ON THE SPECTRAL THEORY OF 1-D SCHRÖDINGER OPERATOR WITH SPARSE RANDOM POTENTIAL

by

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ABSTRACT

THOMAS A. COOK. On the spectral theory of 1-D Schrödinger operator with sparse random potential. (Under the direction of DR. STANISLAV MOLCHANOV)

The goal of this dissertation is to develop a spectral theory for the Schrödinger operator with sparse random potential. To do this, we will first reformulate theories for sparse deterministic potentials. This includes a general development of the spectral measure μ and the use of a generalized Fourier transform for the development of μ will also be discussed. The interpretation would be that the support of μ is the spectrum Σ. The development of a unitary operator known as the monodromy operator will be discussed as well as the fascinating connection to the spectrum of the Schrödinger operator. We will construct an example to show that for sparse potentials the Bargmann estimate is too "rough" of an estimate for the number of negative eigenvalues. Lastly, we will show that there is a spectral transition from singular continuous to pure point spectrum of certain Schrödinger operators with random sparse potentials.

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CHAPTER 1: DETERMINISTIC SPARSE POTENTIALS

1.1 A General Summary

The Schrödinger operator is one of the most studied topics in mathematical physics. Here, we are interested in the spectral properties of the Schrödinger operator,

$$H\psi(x) = -\psi''(x) + V(x)\psi(x) = \lambda\psi(x). \tag{1}$$

The goal is to study the phenomenon of multiscattering on the so-called sparse potentials either on the full axis or the half-axis. If we consider the problem on the half-axis, we must impose the boundary conditions at $x = 0$:

$$\psi(0)\sin\theta_0 - \psi'(0)\cos\theta_0 = 0,\ \ \theta_0 \in [0, \pi)$$

where θ_0 is called the boundary phase of the solution $\psi(\lambda, x)$ of the problem $H\psi = \lambda\psi$, with initial data $\psi(0) = \cos\theta_0,\ \ \psi'(0) = \sin\theta_0$.

In the classical scattering theory case, the potential $V(x)$ is bounded from below and tends to 0 if $|x| \to \infty$. We have to understand the boudedness of the generalized potential $V(x) = \sum_{i=1}^{\infty} \sigma_i \delta(x - x_i)$ in Birman's sense

$$\int_x^{x+1} |V(z)|dz \to 0, \quad |x| \to \infty \tag{2}$$

Under the condition 2 the operator H is essentially self-adjoint in either $L^2(\mathbb{R})$ or $L^2(\mathbb{R}_+)$, where $\mathbb{R}_+$ is the positive half-axis. This means that there is a unique spectral measure $\mu_H(d\lambda)$. The spectrum $\Sigma(H)$ is the support of the spectral measure (i.e. the minial closed set whose complement has zero μ measure) which contains, in our case, the continuous spectrum $[0,\infty)$ and the discrete spectrum $\lambda_i < 0$ with possible accumulation point $\lambda = 0$. It should be noted that the term "continuous spectrum" is misleading: the spectral measure for a slowly decreasing potential $V(x)$ can be pure point, i.e. there exists the dense set on $[0,\infty)$ of eigenvalues $\lambda_i > 0$ with eigenfunctions $\psi_i(x) \in L^2$.

However, if $V(x) \in L^1(\mathbb{R})$ or $L^2(\mathbb{R}_+)$, then the spectral measure $\mu_H(d\lambda)$ is purely absolutely continuous on $[0,\infty)$ (in the case $V \in L^2$ for a.e. $\theta_0 \in [0,\pi)$). To guarantee that the discrete negative spectrum is finite, we'll assume that

$$\int_{\mathbb{R}} (|x|+1)|V(x)|\; dx < \infty, \tag{3}$$

which is known as Bargmann's condition. Let's introduce the basic definitions and formulas of the spectral theory, first on the half-axis $[0,\infty)$ and then on the Hilbert space $\mathcal{H} = L^2([0,\infty), dx)$.

For a given λ and $\theta_0 \in [0,\pi)$ let's define the kernel $\psi_\lambda(x)$:$-\psi''_\lambda + V\psi = \lambda\psi,\ x \geq 0,\ \psi_\lambda(0) = \cos\theta_0,\ \psi'_\lambda(0) = \sin\theta_0$. Here θ_0 is the initial phase of the solution $\psi_\lambda(x)$, introduced above. This solution can be presented in the polar coordinates of the form

$$\psi_\lambda(x) = \rho_\lambda(x)\sin\theta_\lambda(x),\ \psi'_\lambda(x) = \rho_\lambda(x)\cos\theta_\lambda(x).$$

It is well known that phase $\theta_\lambda(\cdot)$ and magnitude $\rho_\lambda(\cdot)$ satisfy the equations

$$\frac{d\theta_\lambda}{dx} = \cos^2\theta_\lambda(x) + (\lambda - V(x))\sin^2\theta_\lambda(x), \tag{4}$$

$$\frac{d\rho_\lambda}{dx} = \frac{1}{2}\sin 2\theta_\lambda\,(1 + \lambda - V(x))\,. \tag{5}$$

Consider in the beginning the spectral problem

$$(H\psi)(x) = -\psi'' + V(x)\psi = \lambda\psi \tag{6}$$

on $[0, L]$ with the boundary conditions

$$\psi(0)\cos\theta_0 - \psi'(0)\sin\theta_0 = 0,$$

$$\psi(L)\cos\theta_L - \psi'(L)\sin\theta_L = 0.$$

Then the spectral measure of this problem (which depends on $\theta_0,\ \theta_L \in [0.\pi)$) is given by the formula

$$\mu_L^{(\theta_0,\theta_L)}(d\lambda) = d\lambda \sum_n \frac{\delta(\lambda - \lambda_{n,L})}{\int_0^L \psi^2_{\lambda_{n,L}}(x)dx}. \tag{7}$$

Here $\lambda_{n,L}$ are the roots of the equation $\theta_\lambda(L) \mod \pi = \theta_L$. If $L \to \infty$ then under mild conditions (say, $\int_x^{x+1} V(x)dx \geq C_0$, that is Birman's condition) the spectral measure $\mu_L^{(\theta_0,\theta_L)}(d\lambda)$ converges weakly (independently of θ_L!) to the limiting spectral measure $\mu^{\theta_0}(d\lambda)$ which is independent on the selection of the sequence $L_n,\ \theta_{L_n} \in [0, \pi)$.

The generalized Fourier transform $\hat{f}(\lambda) = \int_0^\infty \psi_\lambda(x)f(x)dx$ can be defined in the beginning for the compactly supported function $f(\cdot) \in \mathcal{C}_0^\infty([0, \infty))$. For such functions one can also introduce the inverse generalized Fourier transform $f(x) = \int_{\mathbb{R}} \psi_\lambda(x)\hat{f}(\lambda)\mu^{\theta_0}(d\lambda)$

and later these definitions can be extended to $L^2([0,\infty), dx)$ and $L^2(\mathbb{R}, \mu^{\theta_0}(d\lambda))$ like in the usual Plansherel form of the Fourier analysis on $L^2(\mathbb{R})$. The isomorphism between $L^2([0,\infty), dx)$ and $L^2(\mathbb{R}, \mu^{\theta_0}(d\lambda))$ is given by Parseval's identity:

$$\int_0^\infty f(x)g(x)dx = \int_{\mathbb{R}} \hat{f}(\lambda)\hat{g}(\lambda)\mu^{\theta_0}(d\lambda). \tag{8}$$

In the future, we will use two general theorems:

Theorem 1. If $V \in L^1([0,\infty), dx)$, i.e. $\int_0^\infty |V|dx < \infty$, then for arbitrary $\theta_0 \in [0,\pi)$ and $\lambda = k^2 > 0$

$$\psi_\lambda(x) = \alpha(\lambda)\cos kx + \beta(\lambda)\sin kx + O(1), x \to \infty \tag{9}$$

and

$$\mu^{\theta_0}(d\lambda) = \frac{d\lambda}{\pi\sqrt{\lambda}(\alpha^2(\lambda) + \beta^2(\lambda))}. \tag{10}$$

That is, the spectral measure is absolutely continuous for $\lambda = k^2 > 0$.

If $\lambda = -k^2 < 0$, then the spectrum is discrete and bounded below and can contain only on accumulation point $\lambda_0 = 0$.

If

$$\int_0^\infty x|V(x)|dx, \tag{11}$$

(Bargmann's bound, which is much stronger than the condition $V \in L^1$) then $\mathcal{N}_0(V) = \#\{\lambda_i < 0\} < \infty$ and furthermore $\forall(\theta_0 \in [0.\pi)),\ \mathcal{N}_0(V) \leq 1 + \int_0^\infty x|V(x)|dx$

In the case of the full axis $L^2(\mathbb{R}, dx)$, the equation $H\psi(x) = -\psi''(x) + V(x)\psi(x) = \lambda\psi(x)$ has two solutions (similar to $\cos\sqrt{\lambda}x$ and $\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$) $\psi_1(\lambda, x),\ \psi_2(\lambda, x)$ with conditions at $x = 0$:

$$\psi_1(\lambda, 0) = 1 \quad \psi_1'(\lambda, 0) = 0$$

$$\psi_2(\lambda, 0) = 0 \quad \psi_2'(\lambda, 0) = 1.$$

Using these two solutions, one can define the (vector-valued) generalized Fourier transform in this way: For any function $f(x) \in \mathcal{C}_0^2$, we define the Fourier transforms

$$\hat{f}^{(1)}(\lambda) = \int_{\mathbb{R}} \psi_1(\lambda, x) f(x) dx$$
$$\hat{f}^{(2)}(\lambda) = \int_{\mathbb{R}} \psi_2(\lambda, x) f(x) dx.$$

There exists a matrix-valued spectral measure

$$\begin{bmatrix} \mu_{11}(d\lambda) & \mu_{12}(d\lambda) \\ \mu_{21}(d\lambda) & \mu_{22}(d\lambda) \end{bmatrix} = \mu(d\lambda). \tag{12}$$

Note that the spectral measure is not in general unique. However, if the potential is bounded from below or if $\displaystyle\int_x^{x+1} V_-(x)dx \leq C_0 < \infty$, then

$$f(x) = \int_{\mathbb{R}} (\psi_1(\lambda, x), \psi_2(\lambda, x)) \mu(d\lambda) \hat{\mathbf{f}}(\lambda), \tag{13}$$

where

$$\hat{\mathbf{f}}(\lambda) = \begin{bmatrix} \hat{f}^{(1)}(\lambda) \\ \hat{f}^{(2)}(\lambda) \end{bmatrix}.$$

This formula is not very efficient, except when the potential is even. In this case, we are able to write $f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$ where, for $x \geq 0$,

$$f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}, \tag{14}$$

$$f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2}. \tag{15}$$

But $\psi_1(\lambda, x)$ is odd and $\psi_2(\lambda, x)$ is even. This means that

$$\hat{f}^{(1)}(\lambda) = \int_{\mathbb{R}} \psi_1(\lambda, x) f(x) dx = \int_{\mathbb{R}} \psi_1(\lambda, x) f_{\text{odd}} dx \tag{16}$$

$$\hat{f}^{(2)}(\lambda) = \int_{\mathbb{R}} \psi_2(\lambda, x) f(x) dx = \int_{\mathbb{R}} \psi_2(\lambda, x) f_{\text{even}} dx \tag{17}$$

and the spectral measure becomes

$$\mu(d\lambda) = \begin{bmatrix} \mu^{(N)}(d\lambda) & 0 \\ 0 & \mu^{(D)}(d\lambda) \end{bmatrix},$$

if $\int_{\mathbb{R}} |V| dx < \infty,\ V(x) = V(-x)$. See details in [4].

