(f(t,x)) = \sum_{z\neq 0} a(z)[f(t,x+z)-f(t,x)]$, we obtain

$$\begin{cases} \dfrac{\partial m_1(t,x)}{\partial t} & = (\beta(t,x)-\mu(t,x))m_1(t,x) + k(t,x) + \mathcal{L}_a m_1(t,x) \\ m_1(0,x) & = 0 \end{cases} \tag{25}$$

We consider two cases. The first is where the duplication and mortality rates are equal, $\beta(t,x) = \mu(t,x)$. Because of the immigration rate bounded above 0, we find that the expected population size at each site tends to infinity. In the second case, to simplify, we consider $\beta(t,x)$ and $\mu(t,x)$ to be stationary in time, and assume the mortality rate to be greater than the duplication rate everywhere by at least a minimal amount. Here, we show that the interplay between the excess mortality and the positive immigration results in a finite positive expected population size at each site.

4.2.1 Case I

If $\beta(t,x) = \mu(t,x)$

$$\begin{cases} \dfrac{\partial m_1(t,x)}{\partial t} & = k(t,x) + \mathcal{L}_a m_1(t,x) \\ m_1(0,x) & = 0 \end{cases}$$

Taking Fourier transforms,

$$\begin{cases} \dfrac{\partial \widehat{m_1}(t,v)}{\partial t} & = \widehat{k}(t,v) + \widehat{\mathcal{L}_a}(v)\widehat{m_1}(t,v) \\ \widehat{m_1}(0,x) & = 0 \end{cases}$$

$$\frac{\partial}{\partial t}(e^{-\widehat{\mathcal{L}_a}(v)t}\widehat{m_1}) = -\widehat{\mathcal{L}_a}(v)e^{\widehat{\mathcal{L}_a}(v)t}\widehat{m_1} + e^{-\widehat{\mathcal{L}_a}(v)t}\frac{\partial \widehat{m_1}}{\partial t} = e^{\widehat{\mathcal{L}_a}(v)t}\widehat{k}(t,v)$$

$$e^{-\kappa\widehat{\mathcal{L}_a}(v)t}\widehat{m_1}(t,v) = \int_0^t e^{-\widehat{\mathcal{L}_a}(v)s}\widehat{k}(s,v)ds$$

$$\widehat{m_1}(t,v) = \int_0^t e^{-(s-t)\widehat{\mathcal{L}_a}(v)}\widehat{k}(s,v)ds$$

Taking the inverse Fourier transform,

$$\begin{aligned} m_1(t,x) &= \frac{1}{(2\pi)^d}\int_{T_d}\int_0^t e^{-(s-t)\widehat{\mathcal{L}}_a(v)}\widehat{k}(s,v)ds e^{-i(v,x)}dv \\ &= \int_0^t ds \sum_{y\in Z^d} k(s,y)p(t-s,x-y,0) \geq \int_0^t \delta_1 ds = \delta_1 t \end{aligned}$$

where

$$p(t,x,y) = \frac{1}{(2\pi)^d}\int_{T^d} e^{-t\widehat{\mathcal{L}}_a(v)-i(v,x-y)}dv = \frac{1}{(2\pi)^d}\int_{T^d} e^{-t\sum\limits_{j=1}^{d}(\cos(v_j)-1)-i(v,x-y)}dv$$

As $t \to \infty$, $\delta_1 t \to \infty$. Thus, when the birth rate equals the death rate, the expected population at each site $x \in \mathbb{Z}^d$ will go to infinity as $t \to \infty$.

4.2.2 Case II

Here, $\beta(t,x) \neq \mu(t,x)$. For simplification we assume that only immigration, $k(t,x)$, is not stationary in time. In other words, we us assume that the duplication and mortality rates *are* stationary in time and depend only on position: $\beta(t,x) = \beta(x)$, $\mu(t,x) = \mu(x)$ and $\mu(x) - \beta(x) \geqslant \delta_1 > 0$. From Eq. 25, we get

$$\begin{cases} \dfrac{\partial m_1(t,x)}{\partial t} &= k(t,x) + \mathcal{L}_a m_1(t,x) + (\beta(t,x) - \mu(t,x))m_1(t,x) \\ m_1(0,x) &= 0 \end{cases}$$

This has the solution

$$m_1(t,x) = \int_0^t ds \sum_{y\in Z^d} k(s,y)q(t-s,x,y)$$

where $q(t-s,x,y)$ is the solution for

$$\begin{cases} \dfrac{\partial q}{\partial t} &= \mathcal{L}_a q + (\beta(t,x) - \mu(t,x))q \\ q(0,x,y) &= \delta(x-y) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases} \end{cases}$$

By the Feynman-Kac formula,

$$\begin{aligned}
q(s,x,y) &= E_x[e^{\int_0^s(\beta(x_u)-\mu(x_u))du}\delta(x_s-y)] \\
&= E[e^{\int_0^s(\beta(x_u)-\mu(x_u)du}\delta(x_s-y))|x_0=x] \\
&= E[E[e^{\int_0^s(\beta(x_u)-\mu(x_u)du}\delta(x_s-y))|x_0=x,x_s=y]|x_0=x] \\
&= P(x_s=y|x_0=x)E_{x\to y}[e^{\int_0^s(\beta(x_u)-\mu(x_u)du}] \\
&= p(s,x,y)E_{x\to y}[e^{\int_0^s(\beta(x_u)-\mu(x_u)du}]
\end{aligned}$$

where

$$p(t,x,y) = \frac{1}{(2\pi)^d}\int_{T^d} e^{-t\widehat{\mathcal{L}}_a(v)-i(v,x-y)}dv.$$