Let's describe the classical scattering theory. If $V(x) \in L^1(\mathbb{R})$, then one can prove that for $\lambda = k^2 > 0$, there exists the fundamental system of solutions $\psi_{1,2}(\cdot)$ of the equation

$$H\psi = -\psi'' + V(x)\psi = \lambda\psi$$

such that for $x \to -\infty,\ \psi_1(k, x) = e^{ikx} + o(1)$ and $\psi_2(k, x) = \overline{\psi_1(k, x)} = e^{-ikx} + o(1)$. In addition, $\psi_1'(k, x) = ike^{ikx} + o(1)$ and $\psi_2'(k, x) = -ike^{-ikx} + o(1)$. For positive $x \to \infty$

$$\begin{aligned} \psi_1(k, x) &= a(k)e^{ikx} + b(k)e^{-ikx} + o(1) \\ \psi_2(k, x) &= \overline{a}(k)e^{-ikx} + \overline{b}(k)e^{ikx} + o(1) \end{aligned}, \tag{18}$$

and a similar formula for $\psi_{1,2}'(k, x)$,

$$\begin{aligned} \psi_1'(k, x) &= ika(k)e^{ikx} - ikb(k)e^{-ikx} + o(1) \\ \psi_2'(k, x) &= -ik\overline{a}(k)e^{-ikx} + ik\overline{b}(k)e^{ikx} + o(1) \end{aligned} \tag{19}$$

The Wronskian of the system $\psi_1(k,x),\ \psi_2(k,x)$

$$\mathcal{W}(k) = \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1' & \psi_2' \end{vmatrix}$$

is independent of x. For $x \to -\infty$, $\mathcal{W}(k) = -2ik$. But for $x \to \infty$, we have

$$\mathcal{W}(k) = \begin{vmatrix} a(k)e^{ikx} + b(k)e^{-ikx} + o(1) & \overline{a}(k)e^{-ikx} + \overline{b}(k)e^{ikx} + o(1) \\ ika(k)e^{ikx} - ikb(k)e^{-ikx} + o(1) & -ik\overline{a}(k)e^{-ikx} + ik\overline{b}(k)e^{ikx} + o(1) \end{vmatrix}, \quad (20)$$

$$= -2ik(|a|^2 - |b|^2) + o(1),$$

i.e. $|a|^2 = 1 + |b|^2$. This is one of the forms of the conservation of energy law. Similarly, one can find the system of the solutions $\varphi_1(k,x),\ \varphi_2(k,x)$ such that, for $x \to \infty$

$$\begin{aligned} \varphi_1(k,x) &= e^{ikx} + r(k)e^{-ikx} + o(1), \quad \varphi_2(k,x) = t(k)e^{ikx} + o(1) \\ \varphi_1'(k,x) &= ike^{ikx} - ikr(k)e^{-ikx} + o(1), \quad \varphi_2'(k,x) = ikt(k)e^{ikx} + o(1). \end{aligned} \quad (21)$$

We call $r(k)$ the reflection coefficient and $t(k)$ the transmission coefficient for the given frequency $k = \sqrt{\lambda} > 0,\ \lambda > 0$. Then calculation of the Wronskian for $x \to \pm\infty$ gives

$$|r(k)|^2 + |t(k)|^2 = 1. \quad (22)$$

The functions $r(k)$ and $t(k)$ are meromorphic on the upper half plane $\Im(k) > 0$. The functions $a(k)$ and $b(k)$ above are also the transmission and reflection coefficients in the Jost form and

$$r(k) = \frac{b(k)}{a(k)}, \quad t(k) = \frac{1}{a(k)}. \quad (23)$$

For $\lambda = -k^2 < 0$, we can find (under Bargmann condition) at most finitely many eigenvalues and corresponding eigenfunctions. If the potential $V(x)$ is non-negative, then there are no eigenvalues. If $V(x) < 0$ then there is at least one negative eigenvalue. For potentials with positive and negative values, the negative eigenvalues are not always present. This is known as the screening phenomenon. The following proposition 2 illustrates this fact.

Proposition 2. Consider the equation $-\psi''(x) - \sigma\delta(x)\psi(x) = \lambda\psi(x)$. In the case of the full axis, there exists one negative eigenvalue. Let $\lambda_0 = -k^2 < 0$, then $\psi_k(x) = e^{-|k|x}$, and from the gluing condition

$$\psi'(-0) - \psi'(+0) = -\sigma\psi(0)$$

we will get $\lambda_0 = -\dfrac{\sigma^2}{4}$. In the case of the half-axis, then answer depends on the boundary conditions at $x = 0$. If $\sigma < 0$, then for Neumann boundary conditions there is one negative eigenvalue. For Dirichlet boundary conditions, then

- if $\sigma a \leq 1$, there are no negative eigenvalues.
- if $\sigma a > 1$, then there is one negative eigenvalue.

Lets illustrate the above statements by examples:

Example 1 (Single Positive Bump on half-axis). Let $V(x) = \sigma\delta(x-a)$, $x \in [0, \infty)$, $\sigma > 0$. For this potential, the operator H is positive definite, hence we have no negative eigenvalues. It follows from the variational principle: if $H\psi = \lambda\psi$, $\lambda < 0 \Rightarrow (H\psi, \psi) = \lambda < 0$. But

$$(H\psi, \psi) = \int_{\mathbb{R}} (\psi')^2 dx + \int_{\mathbb{R}} V\psi^2 dx > 0.$$

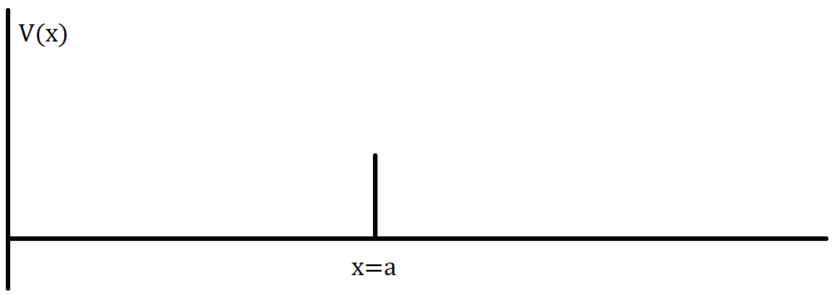

Figure 1: Single positive potential on the positive half-axis.

This leads to a contradiction. Lets study the absolutely continuous spectrum for $\lambda = k^2 > 0$. Consider the Dirichlet boundary condition: $\psi(0) = 0,\ \psi'(0) = 1$. For $x \leq a,\ \psi_\lambda(x) = \frac{\sin kx}{k}$ and for $x > a,\ \psi_\lambda(x) = c_1 \frac{\sin k(x-a)}{k} + c_2 \cos k(x-a)$. We glue the two solutions together to get

$$c_1 = \cos ka + \frac{\sigma \sin ka}{k}, \tag{24}$$

$$c_2 = \frac{\sin ka}{k}. \tag{25}$$

We are then able to write the solution for $x > a$ as

$$\begin{aligned} \psi_\lambda(x) &= \frac{\sin k(x-a)}{k}\left(\cos ka + \frac{\sigma \sin ka}{k}\right) + \cos k(x-a)\left(\frac{\sin ka}{k}\right), \\ &= \alpha(\lambda)\cos kx + \beta(\lambda)\sin kx \end{aligned} \tag{26}$$

where

$$\alpha(\lambda) = -\frac{\sigma \sin^2 \sqrt{\lambda}a}{\lambda}$$

and

$$\beta(\lambda) = \frac{1}{\sqrt{\lambda}} + \frac{\sigma \cos \sqrt{\lambda}a \sin \sqrt{\lambda}a}{\lambda}.$$

The spectral measure is given by

$$\mu^{(D)} = \frac{d\lambda}{\pi\sqrt{\lambda}\,(\alpha^2(\lambda) + \beta^2(\lambda))}, \tag{27}$$

where

$$\alpha^2(\lambda) + \beta^2(\lambda) = \frac{1}{\lambda} + 2\frac{\sigma}{\lambda^{\frac{3}{2}}}\cos\sqrt{\lambda}a\sin\sqrt{\lambda}a + \frac{\sigma^2\sin^2\sqrt{\lambda}a}{\lambda^2}.$$

Example 2 (Single positive bump on full axis). Now consider the same problem with Nuemann boundary condition: $\psi(0) = 1,\ \psi'(0) = 0$. For this problem, we have the two equations: for $x \leq a,\ \psi_\lambda(x) = \cos kx$ and for $x > a,\ \psi_\lambda(x) = c_1\cos k(x-a) + c_2\sin k(x-a)$. As before, we glue the two solutions together to obtain the coefficients:

$$c_1 = \cos ka, \tag{28}$$

$$c_2 = \sin ka + \frac{\sigma}{k}\cos ka. \tag{29}$$

We can write the solution for $x > a$ as

$$\begin{aligned}\psi_\lambda(x) &= \cos k(x-a)\cos ka + \sin k(x-a)\left(\sin ka + \frac{\sigma}{k}\cos ka\right),\\ &= \alpha(\lambda)\cos kx + \beta(\lambda)\sin kx,\end{aligned} \tag{30}$$

where

$$\alpha(\lambda) = \cos^2\sqrt{\lambda}a - \sin^2\sqrt{\lambda}a - \frac{\sigma}{\sqrt{\lambda}}\sin\sqrt{\lambda}a\cos\sqrt{\lambda}a$$

and

$$\beta(\lambda) = 2\cos\sqrt{\lambda}a\sin\sqrt{\lambda}a + \frac{\sigma}{\sqrt{\lambda}}\cos^2\sqrt{\lambda}a.$$

The spectral measure is

$$\mu^{(N)} = \frac{d\lambda}{\pi\sqrt{\lambda}\,(\alpha^2(\lambda) + \beta^2(\lambda))}, \tag{31}$$

where

$$\alpha^2(\lambda) + \beta^2(\lambda) = 1 + \frac{\sigma^2}{\lambda}\cos^2\sqrt{\lambda}a + 2\frac{\sigma}{\sqrt{\lambda}}\sin\sqrt{\lambda}a\cos\sqrt{\lambda}a.$$

For the full axis problem, we consider the transmission and reflection coefficients. In this regard, we have the solutions for $x < 0$

$$\psi_1(k, x) = e^{-ikx} + b(k)e^{ikx},$$

and for $x > 0$,

$$\psi_2(k, x) = c(k)e^{-ikx}.$$

Because the potential $V(x)$ is the generalized function, we have a discontinuous first derivative of the solution $\psi(k, x)$. After integration, we obtain,

$$\psi'(-\epsilon) - \psi'(\epsilon) = -\sigma\psi(0),$$

and then consider the behavior of the solution as $\epsilon \to 0$. The problem now becomes finding the coefficients $b(k)$ and $c(k)$ that satisfies our conditions. It boils down to solving the system

$$\begin{aligned} 1 + b(k) - c(k) &= 0, \\ ik - ikb(k) - ikc(k) &= -\sigma. \end{aligned} \tag{32}$$

We calculate the following scattering data:

$$\begin{aligned} b(k) &= \frac{i\sigma}{2k} & c(k) &= 1 + \frac{i\sigma}{2k} \\ r(k) &= \frac{\sigma}{\sqrt{4k^2 + \sigma^2}} & t(k) &= \sqrt{\frac{4k^2}{4k^2 + \sigma^2}} \end{aligned}$$

Example 3. Consider first the problem

$$H\psi = -\psi'' - \sigma\delta(x - a)\psi = \lambda\psi, \tag{33}$$

with $a > 0$ with Dirichlet boundary conditions; $\psi(0) = 0$, $\psi'(0) = 1$. Then for $\lambda = k^2 > 0$,

$$\psi_\lambda(x) = \frac{\sin kx}{k}, \tag{34}$$

$$\psi_\lambda(x) = c_1 \frac{\sin kx}{k} + c_2 \cos kx. \tag{35}$$

When we glue together the solutions, and noting that the derivative is not continuous and has a jump $\psi'(a-\epsilon) - \psi'(a+\epsilon) = \sigma$, we get the coefficients

$$c_2 = \frac{\sin ka}{k}$$

$$c_1 = \cos ka - \sigma \frac{\sin ka}{k}.$$

That is, we can write the solution for $x > a$ as

$$\begin{aligned} \psi_\lambda(x) &= \frac{\sin k(x-a)}{k} \left(\cos ka - \sigma \frac{\sin ka}{k} \right) + \frac{\sin ka}{k} \cos k(x-a), \\ &= \sin kx \left(\frac{1}{k} - \frac{\sigma \cos ka \sin ka}{k^2} \right) + \cos kx \left(\frac{\sigma \sin^2 ka}{k^2} \right), \end{aligned} \tag{36}$$

where

$$\alpha(\lambda) = \frac{1}{k} - \frac{\sigma \cos ka \sin ka}{k^2},$$

and

$$\beta(\lambda) = \frac{\sigma \sin^2 ka}{k^2}.$$

Then the spectral measure is

$$\mu(d\lambda) = \frac{d\lambda}{\pi\sqrt{\lambda}(\alpha^2(\lambda) + \beta^2(\lambda))}, \tag{37}$$

with

$$\alpha^2(\lambda)+\beta^2(\lambda)=\left(\frac{1}{k}-\frac{\sigma\cos ka\sin ka}{k^2}\right)^2+\left(\frac{\sigma\sin^2 ka}{k^2}\right)^2,$$
$$=\frac{1}{\lambda}-\frac{\sigma\sin 2\sqrt{\lambda}a}{\lambda^{\frac{3}{2}}}+\frac{\sigma^2\sin^2\sqrt{\lambda}a}{\lambda^2}.$$

Now consider negative energies $\lambda=-k^2<0$.

$$-\psi'(a+\epsilon)+\psi'(a-\epsilon)=\sigma\psi(a). \tag{38}$$

We have the solutions

$$\psi_\lambda(x)=\frac{\sinh kx}{k}, x\le a, \tag{39}$$
$$\psi_\lambda(x)=\frac{\sinh ka}{k}\exp\left[-k(x-a)\right],\ \ x>a. \tag{40}$$

We consider the solution of the equation under certain conditions

$$k\frac{\cosh ka}{\sinh ka}+k=\sigma. \tag{41}$$

We rearrange

$$\frac{2e^{ka}}{e^{ka}-e^{-ka}}=\frac{\sigma a}{ka}.$$

We let $ka=x$ and $\sigma a=A$. So we rewrite the equation as

$$\frac{e^x-e^{-x}}{2e^x}=\frac{x}{A}, \tag{42}$$

$$\Rightarrow 1-e^{-2x}=\frac{2x}{A}.$$

Let $y=2x$ and rewrite the equation as

$$e^{-y}=1-\frac{y}{A}. \tag{43}$$

We look for the conditions when the equation is satisfied, since that gives us the conditions of when a negative eigenvalue exists. When $\frac{1}{A} \geq 1$, then we would have no solution and hence no negative eigenvalue. However, if $\frac{1}{A} < 1 \Rightarrow A > 1$, then we would have a solution and hence the existence of a negative eigenvalue. Note that since $A = \sigma a$ that this agrees with the Bargmann estimate as calculated in equation 11. We just proved the lemma

Lemma 1. Consider the eigenvalue problem

$$-\psi''(x) - \sigma\delta(x-a)\psi(x) = \lambda\psi(x),$$
$$\lambda = -k^2 < 0, \quad x \in [0, \infty) \tag{44}$$

with solutions that satisfy the Dirichlet boundary conditions at the origin $\psi(0) = 0$ and $\psi(a) = 1$.

1. If $\sigma a < 1$, then there is no negative eigenvalue.

2. If $\sigma a \geq 1$, then there is a negative eigenvalue.

We have the same type of conditions on the interval $[0, L]$ when we have following the solutions $\psi_1(x) = \frac{\sinh kx}{\sinh ka}$, $\psi_2(x) = \frac{\sinh k(A-x)}{\sinh k(A-a)}$. We would need the following equation satisfied

$$\frac{\coth^2 x - 1}{\coth x - \coth B} = \frac{A}{x}, \tag{45}$$

where $x = ka$, $A = \sigma a$, and $B = kL$. Just as in the lemma, we have a solution when $A > 1$, hence the existence of a negative eigenvalue. We can see easily that $\lambda_0 = -\frac{\sigma^2}{4}$. However, when $A < 1$, there is no solution and hence, no negative eigenvalue.

Now we consider the same equation 44 with Neumann boundary conditions. In this case, we have the solutions

$$\psi_1(x) = \cosh kx, \quad \psi_2(x) = \cosh(ka)e^{-k(x-a)}.$$

From a similar calculation as earlier, we have the equation

$$k + k \tanh ka = \sigma. \tag{46}$$

After manipulations and letting $y = ka,\ A = \sigma a$, we obtain the following equation:

$$\frac{e^y}{\cosh y} = \frac{A}{y}. \tag{47}$$

A solution exists to equation (47) if $A > 0$, thus illustrating proposition 2.

We now turn our attention to the same potential on the whole axis. We wish to know the reflection and transmission coefficients of the problem

$$-\psi'' - \sigma\delta(x) = -k^2\psi.$$

Without loss of generality, we may shift the potential back to the origin. Then we find the reflection and transmission coefficients through solving the system:

$$\begin{aligned} 1 + b(k) - c(k) &= 0, \\ ik - ikb(k) - ikc(k) &= \sigma. \end{aligned} \tag{48}$$

We solve this through very similar calculations as before. We obtain the reflection and transmission coefficients

$$\begin{aligned} b(k) &= \frac{-i\sigma}{2k} & c(k) &= 1 - i\frac{\sigma}{2k} \\ r(k) &= \frac{\sigma}{\sqrt{4k^2+\sigma^2}} & t(k) &= \sqrt{\frac{4k^2}{4k^2+\sigma^2}}. \end{aligned}$$

Example 4. Let $V(x) = \sigma_1\delta(x + x_1) + \sigma_1\delta(x - x_1),\ \sigma_1 > 0$ on $\mathbb{R}$ and $\lambda = -k^2 < 0$.

Let also $x_1 > 0$. We are interested in understanding the conditions on x_1 and σ_1 that gives a negative eigenvalue or none at all. Firstly, it can easily be seen that the potential is even. This means that the eigenfunctions have the property of parity, i.e. the solutions can be separated into even functions and odd functions. So we consider the problem on the positive half line in which we only consider the bump located at $x = x_1$; hence, the problem reduces to that of the previous example.

Example 5. Now we consider the same form of the potential on $\mathbb{R}_+$ with $V(x) = -\sigma_1\delta(x - x_1) - \sigma_1\delta(x - x_2)$. The goal now is to show that there are three possibilities for the number of negative eigenvalues on the half axis. Our tool for this analysis will be the the number of zeros of $\psi_0(x)$, i.e. the solutions to the eigenvalue problem

$$H\psi = -\psi''(x) - \sigma\delta(x - x_1)\psi(x) - \sigma\delta(x - x_2)\psi(x) = 0. \tag{49}$$

In this case, the solutions are linear functions and we are interested in the function values of these functions at the δ-potentials $\{x_i\}$. We come to the following conclusions:

1. If $\dfrac{x_2}{x_2 - x_1} > \sigma$ and $x_1,\ x_2$ is small, then there will be no negative eigenvalues.
2. If $\sigma < 1$, and x_2 is large, then there is one eigenvalue.
3. If $\dfrac{x_2}{x_2 - x_1} < \sigma$ and $x_2 - x_1$ is large, then there are two eigenvalues.

Thus, for two potential wells it is possible to obtain either zero, one, or two eigenvalues. The tools used here will be very helpful when constructing certain potentials with properties that will be extended from this example. In fact, the zeros of the solution $\psi_0(x)$ will be considered in more detail in the next section. First, however, is another interesting example of an even potential.

Example 6. Let $V(x) = \sigma_1\delta_0(x+1) + \sigma_1\delta_0(x-1) - \sigma\delta_0(x)$ on $\mathbb{R}$. In this situation the calculations are a little more involved. We now have four regions to consider. We have $x < -1,\ -1 < x < 0,\ 0 < x < 1,$ and $x > 1$. So we have the coefficients that represent the reflection and transmission coefficients from each of the δ-potentials. We will consider the eigenvalue problem of each bump,

$$H_n\psi = -\psi''(x) + V_n(x)\psi(x),$$

with $n = 1, 2, 3$. For each bump, the reflection and transmission coefficients were calculated. For instance, for the i^{th} potential, $b_i(k)$ is the reflection coefficient and $c_i(k)$ is the transmission coefficient. To solve each equation, we scale the solutions to be equal to unity at each boundary. Again, we follow a very similar method of calculation from the previous two examples. We obtain the following scattering data

$$\begin{aligned}
b_1(k) &= i\frac{\sigma_1}{2k} & c_1(k) &= 1 + i\frac{\sigma_1}{2k} \\
b_2(k) &= -i\frac{\sigma}{2k} & c_2(k) &= 1 - i\frac{\sigma_1+\sigma}{2k} \\
b_3(k) &= i\frac{\sigma_1}{2k} & c_3(k) &= 1 + i\frac{2\sigma_1+\sigma}{2k} \\
r_1(k) &= \frac{\sigma_1}{\sqrt{4k^2+\sigma_1^2}} & t_1(k) &= \sqrt{\frac{4k^2}{4k^2+\sigma_1^2}} \\
r_2(k) &= \frac{\sigma}{\sqrt{4k^2+(\sigma_1+\sigma)^2}} & t_2(k) &= \sqrt{\frac{4k^2}{4k^2+(\sigma_1+\sigma)^2}} \\
r_3(k) &= \frac{\sigma_1}{\sqrt{4k^2+(2\sigma_1+\sigma)^2}} & t_3(k) &= \sqrt{\frac{4k^2}{4k^2+(2\sigma_1+\sigma)^2}}
\end{aligned}$$

The solution for $x < -1$ being $\psi(k,x) = e^{ikx} - \dfrac{-i\sigma_1}{2k}e^{-ikx}$ and the solution for $x > 1$ being $\psi(k,x) = \left(1 - i\dfrac{2\sigma_1+\sigma}{2k}\right)e^{ikx}$. Consider the conditions on which negative eigenvalues exist. If the operator $H\psi = -\psi'' + (\sigma_1\delta_0(x+1) + \sigma_1\delta_0(x-1) - \sigma\delta_0(x))\psi$ is positive definite, i.e. $(H\psi, \psi) > 0$, then there are no negative eigenvalues. This means that if $2\sigma_1 > \sigma$, then there are no negative eigenvalues despite a negative component to the potential. This is an interesting case physically. In this case, we have a screening effect on the quantum particle. This results in the absence of a localized state. And even though there is a negative component in $V(x)$, we have a scattering state of the quantum particle. On the other hand, if

$$\int_{\mathbb{R}} (\psi')^2 \leq \sigma - 2\sigma_1$$

and $|\sigma| > 1$, then we have at least one negative eigenvalue and hence, a localized state.