Finally

$$\lim_{t\to\infty} m_1(t,x) = \lim_{t\to\infty}\int_0^t ds\sum_{y\in Z^d} k(s,y)E_{x\to y}[e^{\int_0^{t-s}(\beta(x_u)-\mu(x_u)du}]p(t-s,x,y)$$

and letting $w=t-s$

$$\begin{aligned}
&\leq \lim_{t\to\infty}\int_0^t dw\|k\|_\infty E_{x\to y}[e^{\int_0^w(\beta(x_u)-\mu(x_u)du}] \\
&\leq \|k\|_\infty\int_0^\infty e^{-\delta_1 w}dw \quad \text{since } \beta(x)-\mu(x)\leq-\delta_1<0 \\
&= \frac{\|k\|_\infty}{\delta_1}.
\end{aligned}$$

Thus, when $\mu(x)-\beta(x)>0$, $\lim\limits_{t\to\infty} m_1(t,x)$ is bounded by 0 and $\frac{\|k\|_\infty}{\delta_1}$, so this limit exists and is finite.

CHAPTER 5: SPATIAL PROCESSES WITH IMMIGRATION ON Z^D IN HOMOGENEOUS ENVIRONMENT FOR MULTIPLE OFFSPRING

5.1 Description of the Model

We now consider our process with birth, death, migration, and immigration on a countable space, specifically the lattice $\mathbb{Z}^d$. We denote $N(t,x)$, $t \leq 0$, $x \in Z^d$ as the global population at time t in the position $y \in Z^d$ and denote $n(t,x,y)$ as the subpopulation generated by one particle at the initial time 0 in the position $y \in Z^d$. Those subpopulations are independent and

$$N(t,x) = \sum_{x \in Z^d} n(t,x,y),\ N(0,x) = 1$$

Notations:

(i): the birth rate of $j-1$ particles: $\beta_j > 0$, $j = 2,3,4,5,\cdots$, the rate of birth that one particle will split into j particles at moment t, in other words, the birth rate that each parent particle will generate $j-1$ offsprings independently.During a period of time $(t, t+dt)$, the probability that one particle will split into j particles is $\beta_j dt$. Let us introduce the corresponding infinitesimal generating function $F(z) = \mu - (\mu + \sum\limits_{j\geq 2} \beta_j)z + \sum\limits_{j\geq 2} \beta_j z^j$. We will assume that $F(z)$ is an analytic function in the circle $|z| < 1+\delta$, $\delta > 0$, i.e the rate of birth β_j as a function of j is exponentially decreasing. And we also assume that the new offsprings start their evolution from the same birth place independently on others, like in the classcial paper of Kolmogorov, Petrovski and Piskunov(1937)[5]

(ii):the rate of death:$\mu > 0$. During a period of time $(t, t+dt)$, the probability that one particle will die is μdt.

(iii):the rate of immigration: $k > 0$. During a period of time $(t, t+dt)$, the probability that one new particle outside of the system appears in the site $x \in Z^d$ is kdt. The appearance of a new particle in the system is uninfluenced by the presence of other particles.

(iv): the rate of particles' migration: $\kappa > 0$. Migration of the parent particle depends on the probability kernel $a(z)$, $a(0) = -1$, $\sum_{z \neq 0} a(z) = 1$,$z \in Z^d$, here $a(z)$ is the transition rate of parent particle from x to $x + z$, $z \in Z^d$ at t. The parent particle jumps from site x to $x + z$ during a period of time $(t, t+dt)$ with probability $\kappa a(z)dt$. The generator of the corresponding (underlying) random walk is the discrete or lattice Laplacian $\mathcal{L}_a\psi(x) = \kappa \sum_{z \neq 0} [\psi(x+z) - \psi(x)]a(z)$.

This model is similar to the well-known Kolmogorov-Petrovski-Piskunov (KPP) model(1937)[5]. However, first, for the KPP model, the state space is continuous state space R^d instead of discrete state space Z^d and the underlying process is Brownian motion instead of a random walk. These two differences are rather essential technical points. In the KPP case, the study of stead states was developed by the ideas of

R. L. Dobrushin [2], who applied a technique involving partial differential equations. In the case of continuous contact model, in the terminology in Kondratiev, Kutoviy and Pirogov (2008) [6], there is no immigration (i.e k=0) and the birth rate equals the death rate (i.e $\beta_2 = \mu$), They applied forward Kolmogrov equation to prove the existence of the steady states (the limit of the total number of population as $t \to \infty$). But with the presence of the immigration, we have to use backward Kolmogrov equation.

For simplicity, we denote $n(t,x,y)$ as $n(t,x)$, the number of particles at x at time t, the equation for this process is given by $n(t+dt,x) = n(t,x) + \xi(dt,x)$, where

$$\xi(dt,x) = \begin{cases} 1 & \text{w. pr. } n(t,x)\beta_2 dt + kdt + \kappa \sum\limits_{z\neq 0} a(z) n(t,x+z)dt \\ j & \text{w. pr. } n(t,x)\beta_{j+1}dt \;\; j \geq 2 \\ -1 & \text{w. pr. } n(t,x)(\mu+\kappa)dt \\ 0 & \text{w. pr. } 1 - \sum\limits_{j=2}^{\infty} n(t,x)\beta_j dt - n(t,x)(\mu+\kappa)dt \\ & \qquad -\kappa \sum\limits_{z\neq 0} a(z)n(t,x+z)dt - kdt \end{cases} \tag{26}$$

Note that $\xi(dt,x)$ is independent on $\mathcal{F}_{\leqslant t}$ (the σ-algebra of events before or including t) and only one event can happen during $(t, t+dt)$. Thus

a) $E[\xi(dt,x)|\mathcal{F}_{\leqslant t}] = \sum\limits_{j=1}^{\infty} jn(t,x)\beta_{j+1}dt + kdt + \kappa \sum\limits_{z\neq 0} a(z)n(t,x+z)dt - n(t,x)(\mu + \kappa)dt$.

b) $E[\xi^2(dt,x)|\mathcal{F}_{\leqslant t}] = \sum\limits_{j=1}^{\infty} j^2 n(t,x)\beta_{j+1}dt + kdt + \kappa \sum\limits_{z\neq 0} a(z)n(t,x+z)dt + n(t,x)(\mu + \kappa)dt$.

c) $E[\xi(dt,x)\xi(dt,y)|\mathcal{F}_{\leqslant t}] = \kappa a(x-y)n(t,x)dt + \kappa a(y-x)n(t,y)dt$.