1.2 Borderline examples

In this section, we will present the series of example which will illustrate the transition from the empty negative spectrum to the infinite one. In all examples, the δ-like potentials are negative, tends to zero near $+\infty$ and

$$\int_0^\infty |V(x)|dx = \infty. \tag{50}$$

The examples illustrate the point that (at least for the sparse potentials) the classical Bargmann's estimate is too rough. We'll use mainly the Dirichlet boundary conditions and the study of the funtion $\psi_0(x)$: the solution of equation

$$-\psi''(x) - V(x)\psi = 0,$$

$$V(x) = \sum_{i=1}^{\infty} \sigma_i \delta(x - x_i), \ \sigma_i \geq 0,$$

$$\psi_0(0) = 0, \ \psi_0'(0) = 1.$$

The number of positive zeros is exactly equal to $\mathcal{N}_0(V) = \#\{\lambda_i < 0\}$.

Solution $\psi_0(x)$ is a piesewise continuous linear function and the gluing condition for each bump at the point x_i has the following form

$$-\psi'(x_i + \epsilon) + \psi'(x_i - \epsilon) = \sigma_i \psi(x_i).$$

Let's denote $\psi_0'(x_i + 0) = k_i$ (the slope of the finction $\psi_0(x)$ on the interval (x_i, x_{i+1})).

In all "solvable" examples below, we have the following conditions:

1. $\psi(x_j) = c_j$, $x_n = q^n, \ n \geq 1 (x_0 = 0), \ q > 1,$
2. $x_n - x_{n-1} = q^{n-1}(q - 1),$
3. $k_n = \frac{c_{n+1} - c_n}{q^n(q-1)} = \psi_0'(x_n + 0).$

The gluing condition gives

$$k_{n-1} - k_n = \psi_0'(x_n - 0) - \psi_0'(x_n + 0) = \sigma_n \psi_0(x_n) = \sigma_n c_n$$

For our potential, the Bargmann bound is

$$N_0 \leq \int_0^{\infty} x|V(x)|dx = \sum_{n=1}^{\infty} x_n \sigma_n = \sum_{n=1}^{\infty} q^n \sigma_n. \tag{51}$$

We will show that this bound is "too rough". We calculate the slope as

$$\frac{c_n - c_{n-1}}{x_n - x_{n-1}} = \alpha_n.$$

Using

$$-\psi'(x_i + \epsilon) + \psi'(x_i - \epsilon) = \sigma,$$

we derive the second order difference equation

$$(q+1)c_n - c_{n+1} - qc_{n-1} = q^n(q-1)\sigma_n c_n. \tag{52}$$

Put $\sigma_n = \dfrac{\alpha_n}{q^n}$. This gives

$$c_{n+1} - (q + 1 - \alpha_n(q-1))c_n + qc_{n-1} = 0,\ c_0 = 0,\ c_1 = 1. \tag{53}$$

First, let $\alpha_n \equiv \beta > 0$. Then we have the equation

$$c_{n+1} - (q + 1 - \beta)c_n - qc_{n-1} = 0. \tag{54}$$

Suppose that $c_n = \rho^n$, then we will have the characteristic equation

$$\rho^2 - (q + 1 - \beta(q-1))\rho + q = 0, \tag{55}$$

which has the solutions

$$\rho_{1,2} = \frac{(q + 1 - \beta(q-1)) \pm \sqrt{(q + 1 - \beta(q-1))^2 - 4q}}{2}. \tag{56}$$

Put $\Delta = (q-1)^2(1+\beta^2) - 2\beta(q+1)(q-1)$. The general solution of equation (55) if $\Delta \neq 0$ equals

$$c_n = \frac{\rho_1^n - \rho_2^n}{\rho_1 - \rho_2},\ n \geq 0 \tag{57}$$

Lets study the discriminate Δ as a function of β for fixed $q > 0$. The equation $\Delta(\beta) = 0$. The equation $1 + \beta^2 + 2\beta\frac{q+1}{q-1} = 0$ has two roots:

$$\beta_{1,2} = \frac{q+1}{q-1} \pm \sqrt{\left(\frac{q+1}{q-1}\right)^2 - 1}$$
$$= \frac{q+1}{q-1} \pm \frac{2\sqrt{q}}{q-1}.$$

That is,

$$\beta_1 = \frac{(1+\sqrt{q})^2}{q-1} = \frac{\sqrt{q}+1}{\sqrt{q}-1} > 1,$$
$$\beta_2 = \frac{1}{\beta_1} = \frac{\sqrt{q}-1}{\sqrt{q}+1} < 1.$$

If $\beta > \beta_1$ or $\beta < \beta_2$, then the roots $\rho_{1,2}$ are real and have the same sign. Note that

$$\rho_1 + \rho_2 = q + 1 - \beta(q-1) > 0,$$

for $\beta < \frac{q+1}{q-1}$ and $\rho_1 + \rho_2 < 0$ for $\beta > \frac{q+1}{q-1}$. However,

$$\frac{q+1}{q-1} = \frac{1}{2}(\beta_1 + \beta_2) = \beta_0.$$

As a result, for small $\beta < \beta_2 = \frac{\sqrt{q}-1}{\sqrt{q}+1}$, the roots $\lambda_{1,2}$ are positive. For $\beta > \beta_1 = \frac{\sqrt{q}+1}{\sqrt{q}-1}$ the roots are negative. Finally, for $\beta_2 < \beta < \beta_1$, the roots are complex and conjugated. Lets consider all possibilities:

1. If $\rho_1 > \rho_2 > 0$ (that is $\beta < \frac{\sqrt{q}-1}{\sqrt{q}+1}$) then

$$\rho_1 = \frac{(q+1-\beta(q-1)) + \sqrt{(q+1-\beta(q-1))^2 - 4q}}{2},$$
$$\rho_2 = \frac{q}{\rho_1}. \tag{58}$$

At the same time $\rho_1\rho_2 = q \Rightarrow \rho_1^2 = q,\ \rho_1 = \sqrt{q}$. The solution

$$c_n = \frac{\rho_1^n - \rho_2^n}{\rho_1 - \rho_2} > 0$$

for all n. This gives $\mathcal{N}_0(V) = 0$ since there are no zeros of the solution (see figure 2). Lets call $\beta_2 = \frac{\sqrt{q}-1}{\sqrt{q}+1} = \beta_{\text{cr}}^{(1)}$. If $\beta = \beta_{\text{cr}}^{(1)}$, then $\rho_1 = \rho_2 = \sqrt{q}$ and the

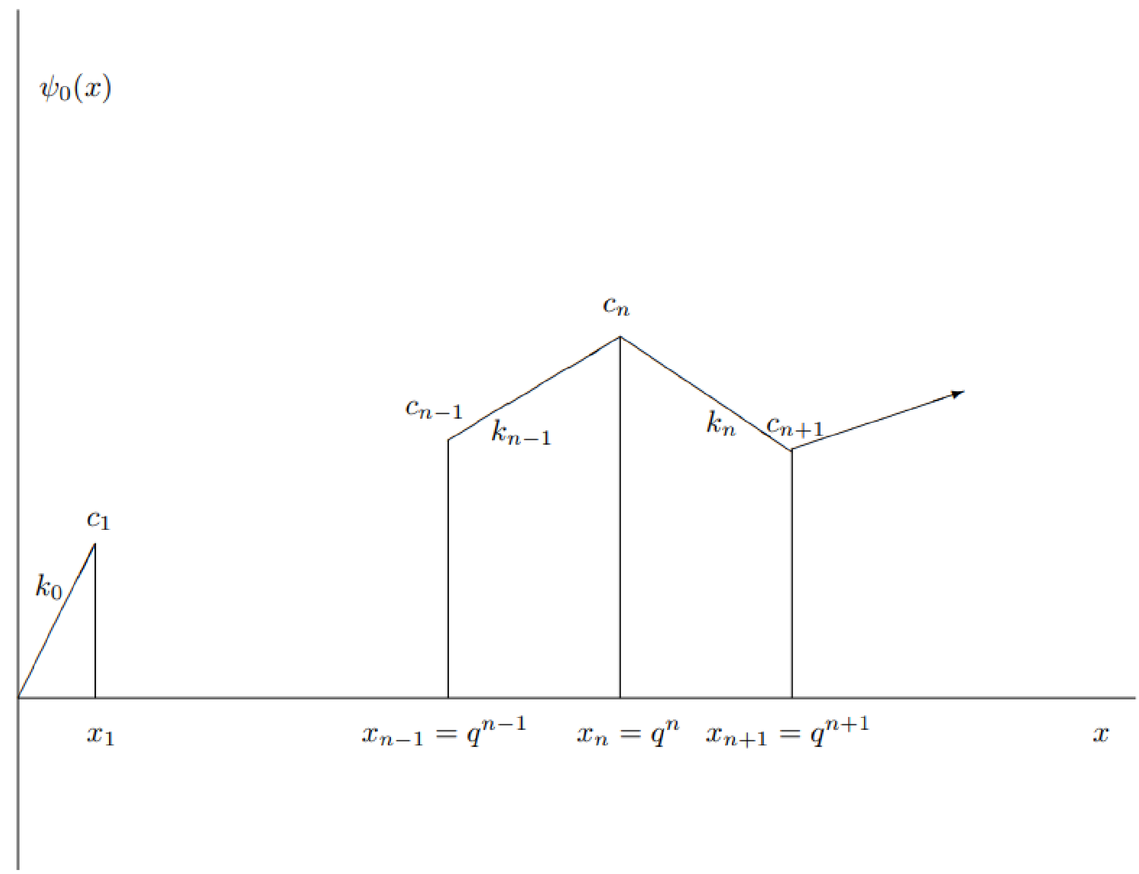

Figure 2: The graph of $\psi_0(x)$ where $\mathcal{N}_0(V) = 0$.

general solution of (55) has the form

$$c_n = a_1\rho_1^n + a_2 n\rho_1^n. \tag{59}$$

The particular solution for $c_0 = 0,\ c_1 = 1$ equals

$$c_n = n(\sqrt{q})^{n-1}. \tag{60}$$

Again, in this case, $\mathcal{N}_0(V) = 0$.

2. If $\beta > \frac{\sqrt{q}+1}{\sqrt{q}-1} = \beta_{\rm cr}^{(2)}$ then for $\rho_1,\ \rho_2 < 0$,

$$c_n = \frac{\rho_1^n - \rho_2^n}{\rho_1 - \rho_2},\ |\rho_1| > |\rho_2| \tag{61}$$

This sequence has alternating signs in any interval $[q^n, q^{n+1}]$. If

$$\beta = \frac{\sqrt{q}+1}{\sqrt{q}-1}$$

then $c(n) = n(-\sqrt{q})^{n-1}$ and we have for $c(n)$ alernating signs (figure 3).

3. If $\beta_{\rm cr}^{(1)} < \beta < \beta_{\rm cr}^{(2)}$ then

$$c_n = q^{\frac{n-1}{2}} \frac{\sin n\phi}{\sin\phi},\ \phi \in (0, \pi) \tag{62}$$

Assume that $\frac{\phi}{2\pi}$ is a irrational number. Then the sequence $\theta_n = n\phi \mod \pi$ is uniformly distributed on $[0, 2\pi)$, a classical result due to Wyel. The sequence $\dfrac{\sin\theta_n}{\sin\phi}$ changes sign if $\pi - \phi < \theta_n < \pi$ or $2\pi - \phi < \theta_n < 2\pi$, i.e. fraction of n such that $\sin n\phi \sin{(n+1)\phi} < 0$ equals $\frac{\phi}{2\pi}$. If $\beta < \beta_{\rm cr}^{(1)}$ and it is close to $\beta_{\rm cr}^{(1)}$, then this fraction is small (the situation is close to the absence of negative eigenvalues). If, on the other hand, $\beta > \beta_{\rm cr}^{(2)}$ and β is close to $\beta_{\rm cr}^{(2)}$ then the function $\psi_0(x)$ changes sign for the majority of the intervals (q^n, q^{n+1}) (see figure 4).

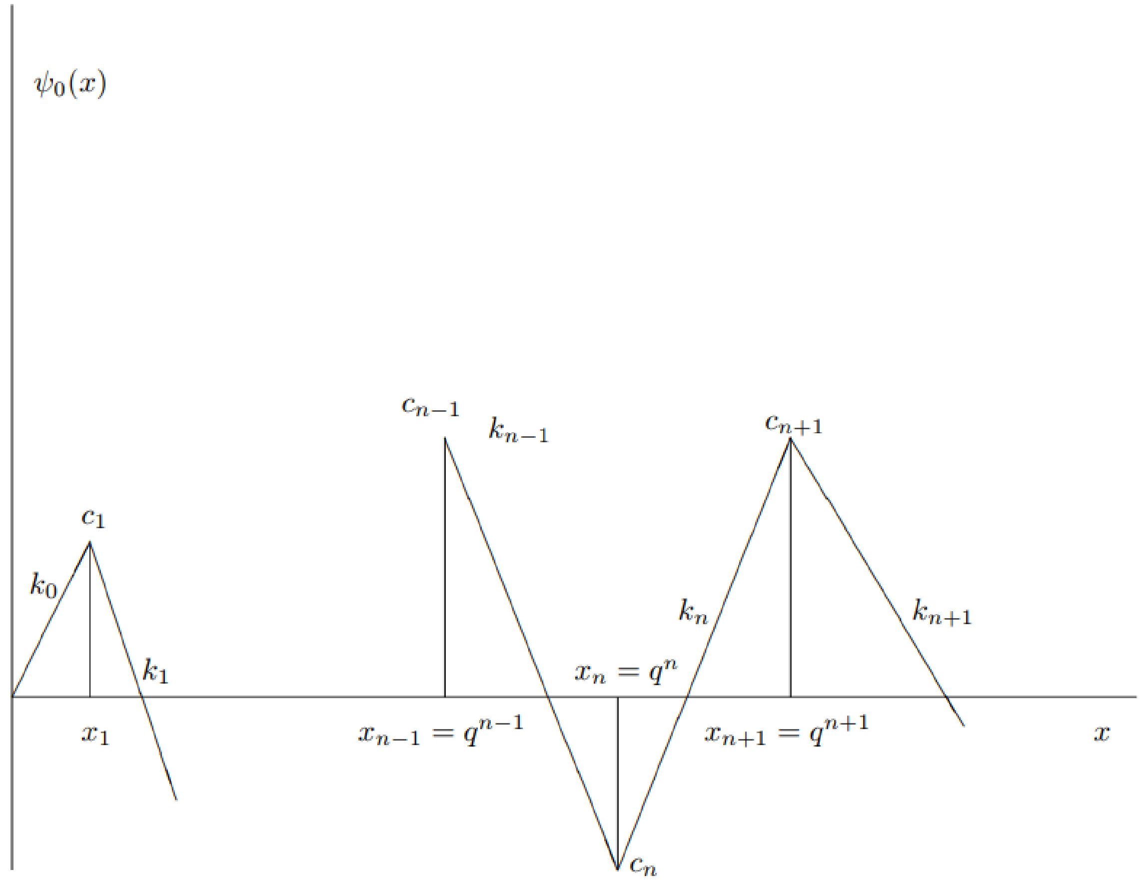

Figure 3: Graph of $\psi_0(x)$ when $\mathcal{N}_0(V) = \infty$.

However, the Bargmann estimate in each of these cases gives

$$\sum_{n=1}^{\infty} \frac{\beta q^n}{q^n} = \infty. \tag{63}$$

In the third case, we note that the number of sign changes of c_n is highly dependent on the argument ϕ. Using the variational principle, we can formulate the following corollaries

Proposition 3. 1. Let $V(x) = \sum_{n=1}^{\infty} \frac{\alpha_n}{q^n} \delta(x - q^n) +$ (Dirichlet boundary conditions) and $\alpha_n < \beta_{\text{cr}}^{(1)}$. Then $\mathcal{N}_0(V) = 0$.

2. If $\alpha_n > \beta_{\text{cr}}^{(1)} + \epsilon$ for $n \geq n_0(\epsilon),\ \epsilon > 0$ then $\mathcal{N}_0(V) = \infty$.

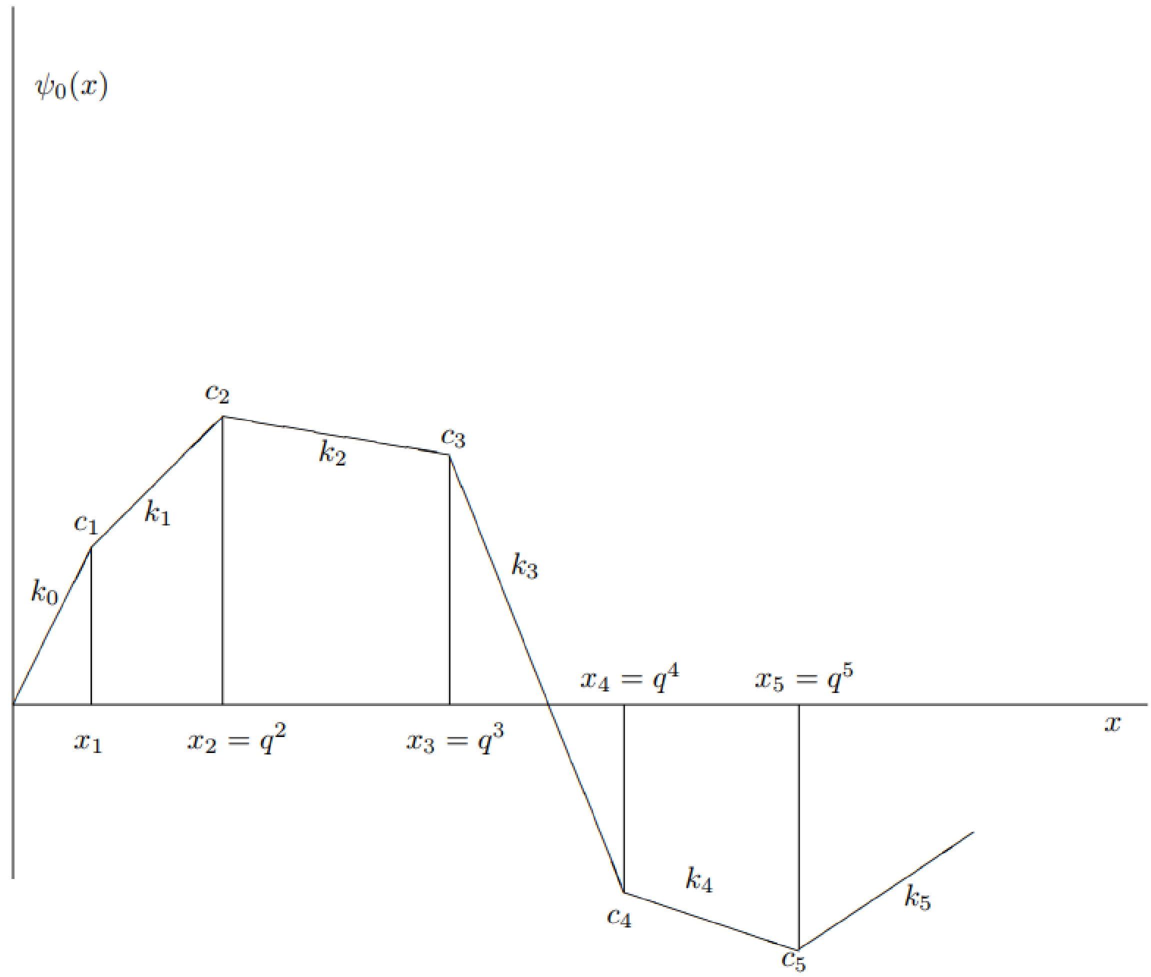

Figure 4: The case where $\beta_{\text{cr}}^{(1)} < \beta < \beta_{\text{cr}}^{(2)}$.