A single particle jumps from x to y or from y to x. Other possibilities have probability $O((dt)^2) \approx 0$. Here, of course, $x \neq y$.

d) If $x \neq y$, $y \neq z$, and $x \neq z$, then $E[\xi(dt,x)\xi(dt,y)\xi(dt,z)] = 0$.

We will not use property d) in this paper but it is crucial for the analysis of moments of order greater or equal to 3.

From here on, we concentrate on the first two moments.

5.2 First Moment

Setting $m_1(t,x) = E[n(t,x)]$, we have

$$m_1(t+dt,x) = E[E[n(t+dt,x)|\mathcal{F}_t]] = E[E[n(t,x) + \xi(t,x)|\mathcal{F}_t]]$$

$$= m_1(t,x) + \sum_{j=1}^{\infty} j m_1(t,x)\beta_{j+1}dt + kdt - \mu m_1(t,x)dt + \kappa \sum_{z\neq 0} a(z)[m_1(t,x+z) - m_1(t,x)]dt \tag{27}$$

Defining the operator $\mathcal{L}_a(f(t,x)) = \sum\limits_{z\neq 0} a(z)[f(t,x+z) - f(t,x)]$, then, from Eq. 27 we get the differential equation

$$\begin{cases} \dfrac{\partial m_1(t,x)}{\partial t} & = (\sum\limits_{j=1}^{\infty} j\beta_{j+1} - \mu)m_1(t,x) + k + \mathcal{L}_a m_1(t,x) \\ m_1(0,x) & = E[n(0,x)] \end{cases} \tag{28}$$

Because of spatial homogeneity, $\mathcal{L}_a m_1(t,x) = 0$, giving

$$\begin{cases} \dfrac{\partial m_1(t,x)}{\partial t} & = (\sum\limits_{j=1}^{\infty} j\beta_{j+1} - \mu)m_1(t,x) + k \\ m_1(0,x) & = E[n(0,x)] \end{cases}$$

Let $\beta = \sum\limits_{j=1}^{\infty} j\beta_{j+1}$. When $\beta = \beta(x)$, $\mu = \mu(x)$, $k = k(x)$ are bounded functions on the lattice Z^d, we will have exactly same equation as Equation 28. When we have all parameters β,k,μ constants, we can solve this equation and get the following result:

$$m_1(t,x) = \frac{k}{\mu-\beta} - \frac{k}{\mu-\beta}e^{(\beta-\mu)t} + E[n(0,x)]e^{(\beta-\mu)t}$$

.

Thus if $\beta \geq \mu$, $m_1(t,x) \to \infty$, and if $\mu > \beta$,

$$\lim_{t\to\infty} m_1(t,x) = \frac{k}{\mu-\beta}$$

independently on the initial conditions. The next result presents the Lyapunov stability of the first moment.

Theorem 5. Let coefficients $\beta_n(x), n \geq 2$, $\mu(x)$, $k(x)$, $x \in Z^d$ are bounded and $\mu(x) - \beta(x) \geq \delta_1 > 0, k(x) \geq \delta_2 > 0$. Then for the bounded initial condition, there exists

$$m_1(\infty,x) = \lim_{t\to\infty} m_1(t,x)$$

Let us stress that in the contact model (see [5] and [8]),the limiting steady states exists only in the critical case when $\mu(x) = \beta(x)$ and this state is unstable with respect to any sufficiently small in L^∞-norm perturbations(including random perturbations) of the parameters of the model.

5.3 Second Moment

We derive differential equations for the second correlation function $m_2(t,x,y)$ for $x = y$ and $x \neq y$ separately, then combine them and use a Fourier transform to prove a useful result concerning the covariance.

I. $x = y$

$$m_2(t+dt,x,x) = E[E[(n(t,x)+\xi(dt,x))^2|\mathcal{F}_{\leqslant t}]]$$

$$= m_2(t,x,x) + 2E[n(t,x)[\sum_{j=1}^{\infty} jn(t,x)\beta_{j+1}dt + kdt + \kappa \sum_{z\neq 0} a(z)n(t,x+z)dt$$

$$- n(t,x)(\mu+\kappa)dt]] + E[\sum_{j=1}^{\infty} n(t,x)\beta_{j+1}dt + kdt + \kappa \sum_{z\neq 0} a(z)n(t,x+z)dt +$$

$$n(t,x)(\mu+\kappa)dt]$$

$$= m_2(t,x,x) + \sum_{j=1}^{\infty} j^2\beta_{j+1}m_1(t,x)dt + kdt + \kappa \sum_{z\neq 0} a(z)[m_1(t,x+z)$$

$$- m_1(t,x)]dt + m_1(t,x)(\mu+2\kappa+2k)dt + 2\sum_{j=1}^{\infty} jm_2(t,x,x)\beta_{j+1}dt$$

$$+ 2\kappa \sum_{z\neq 0} a(z)[m_2(t,x,x+z) - m_2(t,x,x)]dt - 2m_2(t,x,x)\mu dt$$

Denote $\mathcal{L}_{ax}m_2(t,x,y) = \sum_{z\neq 0} a(z)(m_2(t,x+z,y) - m_2(t,x,y))$.