As mentioned above, if ϕ is small, then n would have to be large for there to be a sign change. From the characteristic equation of the second order difference equation, our first root would be $\rho_1 = r^{n-1}e^{i\phi}$, with $r = 2\sqrt{q}$ and

$$\tan\phi = \frac{\sqrt{4q - (q+1-\beta)^2}}{q+1-\beta}.$$

When $\sin(n\pi\phi) = 0$, then $n\phi = k$, with $n,\ k \in \mathbb{Z}$. So this requires

$$n = \frac{k}{\tan^{-1}\left(\frac{\sqrt{4q-(q+1-\beta)^2}}{q+1-\beta}\right)}.$$

If n is large, then $\tan^{-1}\left(\frac{\sqrt{4q-(q+1-\beta)^2}}{q+1-\beta}\right)$ is small, i.e. when the expression

$$\frac{\sqrt{4q-(q+1-\beta)^2}}{q+1-\beta}$$

is small. We formulated the lemma

Lemma 2. Let the solution of the characteristic equation 55 be

$$\rho_{1,2} = \frac{q+1-\beta \pm i\sqrt{4q-(q+1-\beta)^2}}{2}$$

then the general solution is $c_n = r^{n-1}\dfrac{\sin n\pi\phi}{\sin\pi\phi}$. The number of negative eigenvalues is the integer value of

$$k = \left[n\tan^{-1}\left(\frac{\sqrt{4q-(q+1-\beta)^2}}{q+1-\beta}\right)\right] \tag{64}$$

1.3 General results for the spectral theory for sparse potentials

The multiscattering can be presented in the following naive form: the incident sinusoidal wave (starting from $x = 0$) interacts with the first bump and is split into two parts. The first one is reflected from the bump and the second one is transmitted through the bump. The energy of the wave is distributed between these parts proportionally to $|r(k)|^2$ and $|t(k)|^2$ (if we neglect the interaction with the other bumps which are far away). The transmitted wave interacts with the second bump and so on. It is clear (at least on the level of physical intuition) that the quantum particle will propagate to infinity if the reflection coefficient $r_n(k)$ of the distant n^{th} bump are small. Such propagation is the manifestation of the absolute continuous spectral measure. These sentences can be transformed into mathematical theorems. The next three results are from [2].

Theorem 4. Consider the operator 1 on $L^2(\mathbb{R}_+)$ with the boundary conditions

$$\psi(0)\sin\theta_0 - \psi'(0)\cos\theta_0 = 0,\ \ \theta_0 \in [0,\pi).$$

Assume that the bumps $\varphi_n(\cdot)$ satisfies the condition

$$|\varphi_n(z)| \leq \frac{c}{1+|z|^\alpha},\ \ \alpha > 2,\ \ z \in \mathbb{R},$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{x_{n+1}} < \infty$. Consider the interval Δ on the energy k-axis $k = \sqrt{\lambda} > 0$ and let $\displaystyle\sum_{n=1}^{\infty} |b_n(k)|^2 = \infty$ where $b_n(k)$ is the Jost reflection coefficient for the individual bump $\varphi_n(\cdot)$ (i.e. the reflection coefficient for the scattering problem

$$H_n\psi = -\psi'' + \varphi_n\psi = \lambda\psi,\ \ \lambda = k^2 > 0 \text{ on } L^2(\mathbb{R}).$$

Then the spectral measure $\mu(d\lambda)$ is pure singular continuous.

Theorem 5. If under the same conditions, we have instead $\displaystyle\sum_n |b_n(k)|^2 < \infty$ a.e. then the spectral measure is pure absolutely continuous on Δ for a.e. $\theta_0 \in [0,\pi)$.

Theorem 6. Assume that for $n \geq 1$, the operator H_n with the potential $\varphi_n(\cdot)$ has nonempty set of negative eigenvalues $\lambda_{n,i}$. Then for a.e. $\theta_0 \in [0, \pi)$ operator 1 has infinite set of negative eigenvalues and the corresponding eigenfunctions are exponentially decreasing.

Remark 1. The important assumption that the a.c. of the spectral measure $\mu(d\lambda)$ on $[0, \infty)$ theorem 5 or exponential localization (theorem 6) has the following meaning: the singular continuous spectrum can appear in theorem 5 for some θ_0 from the set of zero measure. Similarly the pure point (p.p.) spectrum from theorem 6 can disappear on the same set of the set of zero measure (in both cases we consider the Lebesgue measure on $[0, \pi)$) (see [2] for details).

At the physical level of intuition one can expect that in the case of the strong enough reflection (theorem 4) that the spectrum of H must be p.p. (i.e. localization). It is not clear how to prove such a result in the deterministic situation (compared with the Fourier analysis case where there were no results behind the lucunar series case up to resolution of the L^2-conjecture).

But we have the possibility to consider the random sparse potentials. Here we expect to find new phase transitions: from the singular continuous measure to p.p. measure or the continuation of the p.p. and singular continuous components of the spectral measure.

It is the main topic of this dissertation. Lets describe our models. Again the central object to this study is the same: operator 1 on the half-axis with the sparse potentials $V(x) = \sum_n \varphi_n(x - x_n)$ where $k \in \mathbb{R}$ (or $\mathbb{R}_+$). However the sequence of the central points of the bumps $\{x_n,\ n \geq 1\}$ now is the random one. Typical assumptions is that $x_{n+1} - x_n = l_n \mathcal{X}_n,\ n \geq 0, x_0 = 0$ where $l_n \to \infty$ and $\mathcal{X}_n$ are i.i.d. r.v. with the piece-wise continuous compactly supported density $p(x)$ such that $\mathrm{Supp}(p(\cdot)) \subset [a,b],\ 0 < a < b < \infty$. For instance, $\{\mathcal{X}_n,\ n \geq 1\}$ are uniformly distributed on $[a,b]$. For $\{L_n,\ n \geq 1\}$, we consider two possibilities.

1. $L_n = e^{\tau n}$, $\tau > 0$. This is the most interesting case. We'll prove several particular situations that instead of pure singular continuous spectral measure, one has for $\lambda \in [0, \infty)$ the coexistence of the p.p. spectral measure (for small $k = \sqrt{\lambda} > 0$) and singular continuous measure (for large k).

2. There could be the situation where $L_n = e^{\tau n^{\tau_1}},\ 0 < \tau_1 < 1$. In this case, for $\lambda = k^2 > 0$, the spectral measure is p.p. This is known as fractional exponential growth.

3. For $\lambda = -k^2 < 0$, like in the deterministic situation, the spectral measure is p.p. for a.e. $\theta_0 \in [0, \pi)$; however, this is a different situation from part 1. For $\lambda > 0$, the eigenfunctions of the p.p. spectrum are decreasing much slower than exponentially (only power decay in the first case). At the same time, for negative λ, the eigenfunctions are decreasing exponentially fast.

Lets now describe the several classes of elementary bumps (potentials) that we will study.

1. The δ-like potentials (compare to examples 4, 5, and 6)

$$V(x) = \sum_{k=1}^{\infty} \mathcal{Y}_k \delta(x - x_k),\ x_k = \mathcal{X}_1 + \cdots + \mathcal{X}_k.$$

Here, the set $\{\mathcal{Y}_k,\ k \geq 1\}$ are i.i.d. r.v. (with positive and negative values, since we want the existence of the non-trivial spectrum for $\lambda < 0$), and the set $\{\mathcal{X}_k,\ k \geq 1\}$ was defined earlier.

2. $V(x) = \sum_{n=1}^{\infty} \varphi(n - x_n),\ x_n = \mathcal{X}_1 + \cdots + \mathcal{X}_n$ and the $\varphi_k(z)$ are 1-soliton reflectionless potentials, say,

$$\varphi_k(z) = \frac{-2\kappa_k^2}{\cosh^2 \kappa_k z}.$$

3. Instead of δ-potentials in 1), we consider compactly supported sums, say

$$V(x) = \sum_{k=1}^{\infty} \mathcal{Y}_k I_{[x_k, x_{k+1}]}(x).$$

The above classes of potentials will be of interest to us in the next chapter where random sparse potentials will be studied. In which case, we will consider the different types of exponential growth of the bumps and how the spectrum of the operator is affected.

CHAPTER 2: RANDOM SPARSE POTENTIALS

2.1 The monodromy operator $\mathfrak{M}_k$

In this chapter, we study the following spectral problem:

$$H\psi = -\psi''(x) + V(x)\psi(x) = \lambda\psi(x), \tag{65}$$

$$V(x) = -\sum_{n=1}^{\infty} \mathcal{X}_n \delta(x - x_n), \tag{66}$$

$$x_{n+1} - x_n = \xi_n L_n, \tag{67}$$

where each $\mathcal{X}_n$ are constant i.i.d. r.v. with the boundary conditions

$$\psi(0)\cos\theta_0 - \psi'(0)\sin\theta_0 = 0,\ \ \theta_0 \in [0, \pi). \tag{68}$$

To investigate the spectral theory for random Schrödinger operators, it is necessary to calculate the Prüffer form of the so-called monodromy operator $\mathfrak{M}_k$ for $\lambda = k^2 > 0$. The monodromy operator is also known as the fundamental matrix. It is a unimodular 2×2 matrix function where, if $x_2 > x_1$, we have

$$-\frac{\partial^2 \mathfrak{M}_k}{\partial x_2^2} + V(x_2)\mathfrak{M}_k = \lambda \mathfrak{M}_k,\ \text{where } \mathfrak{M}_k(x_1, x_1) = I_2. \tag{69}$$

The monodromy operator will be of the form

$$\mathfrak{M}_k(0, x_n) = \begin{bmatrix} \psi_1(x_n) & \frac{\psi_1'(x_n)}{k} \\ \psi_2(x_n) & \frac{\psi_2'(x_n)}{k} \end{bmatrix}. \tag{70}$$

Note that for k-space, we have the property $\mathfrak{M}_k(x_1, x_2)\mathfrak{M}_k(x_2, x_3) = \mathfrak{M}_k(x_1, x_3)$ and $\mathfrak{M}_k(x_1, x_1) = I_2$. More specifically,

$$\begin{pmatrix} \psi'(x) \\ \psi(x) \end{pmatrix} = \mathfrak{M}_k(0, x) \begin{pmatrix} \psi'(0) \\ \psi(0) \end{pmatrix} \tag{71}$$

An important aspect of the calculations is the use of Iwasawa decompostion $\mathfrak{M} = \mathcal{O}T$, where $\mathcal{O}$ is an orthongoal matrix and T is an upper triangular matrix. The next lemma is a fundamental result.

Lemma 3. If ξ_n are independently identically uniformly distributed random variable with continuous distribution density $p(\xi),\ \xi \in [a, b],\ 0 < a < b < \infty$ and $x_{n+1} - x_n = L_n\xi_n$ then the matrix

$$\mathcal{O}_n = \begin{bmatrix} \cos kL_n\xi_n & -\sin kL_n\xi_n \\ \sin kL_n\xi_n & \cos kL_n\xi_n \end{bmatrix} = \begin{bmatrix} \cos\varphi_n & -\sin\varphi_n \\ \sin\varphi_n & \cos\varphi_n \end{bmatrix} \tag{72}$$

have the densities of $\pi_n(\varphi)$ which tends to $\dfrac{1}{2\pi}$ if $n \to \infty$.

We put $\psi_k(x) = r_k(x)\cos t_k(x) \Rightarrow \psi'_k(x) = kr_k(x)\sin t_k(x)$. The function $r_k^{\pm}(n) = r_k(x_n \pm 0)$ is given as $r_k(x) = r_k(0)\exp\left(\dfrac{1}{2k}\displaystyle\int_0^x \sin(2t_k(z))V(z)dz\right)$. We note that due to lemma, we also have $r_k^-(n) = r_k^+(n-1)$ and

$$\psi_k(x_n^-) = r_k^+(n-1)\cos\varphi_n \tag{73}$$

$$\frac{\psi'_k(x_n^-)}{k} = r_k^+(n-1)\sin\varphi_n, \tag{74}$$

where φ_n is asymptotically uniformly distributed on $[0, 2\pi]$.

2.2 $\mathfrak{M}_k$ inside δ-potential

Here, we solve the problem (65) inside the potential. We come to the following lemma.

Lemma 4. The monodromy $\mathfrak{M}_k(x_n^-, x_n^+)$ has the form

$$\begin{bmatrix} 1 & -\dfrac{2\mathcal{X}_n}{k} \\ 0 & 1 \end{bmatrix}. \tag{75}$$

Proof. Consider the potential in the interval $(-\epsilon, \epsilon)$ and then take $\epsilon \to 0$. For this we get the following monodromy operator

$$\mathfrak{M}_k(-\epsilon, \epsilon) = \begin{bmatrix} \cosh 2\epsilon\sqrt{\frac{\mathcal{X}_n}{\epsilon} - k^2} & \frac{\sqrt{\frac{\mathcal{X}_n}{\epsilon} - k^2}\sinh 2\epsilon\sqrt{\frac{\mathcal{X}_n}{\epsilon} - k^2}}{k}, \\ \frac{k\sinh 2\epsilon\sqrt{\frac{\mathcal{X}_n}{\epsilon} - k^2}}{\sqrt{\frac{\mathcal{X}_n}{\epsilon} - k^2}} & \cosh 2\epsilon\sqrt{\frac{\mathcal{X}_n}{\epsilon} - k^2} \end{bmatrix}. \tag{76}$$

Passing the limit, we obtain the desired result. □

Remark 2. This is the upper triangular matrix from the Iwasawa decomposition mentioned above. The upper off-diagonal entry is the effect of the potential. As we will see, this changes with the sign of the potential of the single δ-potential.

2.3 The free operator $\mathfrak{M}_k$

Now we consider the portion of the interval in between two successive bumps of $V(x)$ in which the electron is free, i.e. $V(x) \equiv 0$ on the interval (x_i^+, x_{i+1}^-). So we have the equation

$$-\psi''(x) = k^2\psi(x) \tag{77}$$

$$\psi_1(0) = 1 \quad \psi_1'(0) = 0 \tag{78}$$

$$\psi_2(0) = 0 \quad \psi_2'(0) = 1 \tag{79}$$

Lemma 5. For equation 77, the monodromy operator takes the form

$$\mathfrak{M}_k(x_{n-1}^+, x_n^-) = \begin{bmatrix} \cos k(x_{n-1}^+, x_n^-) & -\sin k(x_{n-1}^+, x_n^-) \\ \sin k(x_{n-1}^+, x_n^-) & \cos k(x_{n-1}^+, x_n^-) \end{bmatrix}. \tag{80}$$

Proof. By direct calculation, we have two different boundary conditions. We have that $\psi_1(x) = \cos kx$ and $\psi_2(x) = \sin kx$. Then plugging these solutions into the monodromy operator $\mathfrak{M}_k$, we have the form as shown. □

We are interested in the case where $x_{n+1} - x_n = \xi_n L_n \to \infty$, where ξ_n are i.i.d. r.v. (perhaps uniformly distributed) and $L_n \approx \exp(\tau n^{\tau_1})$ with $\tau_1 \in [0, \infty)$.

2.4 The Lyapunov exponent

Kotani (1986) ([5]) showed the connection between the spectrum of an operator and the Lyapunov exponent, $\gamma(k)$. Here, we calculate the Lyapunov exponent of the spectral problem 65 using a theory originally due to Furstenberg. Furstenberg studied the asymptotic behavior of a product of noncommunitive random variables. In our program, these noncommunitive random variables are our monodromy operators from $\mathcal{SL}(2, \mathbb{R})$ with Iwasawa decomposition. We have the important theorem due to Furstenberg (see [1]):

Theorem 7. Let $A_1, \ldots, A_n$ be independent, identically distributed elements of the linear group $\mathcal{SL}(2, \mathbb{R})$ of the real unimodular 2×2 matrices, suppose also the $\|A_n\| < \infty$ (a.s.) and that they have common bounded distribution density with respect to the Haar measure on the group $\mathcal{SL}(2, \mathbb{R})$. Then for fixed initial vector $e_0 \in \mathbb{R}^2$, with probability 1 the quantity

$$\lim_{n \to \infty} \frac{\ln |e_0 A_1 \cdots A_n|}{n} = \gamma > 0, \tag{81}$$

where $e_0 = (\cos \varphi, \sin \varphi)$ is the form of the initial vector.

First, we give two examples to illustrate the importance of each hypothesis. For it is not difficult to show the existence of the limit, rather it is that the limit is strictly positive. Each of the next two examples is from [1].

Example 7. In this example, we weaken the independence of the matrices. Let $B_1, B_2, \ldots, B_n, \ldots$ be independent uniformly bounded elements of $\mathcal{SL}(2, \mathbb{R})$ with bounded density, with bounded support. Define

$$A_1 = B_1 B_2^{-1}, \quad A_2 = B_2 B_3^{-1}, \quad \ldots, \quad A_n = B_n B_{n+1}^{-1}.$$

The matrices $\{A_i\}$ are uniformly bounded, has good distribution density, and stationary. However, each A_i is weakly dependent. We calculate the Lyapunov exponent as

$$x_n = x_0 A_1 A_2 \ldots A_n = x_0 B_1 B_{n+1}^{-1},$$

$$\|x_n\| \leq c_1 < \infty, \qquad \frac{\ln \|x_n\|}{n} \to 0 \text{ as } n \to \infty$$

Example 8. Let $\mathcal{A} \in \mathcal{SL}(2, \mathbb{R})$ be a fixed matrix. Suppose that each A_i are independent and takes the values $\mathcal{A}$ and $\mathcal{A}^{-1}$ with probability $\frac{1}{2}$. Then,

$$x_n = x_0 A_1 A_2 \dots A_n = x_0 \mathcal{A}^{\nu_n},$$

where ν_n is the difference between the number of "success" and "failures" in n Bernoulli tests with success probability $\frac{1}{2}$. Then we calculate the Lyapunov exponent as

$$\|x_0\| \leq |x_0| \cdot \|\mathcal{A}\|^{\nu_n},$$
$$\frac{\nu_n}{n} \underset{n\to\infty}{\to} 0,$$
$$\frac{\ln \|x_n\|}{n} \underset{n\to\infty}{\to} \gamma = 0$$

This example illustrates the necessity of assuming absolute continuity of the distribution of A_i.

In theorem 7, we can consider the matrix A_1 as the propagation of the solution through the first bump, A_2 has the propagation through the second bump, and so forth to n. Hence, we can write A_1 as the product of $\mathfrak{M}_k(0, x_1^-)\mathfrak{M}_k(x_1^-, x_1^+)$. From the decomposition of Iwasawa and the orthogonality of $\mathfrak{M}_k$ between bumps, we can write A_i as the product $A_i = \mathfrak{M}_k(x_{i-1}^+, x_i^-)\mathfrak{M}_k(x_i^-, x_i^+)$, where $\mathfrak{M}_k(x_{i-1}^+, x_i^-)$ is the orthogonal matrix and $\mathfrak{M}_k(x_i^-, x_i^+)$ is upper triangular. We calculate the operator norm for $e_0 A_1 \cdots A_n$ as:

$$\begin{aligned} \ln |e_0 A_1 A_2 \cdots A_n| &= \ln |e_0| + \sum_{i=1}^{n} \ln |e_i A_i| \\ &= \ln |e_0| + \sum_{i=1}^{n} \ln |e_i \mathfrak{M}_k(x_i^-, x_i^+)| \end{aligned} \tag{82}$$

We need to estimate the norm of the monodormy operator $\mathfrak{M}_k$ within the potentials.

It can be shown, through elementary linear algebra, that for any orthogonal matrix $\mathcal{O}$ and upper triangular matrix T, that

$$\|\mathcal{O}T\|_O = \|T\|_O$$

We consider the norm, for $e_i = (\cos\varphi_i, \sin\varphi_i)$,

$$\|e_i\mathfrak{M}_k(x_i^-, x_i^+)\|^2 = \cos^2\varphi_i + \left(1 + \frac{4\mathcal{X}_i^2}{k^2}\right)\sin^2\varphi_i - \frac{4\mathcal{X}_i}{k}\cos\varphi_i\sin\varphi_i, \tag{83}$$

where $r_i(k) = \|e_i\mathfrak{M}_k(x_i^-, x_i^+)\|$ is the Prüffer transformation of equation 5, that is

$$r_k(x) = \exp\left(\frac{1}{2k}\int_0^x V(z)\sin 2t_k(z)dz\right). \tag{84}$$

We calculate the expectation of $\ln\|e_i\mathfrak{M}_k(x_i^-, x_i^+)\|$ using some basic transformation of the integral

$$\frac{1}{4\pi}\int_0^{2\pi}\ln\left(\cos^2\varphi + \left(1 + \frac{4\mathcal{X}^2}{k^2}\right)\sin^2\varphi - \frac{4\mathcal{X}}{k}\cos\varphi\sin\varphi\right)d\varphi, \tag{85}$$

i.e. the transformation

$$\int_0^{2\pi}\ln(a + b\cos x)dx = \pi\ln\frac{a + \sqrt{a^2 - b^2}}{2}.$$

Then the integral (85) has the following form after transformation,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\ln|e_i\mathfrak{M}_k(x_i^-, x_i^+)| &= \frac{1}{2}\ln\left(\frac{2 + Tr\left[\mathfrak{M}_k(x_i^-, x_i^+)\mathfrak{M}_k^T(x_i^-, x_i^+)\right]}{4}\right) \\ &= \frac{1}{2}\ln\left(1 + \frac{\mathcal{X}_n^2}{k^2}\right)\end{aligned} \tag{86}$$

This is the Lyapunov exponent $\gamma(k)$ in equation (86).