From this follows the differential equation

$$\begin{cases} \dfrac{\partial m_2(t,x,x)}{\partial t} & = 2(\beta-\mu)m_2(t,x,x) + 2\kappa\mathcal{L}_{ax}m_2(t,x,x) + \sum\limits_{j=1}^{\infty} j^2\beta_{j+1}m_1(t,x) + \\ & \quad k + \mathcal{L}_a m_1(t,x) + m_1(t,x)(\mu+2\kappa+2k) \\ m_2(0,x,x) & = E[n^2(0,x)] \end{cases}$$

II. $x \neq y$

Because only one event can happen during dt

$$P\{\xi(dt,x)=1, \xi(dt,y)=1\} = P\{\xi(dt,x)=-1, \xi(dt,y)=-1\} = 0,$$

while the probability that one particle jumps from y to x is

$$P\{\xi(dt,x)=1, \xi(dt,y)=-1\} = \kappa a(x-y)n(t,y)dt,$$

and the probability that one particle jumps from x to y is

$$P\{\xi(dt,x)=-1,\xi(dt,y)=1\}=\kappa a(y-x)n(t,x)dt.$$

Denote $\mathcal{L}_{ay}=\sum_{z\neq 0}a(z)(f(t,x,y+z)-f(t,x,y))$. Thus

$$m_2(t+dt,x,y)=E[E[(n(t,x)+\xi(t,x))(n(t,y)+\xi(t,y))|\mathcal{F}_{\leqslant t}]]$$

$$=m_2(t,x,y)+(\beta-\mu)m_2(t,x,y)dt+km_1(t,y)dt+$$

$$\kappa\sum_{z\neq 0}a(z)(m_2(t,x+z,y)-m_2(t,x,y))dt+(\beta-\mu)m_2(t,x,y)dt+km_1(t,x)dt+$$

$$\kappa\sum_{z\neq 0}a(z)(m_2(t,x,y+z)-m_2(t,x,y))dt+\kappa a(x-y)m_1(t,y)dt+\kappa a(y-x)m_1(t,x)dt$$

$$=m_2(t,x,y)+2(\beta-\mu)m_2(t,x,y)dt+km_1(t,y)dt+km_1(t,x)dt+$$

$$\kappa(\mathcal{L}_{ax}+\mathcal{L}_{ay})m_2(t,x,y)dt+\kappa a(x-y)(m_1(t,x)+\kappa m_1(t,y))dt$$

The resulting differential equation is

$$\begin{aligned}\frac{\partial m_2(t,x,y)}{\partial t}&=\kappa(\mathcal{L}_{ax}+\mathcal{L}_{ay})m_2(t,x,y)+2(\beta-\mu)m_2(t,x,y)\\&+km_1(t,x)+km_1(t,y)+\kappa a(x-y)[m_1(t,x)+m_1(t,y)]\end{aligned}\tag{29}$$

with the initial condition $m_2(0,x,y)=(En(0,x))^2$

Due to the fact that for fixed t,$n(t,x)$ is homogeneous in space, we can write $m_2(t,x,y)=m_2(t,x-y)=m_2(t,u)$. Thus we can combine two cases and recieve the equation for the second moment:

$$\begin{cases} \dfrac{\partial m_2(t,u)}{\partial t} & = 2\kappa\mathcal{L}_{au}m_2(t,u) + 2(\beta-\mu)m_2(t,u) \\ & +\delta_0(u)\Psi(m_1) + 2\kappa a(u)\Phi(m_1) \\ m_2(0,x,y) & = (En(0,x))^2(1-\delta_0(u)) + \delta_0(u)En^2(0,u) \end{cases}$$

Here $x - y = u$, $\Psi(x)$ and $\Phi(x)$ are known functions and depend linearly on the first moment m_1.

Without loss of generality, to simplify the calculation, assume for each site, in the beginning, the number of population is $\frac{k}{\mu-\beta}$, then we can obtain final differential equation:

$$\begin{cases} \dfrac{\partial m_2(t,u)}{\partial t} & = 2\kappa\mathcal{L}_{au}m_2(t,u) + 2(\beta-\mu)m_2(t,u) \\ & +2\dfrac{k^2}{\mu-\beta} - \dfrac{2\kappa k a(u)}{\mu-\beta} + \delta_0(u)\dfrac{k(\sum\limits_{j=1}^{\infty} j(j-1)\beta_{j+1} + 2\mu)}{\mu-\beta} \\ m_2(0,u) & = \frac{k^2}{(\mu-\beta)^2} \end{cases} \tag{30}$$

To solve this equation, we can make a transformation of (30). The solution of equation (30) is $m_2(t,u) = m_{21}(t,u) + \frac{k^2}{(\mu-\beta)^2}$.

$$\begin{cases} \dfrac{\partial m_{21}(t,u)}{\partial t} & = 2\kappa\mathcal{L}_{au}m_{21}(t,u) - \dfrac{2\kappa k a(u)}{\mu-\beta} + \delta_0(u)\dfrac{k(2\mu + \sum\limits_{j=1}^{\infty} j(j-1)\beta_{j+1})}{\mu-\beta} \\ m_{21}(0,u) & = 0 \end{cases} \tag{31}$$

For equation (31),we can apply discrete Fourier transform to $m_{21}(t,u)$:

$$\hat{m}_{21}(t,\theta) = \sum_{u\in Z^d} m_{21}(t,u)e^{i(\theta,u)}, \quad \theta \in [-\pi,\pi]^d$$

and use the following lemma:

Lemma 6. Define $\hat{\mathcal{L}}(\varphi) = \sum_{z\neq 0}(1 - cos(\varphi, z))a(z)$. Then $\widehat{\mathcal{L}f}(\varphi) = -\hat{f}(\varphi)\hat{\mathcal{L}}(\varphi) = (\hat{a}(\varphi) - 1)\hat{\mathcal{L}}(\varphi)$

Proof.

$$\begin{aligned}
\widehat{\mathcal{L}_{ax}f}(\varphi) &= \sum_{x\in Z^d} e^{i(\varphi,x)} \sum_{z\neq 0} a(z)(f(x+z) - f(x)) \\
&= \sum_{z\neq 0} a(z)[e^{-i(\varphi,z)} \sum_{x\in Z^d} e^{i(\varphi,x+z))} f(x+z) - \sum_{x\in Z^d} e^{i(\varphi,x)} f(x)] \\
&= \sum_{z\neq 0} a(z)(e^{-i(\varphi,z)} - 1)\hat{f}(\varphi) \\
&= -\hat{f}(\varphi) \sum_{z\neq 0}(1 - cos(\varphi, z))a(z) \\
&= -\hat{\mathcal{L}}(\varphi)\hat{f}(\varphi)
\end{aligned}$$

And

$$\begin{aligned}
\hat{a}(\varphi) &= \sum_{z\in Z^d} a(z)e^{i(\varphi,z)} \\
&= \sum_{z\in Z^d} a(z)cos(\varphi, z) \text{ since } a(z) \text{ is symmetric} \\
&= \sum_{z\neq 0} a(z)(cos(\varphi, z) - 1)
\end{aligned}$$

Thus $\widehat{\mathcal{L}f}(\varphi) = -\hat{f}(\varphi)\hat{\mathcal{L}}(\varphi) = (\hat{a}(\varphi) - 1)\hat{\mathcal{L}}(\varphi)$. $\square$

After applying Fourier transform at both sides of equation (31), we get

$$\begin{cases} \dfrac{\partial \hat{m}_{21}(t,\theta)}{\partial t} & = 2\kappa(\hat{a}(\theta)-1)\hat{m}_{21}(t,\theta) - \dfrac{2\kappa k \hat{a}(\theta)}{\mu-\beta} + \dfrac{k(2\mu + \sum\limits_{j=1}^{\infty} j(j-1)\beta_{j+1})}{\mu-\beta} \\ \hat{m}_{21}(0,\theta) & = 0 \end{cases} \tag{32}$$

The solution of equation (32) is in the following form:

$$\hat{m}_{21}(t,\theta) = \frac{k(2\mu + \sum\limits_{j=1}^{\infty} j(j-1)\beta_{j+1}) - 2\kappa\hat{a}(\theta)}{2\kappa(1-\hat{a}(\theta))(\mu-\beta)}(1 - e^{2\kappa(\hat{a}(\theta)-1)t})$$

Then we can find the inverse Fourier transform.

$$\begin{aligned} m_{21}(t,u) &= \frac{1}{(2\pi)^d}\int_{T^d} e^{-i(\theta,u)}\hat{m}_{21}(t,\theta)d\theta \\ &= \frac{1}{(2\pi)^d}\int_{T^d} f(\theta)e^{\hat{a}(\theta)t}d\theta \end{aligned}$$

where $T^d = [-\pi,\pi]^d$ and $f(\theta) = e^{-\hat{a}(\theta)t - i(\theta,u)}\hat{m}_{21}(t,\theta)$.

$\hat{a}(\theta)$ is twice continuously differentiable and has a maximum at the point $\theta = 0$, see E. Yarovaya [11]. Then using the Laplace method, we get $m_{21}(t,u)$ has the following asymptotic property:

$$\begin{aligned} m_{21}(t,u) &= e^{t\hat{a}(\theta)}(\frac{2\pi}{t})^{d/2}\frac{f(0)+O(t^{-1})}{\sqrt{|det\hat{a}''_{\theta\theta}(0)|}} \\ &\sim (\frac{2\pi}{t})^d \frac{\dfrac{k(2\mu + \sum\limits_{j=1}^{\infty} j(j-1)\beta_{j+1})}{(2\kappa\mu-\beta)}(1-e^{-2\kappa t}) + O(t^{-1})}{\sqrt{|det\hat{a}''_{\theta\theta}(0)|}} \end{aligned}$$

as $t \to \infty$.

Hence, we have already proved the following theorem:

Theorem 7. Let coefficients $\beta_n(x), n \geq 2$, $\mu(x)$, $k(x)$, $x \in Z^d$ are bounded and $\mu(x) -$

$\beta(x) \geq \delta_1 > 0, k(x) \geq \delta_2 > 0$. Then for the bounded initial condition, there exists

$$m_2(\infty, x, y) = \lim_{t \to \infty} m_2(t, x, y)$$

Let's compare our results with the corresponding results for the critical contact model [3] (where $k = 0$, $\mu = \beta$). In the last case, the limiting distribution for the field $n(t, x)$, $t \geqslant 0$, $x \in \mathbb{Z}^d$, exists if and only if the underlying random walk with generator $\mathcal{L}_a$ is transient. In the recurrent case, we have the phenomenon of clusterization. The limiting correlation function is always slowly decreasing (like the Green kernel of $\mathcal{L}_a$). In the presence of immigration, the situation is much better: the limiting correlation function always exists and we believe that the same is true for all moments.