2.5 Spectral theory of random sparse potentials

We now move to the ultimate goal of this study of the Schrödinger operator. We consider the spectral problem (65). The monodromy operator outside the δ-potential wells becomes

$$\mathfrak{M}_k(x_n^-, x_n^+) = \begin{bmatrix} 1 & -\dfrac{2\mathcal{X}_n}{k} \\ 0 & 1 \end{bmatrix} \tag{87}$$

and the monodromy operator outside the δ-potential wells becomes the orthogonal matrix

$$\mathfrak{M}_k(x_{n-1}^+, x_n^-) = \begin{bmatrix} \cos kL_n & -k\sin kL_n \\ k^{-1}\sin kL_n & \cos kL_n \end{bmatrix}. \tag{88}$$

For some $x \in [x_n^+, x_{n+1}^-]$, we have the operator

$$\mathfrak{M}_k(0, x) = \mathfrak{M}_k(0, x_1^-)\mathfrak{M}_k(x_1^-, x_1^+)\cdots\mathfrak{M}_k(x_n^-, x_n^+)\mathfrak{M}_k(x_n^+, x_{n+1}^-). \tag{89}$$

This is a product of matrices with random entries in each $\mathfrak{M}_k(x_i^-, x_i^+)$ and thus, the theory of Furstenberg is needed ([10]). We let $A_i = \mathfrak{M}_k(x_{i-1}^+, x_i^-)\mathfrak{M}_k(x_i^-, x_i^+)$, and consider the random product

$$\|e_0 A_1 A_2 \cdots A_n\|, \tag{90}$$

with initial vector $\mathbf{e_0}$ describing the boundary conditions of the spectral problem. We can now calculate the coefficient $\ln r_n(k)$ as

$$\begin{aligned}\sum_{n=0}^{\infty} \ln r_n(k) &= \ln\|e_0\| + \sum_{n=1}^{\infty} \ln\|e_n \mathfrak{M}_k(x_n^-, x_n^+)\| \\ &= \sum_{n=1}^{\infty} \ln\left(\cos^2\varphi_n + \left(1 + \frac{4\mathcal{X}_n^2}{k^2}\right)\sin^2\varphi_n - \frac{2\mathcal{X}_n}{k}\sin 2\varphi_n\right)\end{aligned} \tag{91}$$

We estimate the matrix as follows: the orthogonal matrix has the estimation

$$\|\mathfrak{M}_k(x_n^+, x_{n+1}^-)\| \leq \max\{k, k^{-1}\}, \tag{92}$$

and the estimation for the upper triangular matrix is

$$\|\mathfrak{M}_k(x_n^-, x_n^+)\| \leq \exp\left[2n \ln\left(1 + \frac{\mathcal{X}_n^2}{k^2}\right)\right] \tag{93}$$

These two estimations allow us to estimate the matrix product $\mathfrak{M}_k(0, x)$ when $x \in (x_n^+, x_{n+1}^-)$. So we can write:

$$\|\mathfrak{M}_k(0, x)\|^2 \leq [\max\{k, k^{-1}\}]^{n+1} \exp\left[\sum_{l=1}^{n} 2l \ln\left(1 + \frac{\mathcal{X}_l^2}{k^2}\right)\right] \tag{94}$$

From the general theory from Molchanov and Simon, the integral of $\|\mathfrak{M}_k(0, x)\|^{-2}$ determines the spectrum of the problem 65. That is, if we consider the integral

$$\int_0^\infty \|\mathfrak{M}_k(0, x)\|^{-2} dx \tag{95}$$

and it diverges and $\lim_{x\to\infty} \|\mathfrak{M}_k(0, x)\| = \infty$, then the spectrum is singular continuous. However, if the integral converges and $\lim_{x\to\infty} \|\mathfrak{M}_k(0, x)\| = \infty$, then the spectrum is pure point. From the idea of random potentials, it may seem like the pure point spectrum is dominate. However, from [14], there are certain conditions the lend singular continuous spectrum. In the present case, if

$$L_n > \exp\left[\sum_{l=1}^{n} 2l \ln\left(1 + \frac{\mathcal{X}_l^2}{k^2}\right)\right], \tag{96}$$

then the spectrum is singular continuous. Recall that $L_n \approx e^{\tau n^{\tau_1}}$. So if the above inequality (96) holds, then $\tau n^{\tau_1} > \sum_{l=1}^{n} 2l \ln\left(1 + \frac{\mathcal{X}_l^2}{k^2}\right)$, that is, if $\gamma_1 \geq 1$ and $\tau >$

$\ln\left(1+\frac{\mathcal{X}_n^2}{k^2}\right)$. For $\mathcal{X}_n \sim U([a,b])$, then we calculate the expectation of $\mathcal{X}_n$ and relate it to γ. We obtain,

$$\begin{aligned}\tau > \mathbb{E}\left[\ln\left(1+\frac{\mathcal{X}^2}{k^2}\right)\right] &= \frac{k^2}{2(b-a)}\int_{\mathbb{R}} u\frac{e^u}{\sqrt{k^2(e^u-1)}}du \\ &= \frac{1}{b-a}\int_a^b \ln\left(1+\frac{x^2}{k^2}\right)dx\end{aligned} \tag{97}$$

since $\mathbb{E}[\tau]=\tau$. Consider the case where the bumps increase exponentially, i.e. when $\tau_1 = 1$. If $k=\sqrt{\lambda}$ is large enough, then

$$\gamma > \frac{1}{b-a}\int_a^b \ln\left(1+\frac{x^2}{k^2}\right)dx \tag{98}$$

then the integral (95) converges and the spectrum becomes pure point. It can easily be seen that if $\tau_1 \in (0,1)$, then the spectrum is pure point because $L_n < \exp\left[\sum_{l=1}^{n} 2l\ln\left(1+\frac{\mathcal{X}_l^2}{k^2}\right)\right]$ and the integral (95) converges. Thus, we have a transition from pure point spectrum to singular continuous spectrum as $\sqrt{\lambda}=k$ increases for constant τ. We again illustrate the above statements by discussing the behavior of $\gamma(k)$ with each potential mentioned at the end of chapter 1. We now formulate the following theorems and prove each by examples.

Theorem 8. If $L_n \to \infty$ superexponentially, then the spectral problem (65) has singular continuous spectrum for $\lambda = k^2 > 0$ and pure point spectrum for $\lambda = -k^2 < 0$ for a.e. $\theta_0 \in [0,\pi)$. Moreover, the eigenfunctions decay exponentially.

Theorem 9. If $L_n = e^{c\tau^{\tau_1}}$ then the spectral problem (65) has a transition from singular continuous if $\lambda > \lambda_{\text{cr}} > 0$ and pure point spectrum if $\lambda_{\text{cr}} > \lambda > 0$. The eigenfunctions have power decay. For $\lambda = -k^2 < 0$ the spectrum is pure point and eigenfunctions decay exponentially.

Theorem 10. If $L_n = e^{c\tau^{\tau_1}} \to \infty$ subexponentially $(0 < \tau_1 < 1)$, then the spectrum is pure point for all λ.

We will illustrate these theorems through an example. First, however, we give an interesting example of a reflectionless potential. Details of the calculations can be found in [2].

Example 9. Let

$$V(x) = \sum_{n=1}^{\infty} \frac{-\kappa_n^2}{\cosh^2 \kappa_n (x - x_n)}. \tag{99}$$

In this case, we have a 1-soliton which is a solution to the KdV equation $u_t + 6u_x u + x_{xxx} = 0$. The spectral problem of one bump

$$H_n\psi = -\psi'' - \frac{\kappa_n^2}{\cosh^2 \kappa_n (x - x_n)}\psi = \lambda_n \psi, \tag{100}$$

has one negative eigenvalue $\lambda_{n,1} = -\kappa_n^2$. If the set $\{\kappa_n\}$ is dense in, say the interval $[1, 2]$, then the essential spectrum would be on the set $[-2, -1] \cup [0, \infty)$. For some initial phase $\varphi \subset C^0$, it is known that the spectrum on $[-2, -1]$ would be singular continuous and for typical φ the spectrum would be pure point. For positive energies, the spectrum on $[0, \infty)$ is absolutely continuous. Furthermore, the singular continuous spectrum on this interval is completely absent. If the set $\{\kappa_n\}$ are random on the interval, say, $[1, 2]$, then the essential spectrum would be random on the corresponding interval.

Remark 3. The 1-soliton potential has reflectionless properties. Hence, the spectrum is independent on the behavior of the bump centers. From the discussion above, we gave some hint that a spectrum transition occurs depending on the behavior of the bump centers. In the next example, we will illustrate this dependence on the behavior of the set $\{x_n\}$.

Example 10. Now consider the potential

$$V(x) = \sum_{n=1}^{\infty} \mathcal{Y}_n \delta(x - x_n), \tag{101}$$

where each $\mathcal{Y}_n$ are i.i.d. r.v., i.e. uniformly distributed on a finite interval. For $\lambda = -k^2 < 0$, the spectrum will be discrete. For positive $\lambda = k^2 > 0$, we can use the monodromy operator to obtain the spectrum. For simplicity, first consider the value of the solution $\psi(x_n) = e^{-\gamma n}$ and the geometric progression of the bumps $x_n - x_{n+1} \approx c^n(c-1)$. It is well known that since the monodromy operator is a 2×2 matrix then there is a negative Lyapunov exponent. In this case, we calculate the L^2-norm and find conditions for different spectra. We obtain,

$$\sum_n e^{-2\gamma n} c^n (c-1). \tag{102}$$

The convergence of this sum gives pure point spectrum. The condition for convergence means that

$$-2\gamma n + n \ln c < 0,$$
$$\Rightarrow \gamma > \frac{\ln c}{2}.$$

Since $\gamma(k) = \ln\left(1 + \frac{\mathcal{Y}^2}{k^2}\right)$, then for small $\lambda = k^2$ the spectrum is pure point. Since $\|\mathfrak{M}_k(0,x)\| \to \infty$ as $x \to \infty$, there is no absolutely continuous spectrum. So for large values of $\lambda = k^2$, the spectrum becomes singular continuous. Thus, we have a transition of the spectrum from pure point to singular continuous for increasing values of k. It is easily seen that if $L_n \approx e^{\tau n}$, $\tau > 0$, that the condition for convergence of the series (102) becomes

$$\frac{\tau}{2} < \gamma(k) = \ln\left(1 + \frac{\mathcal{Y}^2}{k^2}\right).$$

The same conditions for the spectral transition applies. We have a transition from pure point spectrum for small values of λ to singular continuous spectrum for large values of λ.

From the previous example, we demonstrated the coexistence of spectra that depends on the value of k.

CHAPTER 3: CONCLUSION AND FUTURE WORK

3.1 Conclusion

We have demonstrated the following results, some of which are new.

1. For sparse potentials that increase with geometric progression, the Bargmann estimate is too rough. We constructed a series of potentials that give an unbounded Bargmann estimate on the number of negative eigenvalues. In reality, we have zero negative eigenvalues in some cases. That is, when there are no zeros to the homogeneous spectral problem

$$H\psi(x) = -\psi''(x) + V(x)\psi(x) = 0 \tag{103}$$

2. We considered the case where, despite having a negative potential, there would be no negative eigenvalues. Physically speaking, this is known as quantum screening. Some of the conditions on the strength and position of potentials were discussed.

3. In the case where we had a series of two negative δ-potentials, we demonstrated the conditions where we had either: 1) no negative eigenvalues, 2) only one negative eigenvalue existed, or 3) we had at most two negative eigenvalues. In each case, we discussed the conditions on the strength of the potentials and the distance between each potential and again considered the number of zeros of

the homogeneous solution to the operator H.

4. For random sparse potentials, we demonstrated, through carefully constructed examples, the coexistence of different spectra. This coexistence of different spectra is dependent on the value of k and the behavior of the sequence $\{x_n\}$ of bump centers. It was calculated that for increasing values of k, the spectrum transformed from pure point to singular continuous.

It is important to note that the spectrum of an operator is a set. This means that in the context of a continuous spectrum, it is the measure of $\mu(d\lambda)$ that is continuous and not Σ itself. General theories of the spectrum of an operator gives the energies of a quantum system. The energies that correspond to discrete spectrum correspond to bound states. More information about the general properties of the Schrödinger operator and the spectrum of the operator can be found in Landau and Lifshitz Volume 3 of *A Course in Theoretical Physics*.

3.2 Future Work

We have shown that the Bargmann estimate is too rough for sparse potentials. In the future, I would like to modify the Bargmann bound in order to obtain a useful estimate of the number of negative eigenvalues for sparse potentials.

I would also like to explore spectral theory for nonlinear Schrödinger operators. Ultimately leading to a spectral theory for nonlinear Schrödinger operators with sparse random potentials.

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