CHAPTER 6: LOCAL AND NONLOCAL PERTURBATIONS OF THE HOMOGENEOUS ENVIRONMENT

6.1 Nonlocal Perturbations

Let $m_1(t,x) = En(t,x)$, $m_1(0,x) = 0$, $V(x) = \beta(x) - \mu(x)$.

$$\frac{\partial m_1}{\partial t} = \mathcal{L}_a m_1 + V(x) m_1 + k(x) \tag{33}$$

Let

$$\beta(x) = \beta_0 + \epsilon\xi(x), \ \ |\xi| \le 1$$

$$\mu(x) = \mu_0 + \epsilon\eta(x), \ \ |\eta| \le 1$$

$$k(x) = k_0 + \epsilon\zeta(x), \ \ |\zeta| \le 1$$

and ϵ is a sufficiently small constant. Denote $\Delta = \mu_0 - \beta_0 > 0$

Then

$$0 < k_0 - \epsilon \le k(x) \le k_0 + \epsilon$$

$$-\Delta - 2\epsilon \le V(x) \le -\Delta + 2\epsilon$$

Due to Kac-Feinman Formula,

$m_1(t,x) = E_x \displaystyle\int_0^t k(x_s) e^{\int_0^s V(x_u)du} ds$

First, $m_1(t,x) \le (k_0+\epsilon) \displaystyle\int_0^t e^{(-\Delta+2\epsilon)s} ds \le \frac{k_0+\epsilon}{\Delta - 2\epsilon} \to \frac{k_0+\epsilon}{\mu_0 - \beta_0 - 2\epsilon}$.

Second, $m_1(t,x) \ge (k_0-\epsilon) \int_0^t e^{(-\Delta-2\epsilon)s} ds \to \dfrac{k_0-\epsilon}{\mu_0 - \beta_0 + 2\epsilon}$

Thus $m_1(t,x) \to \frac{k}{\Delta} + O(\epsilon) + O(e^{-\gamma t})$ as $t \to \infty$ uniformly in t. and we proved the Lyapunov stability of the first moment.

6.2 Local Perturbations

We now consider our process with birth, death, migration, and immigration on a countable space, specifically the lattice $\mathbb{Z}^d$. We have $\beta = \beta_0 + \sigma\delta_0(x) > 0$, the rate of duplication at $x \in Z^d$; $\mu = \mu_0 > 0$, the rate of death; and $k > 0$, the rate of immigration. Here, we add migration of the particles with rate $\kappa > 0$ and probability kernel $a(z)$, $z \in \mathbb{Z}^d$, $z \neq 0$, $a(z) = a(-z)$, $\sum_{z\neq 0} a(z) = 1$. That is, a particle jumps from site x to $x+z$ with probability $\kappa a(z)dt$.

For $n(t,x)$ the number of particles at x at time t, the forward equation for this process is given by $n(t+dt,x) = n(t,x) + \xi(dt,x)$, where

$$\xi(dt,x) = \begin{cases} 1 & \text{w. pr. } n(t,x)\beta dt + kdt + \kappa \sum\limits_{z\neq 0} a(z)n(t,x+z)dt \\ -1 & \text{w. pr. } n(t,x)(\mu+\kappa)dt \\ 0 & \text{w. pr. } 1-(\beta+\mu+\kappa)n(t,x)dt - \kappa\sum\limits_{z\neq 0} a(z)n(t,x+z)dt - kdt \end{cases} \tag{34}$$

Note that $\xi(dt,x)$ is independent on $\mathcal{F}_{\leqslant t}$ (the σ-algebra of events before or including t) and

a) $E[\xi(dt,x)|\mathcal{F}_{\leqslant t}] = n(t,x)(\beta-\mu-\kappa)dt + kdt + \sum\limits_{z\neq 0} a(z)n(t,x+z)dt.$

b) $E[\xi^2(dt,x)|\mathcal{F}_{\leqslant t}] = n(t,x)(\beta+\mu+\kappa)dt + kdt + \sum\limits_{z\neq 0} a(z)n(t,x+z)dt.$

c) $E[\xi(dt,x)\xi(dt,y)|\mathcal{F}_{\leqslant t}] = \kappa a(x-y)n(t,x)dt + \kappa a(y-x)n(t,y)dt.$

A single particle jumps from x to y or from y to x. Other possibilities have probability $O((dt)^2) \approx 0$. Here, of course, $x \neq y$.

d) If $x \neq y$, $y \neq z$, and $x \neq z$, then $E[\xi(dt,x)\xi(dt,y)\xi(dt,z)] = 0$.

We will not use property d) in this paper but it is crucial for the analysis of moments of order greater or equal to 3.

6.2.1 First Moment

Theorem 8. There is a critical value of σ, denoted as σ_{cr}. If σ is large enough ($\sigma > \sigma_{cr}$), then $H\psi = \lambda_0\psi$ has positive eigenvalue $\lambda_0(\sigma)$ with positive eigenfunction $\psi(x)$ and

$$\frac{\partial m_1}{\partial t} = \kappa\mathcal{L}_a m_1 + (\beta - \mu)m_1 + \sigma\delta_0(x)m_1 + k \tag{35}$$

has solution $m_1(t,x) = \dfrac{k}{\mu - \beta} + C_0\psi(x)e^{\lambda_0 t} + \cdots$

Proof. Denote $\tilde{m}_1 = \dfrac{k}{\mu - \beta} + m_1$, then

$$\frac{\partial \tilde{m}_1}{\partial t} = \kappa\mathcal{L}_a \tilde{m}_1 + (\beta - \mu)\tilde{m}_1 + \sigma\delta_0(x)\tilde{m}_1 + \frac{\sigma k\delta_0(x)}{\mu - \beta} \tag{36}$$

Denote $\kappa\mathcal{L}_a u + (\beta - \mu)u + \sigma\delta_0(x)u = Hu$, H is the Schrödinger operator.

Applying Fourier transform to $H\psi = \lambda_0\psi$,

$$-\kappa(1 - \hat{a}(\theta))\hat{\psi}(\theta) - \Delta\hat{\psi}(\theta) + \sigma\psi(0) = \lambda_0\hat{\psi}(\theta)$$

$$\sigma\psi(0) = (\lambda_0 + \kappa(1 - \hat{a}(\theta)) + \Delta)\hat{\psi}$$

$$\hat{\psi}(\theta) = \frac{\sigma\psi(0)}{\lambda_0 + \Delta + \kappa(1 - \hat{a}(\theta))}$$

$$\psi(x) = \frac{1}{(2\pi)^d} \int_{T^d} \frac{\sigma\psi(0)}{\lambda_0 + \Delta + \kappa(1 - \hat{a}(\theta))} e^{-i\theta x} d\theta$$

Thus

$$\psi(0) = \frac{1}{(2\pi)^d} \int_{T^d} \frac{\sigma\psi(0)}{\lambda_0 + \Delta + \kappa(1 - \hat{a}(\theta))} d\theta$$

$$\frac{1}{\sigma} = \frac{1}{(2\pi)^d} \int_{T^d} \frac{d\theta}{\lambda_0 + \Delta + \kappa(1 - \hat{a}(\theta))} = I(\lambda_0) \tag{37}$$

This is the equation for λ_0. Since $1-\hat{a}(\theta) > 0$, $\Delta > 0$, thus $I(0) = \dfrac{1}{(2\pi)^d} \displaystyle\int_{T^d} \frac{d\theta}{\Delta + \kappa(1 - \hat{a}(\theta))}$ is positive and finite in any dimensions , and $I(\lambda_0)$ is a decreasing function of λ_0, thus positive eigenvalue λ_0 exists if $\dfrac{1}{\sigma} < I(0)$, that is $\sigma > \sigma_{cr}$, where σ_{cr}=$\dfrac{1}{I(0)}$ and at most one positive eigenvalue exists.

Then we can express $\tilde{m}_1 = \sum\limits_{i=1}^{\infty} C_i(t)\psi_i(x)$, $\frac{\sigma\delta_0(x)k}{\mu-\beta} = \sum\limits_{i=1}^{\infty} a_i\psi(x)$, when we substitute this into 36, we have

$$\sum C_i'(t)\psi_i(x) = \sum \lambda_i C_i(t)\psi(x) + \sum a_i\psi_i(x) \tag{38}$$

Thus $C_i = \lambda_i C_i + a_i$, which leads to $C_i = \dfrac{e^{\lambda_i t + D\lambda_0} - a_0}{\lambda_0}$. Thus $m_1(t,x)$ has the solution $m_1(t,x) = \frac{k}{\mu-\beta} + C_0\psi_0(x)e^{\lambda_0 t} + \cdots$

□

Theorem 9. If $0 < \sigma < \sigma_{cr}$, for the first moment,

$$\begin{cases} \dfrac{\partial m_1(t,x)}{\partial t} & = \mathcal{L}_a m_1(t,x) + (\beta-\mu)m_1 + \sigma\delta_0 m_1 + k \\ m_1(0,x) & = 0 \end{cases} \tag{39}$$

Where $\mu - \beta \geq A_1 > 0$, $0 < k \leq A_2$. Then the solution of 39 is bounded and has a limit if $t \to \infty$

Proof. This has the solution

$$m_1(t,x) = \int_0^t ds \sum_{y \in Z^d} k(s,y) q(t-s,x,y)$$

where $q(t-s,x,y)$ is the solution for

$$\begin{cases} \dfrac{\partial q}{\partial t} & = \mathcal{L}_a q + (\beta - \mu + \sigma\delta_0(x))q \\ q(0,x,y) & = \delta(x-y) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases} \end{cases}$$

By the Feynman-Kac formula,

$$\begin{aligned} q(s,x,y) &= E_x[e^{\int_0^s (\beta(x_u) - \mu(x_u) + \sigma\delta_0(x_u))du} \delta(x_s - y)] \\ &= E[e^{\int_0^s (\beta(x_u) - \mu(x_u)du + \sigma\delta_0(x_u)} \delta(x_s - y))|x_0 = x] \\ &= E[E[e^{\int_0^s (\beta(x_u) - \mu(x_u) + \sigma\delta_0(x_u))du} \delta(x_s - y))|x_0 = x, x_s = y]|x_0 = x] \\ &= P(x_s = y|x_0 = x) E_{x \to y}[e^{\int_0^s (\beta(x_u) - \mu(x_u) + \sigma\delta_0(x_u))du}] \\ &= p(s,x,y) E_{x \to y}[e^{\int_0^s (\beta(x_u) - \mu(x_u) + \sigma\delta_0(x_u))du}] \end{aligned}$$

where

$$p(t,x,y) = \frac{1}{(2\pi)^d} \int_{T^d} e^{-t\widehat{\mathcal{L}}_a(v) - i(v,x-y)} dv.$$

Let $V(x) = \beta - \mu + \sigma\delta_0(x)$ Finally

$$\lim_{t\to\infty} m_1(t,x) = \lim_{t\to\infty}\int_0^t ds \sum_{y\in Z^d} k(s,y)E_{x\to y}[e^{\int_0^{t-s} V(x_u)du}]p(t-s,x,y)$$

and letting $w = t-s$

$$\le \lim_{t\to\infty}\int_0^t dw\|k\|_\infty E_{x\to y}[e^{\int_0^w V(x_u)du}]$$

$$\le \|k\|_\infty \int_0^\infty e^{(-A_1+\sigma_{cr})w}dw \quad \text{since } \beta-\mu+\sigma\delta_0(x) \le -A_1+\sigma_{cr} < 0$$

$$= \frac{A_2}{A_1-\sigma_{cr}}.$$

Thus, when $\mu-\beta > 0$, $\lim_{t\to\infty} m_1(t,x)$ is bounded by 0 and $\frac{A_2}{A_1-\sigma_{cr}}$, so this limit exists and is finite.

□

6.2.2 Second Moment

We derive differential equations for the second correlation function $m_2(t,x,y) = E[n(t,x)n(t,y)]$ for $x = y$ and $x \neq y$ separately, then combine them and use a Fourier transform to prove a useful result concerning the covariance.

I. $x = y$

$$m_2(t+dt,x,x) = E[E[(n(t,x)+\xi(dt,x))^2|\mathcal{F}_{\leqslant t}]]$$

$$= m_2(t,x,x) + 2E[n(t,x)[n(t,x)(\beta-\mu-\kappa)dt + kdt + \kappa\sum_{z\neq 0} a(z)n(t,x+z)]dt]$$

$$+ E[n(t,x)(\beta+\mu+\kappa)dt + kdt + \kappa\sum_{z\neq 0} a(z)n(t,x+z)dt]$$

Denote $\mathcal{L}_{ax}m_2(t,x,y) = \sum_{z\neq 0} a(z)(m_2(t,x+z,y) - m_2(t,x,y))$.

From this follows the differential equation

$$\begin{cases} \dfrac{\partial m_2(t,x,x)}{\partial t} &= 2(\beta(x)-\mu(x))m_2(t,x,x) + 2\kappa\mathcal{L}_{ax}m_2(t,x,x) + \kappa\mathcal{L}_a m_1(t,x) \\ & +(\beta(x)+\mu(x)+2\kappa+2k)m_1(t,x)+k \\ m_2(0,x,x) &= 0 \end{cases}$$

II. $x \neq y$

Because only one event can happen during dt

$$P\{\xi(dt,x)=1,\xi(dt,y)=1\} = P\{\xi(dt,x)=-1,\xi(dt,y)=-1\} = 0,$$

while the probability that one particle jumps from y to x is

$$P\{\xi(dt,x)=1,\xi(dt,y)=-1\} = \kappa a(x-y)n(t,y)dt,$$

and the probability that one particle jumps from x to y is

$$P\{\xi(dt,x)=-1,\xi(dt,y)=1\} = \kappa a(y-x)n(t,x)dt.$$

Then, similar to above

$$m_2(t+dt,x,y) = E[E[(n(t,x)+\xi(t,x))(n(t,y)+\xi(t,y))|\mathcal{F}_{\leqslant t}]]$$

$$= m_2(t,x,y) + (\beta(x)-\mu(x))m_2(t,x,y)dt + k(x)m_1(t,y)dt +$$

$$\kappa\sum_{z\neq 0} a(z)(m_2(t,x+z,y)-m_2(t,x,y))dt + (\beta(y)-\mu(y))m_2(t,x,y)dt + k(y)m_1(t,x)dt +$$

$$\kappa\sum_{z\neq 0} a(z)(m_2(t,x,y+z)-m_2(t,x,y))dt + \kappa a(x-y)m_1(t,y)dt + \kappa a(y-x)m_1(t,x)dt$$

$$= m_2(t,x,y) + (\beta(x)-\mu(x)+\beta(y)-\mu(y))m_2(t,x,y)dt + k(x)m_1(t,y)dt + k(y)m_1(t,x)dt +$$

$$+\kappa a(x-y)(m_1(t,x)+\kappa m_1(t,y))dt$$

The resulting differential equation is

$$\frac{\partial m_2(t,x,y)}{\partial t} = \kappa(\mathcal{L}_{ax} + \mathcal{L}_{ay})m_2(t,x,y) + (\beta(x) - \mu(x) + \beta(y) - \mu(y))m_2(t,x,y)$$
$$+ k(y)m_1(t,x) + k(x)m_1(t,y) + \kappa a(x-y)[m_1(t,x) + m_1(t,y)] \tag{40}$$

Consider H_x and H_y, two operators applied to x and y respectively. $H_x m_2(t,x,y) = \kappa\mathcal{L}_{ax} m_2(t,x,y) + (\beta(x) - \mu(x))m_2(t,x,y)$, and $H_y m_2(t,x,y) = \kappa\mathcal{L}_{ay} m_2(t,x,y) + (\beta(y) - \mu(y))m_2(t,x,y)$.

From the theorem we proved in the first moment case, we know if $\sigma > \sigma_{cr}$, then H_x and H_y both have at most one positive eigenvalue $\lambda_0(\sigma)$ with positive eigenfunction $\psi(x)$ and $\psi(y)$ respectively. $(H_x + H_y)\psi_i(x)\psi_j(y) = \lambda_p \psi_i(x)\psi_j(y)$ and $\lambda_p = \lambda_i + \lambda_j$,where λ_i and λ_j are eigenvalues of H_x and H_y respectively and $m_2(t,x,y) = \sum_{p=0}^{\infty} e^{\lambda_p t}\psi_i(x)\psi_j(y)$. As $t \to \infty$, $m_2(t,x,y) \sim e^{2\lambda_0}\psi_0(x)\psi_0(y) + e^{(\lambda_0+\lambda_1)t}\psi_0(x)\psi_1(y) + e^{(\lambda_0+\lambda_1)t}\psi_1(x) * \psi_0(y)$, thus $m_2(t,x,y) \sim e^{2\lambda_0 t}[\psi_0(x)\psi_0(y) + e^{(\lambda_1-\lambda_0)t}[\psi_0(x)\psi_1(y) + \psi_1(x)\psi_0(y)]] = e^{2\lambda_0 t}(\psi_0(x)\psi_0(y) + o(1))$

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