OPTIMAL STRATEGIES IN "LOCKS, BOMBS AND TESTING" (LBT) PROBLEM FOR THE CASE OF INDEPENDENT PROTECTION

by

Li Liu

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Approved by:

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Dr. Isaac Sonin

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Dr. Zhiyi Zhang

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Dr. Qingning Zhou

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Dr. I-Hsuan Ethan Chiang

ABSTRACT

LI LIU. Optimal Strategies in "Locks, Bombs and Testing" (LBT) Problem for the Case of Independent Protection. (Under the direction of DR. ISAAC SONIN)

This thesis constructs a Defense/Attack resource allocation model. Defender uses "locks" to protect their boxes from Attacker, and Attacker uses "bombs" to destroy as many boxes as possible. The first models of such type were given by E. Borel (1921). Later such models were extensively analyzed at the initial stage of Game Theory development under the general title (Colonel) Blotto game. Previous LBT model focuses on violence patterns produced by attackers with different levels of capacity to see whether rebel capacity influences how rebels fight (the attack timing). We sought to extend this problem into a situation with an extra setting where rebels can test vulnerability of boxes before placing bombs. In previous problem the goal was to find violence patterns produced by rebels. Here, we are interested in the optimal strategy of placing bombs. Further, our problem discusses the optimal strategy for defenders to allocate locks even when attackers have already applied their best strategy for placing bombs.

After posing the basic problem we then examine several specific cases with dependent and independent, identical and non-identical, locks distribution in valued boxes by using Bayes' Posterior distribution and Monte Carlo simulations.

Key words: Defense/Attack model, Blotto game, Search, Testing, game theory.

Classification

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CHAPTER 1: INTRODUCTION

1.1 Motivation and Goal

The LBT model is motivated by the paper "Rebel Capacity, Intelligence Gathering, and the Timing of Combat Operations", K. Sonin, J. Wilson, A. Wright.(SWW)[1]. Classic counterinsurgency claims rebel forces execute attacks in an unpredictable manner to limit the government's ability to anticipate and defend against them. SWW focuses on the question whether rebel capacity influences how rebels fight (the attack timing). With the help of data on opium production and farmgate prices from Afghanistan, SWW find high capacity rebels produce patterns of violence that are less random and exhibit temporal clustering.

The LBT model inherits this background setting and adds a new feature where rebels being able to test the vulnerability of the government and take action after receiving signals from test. Let's place above background into the following situation. Suppose two parties are in confrontation.

Defenders (Government): Defenders use locks to protect n boxes(sites, cities). Due to limited sources, they can only protect some of these boxes with locks. The probability that a lock can stop explosion of bombs is 1.

Attackers (Rebel): Attackers have m bombs, and the probability of explosion for a single bomb is p ($p \leq 1$). They test n boxes and receive a signal from each box that help determine the existence of locks. The signal can be positive or negative. If the signal is positive, it indicates that a lock probably exists; otherwise it does not exist. Attackers need to decide where to place m bombs, particularly, how many of them should be in the same box.

Remark 1: Attackers can and will test *every* box trying to find boxes without locks. But testing of each box is not perfect: A test can give plus for a box without a lock and minus for a box with a lock. Thus we introduce probability of true positive (sensitivity a) and true negative (specificity b)

Remark 2: The defenders can decide how to distribute the locks. For the case of k locks allocated to n boxes, there is a dependency model $A(n, k)$. The case of locks placed into n boxes independently with a certain probability is model $B(n, \lambda)$. This paper is mainly focusing on the $B(n, \lambda)$ model .

Attackers have the following main goal:

Functional F1: to maximize the expected number of destroyed boxes.

We discuss two models in this paper.

The first is the Symmetric LBT (**S-LBT**) model, where allocation of locks and testing has a strictly symmetrical structure.

The second is the General LBT (**G-LBT**) model. Where some of the various statements about this model remain true when testing is symmetrical but the prior distribution of locks for Defenders can be different from a uniform distribution and there are different kinds of boxes with possibly different values of benefits and costs for Defender and/or Attacker. This is a natural assumption when the importance, the value of different boxes for Defender/Attacker can vary. This immediately transforms the symmetric model into a full-fledged game with equilibrium points defined by randomized strategies, etc. The simplest example of such a game in $A(n,k)$ is a problem where the values of three boxes are $(2,1,1)$ and then, having one lock, Defender will distribute it at random with probabilities $(1-2\alpha,\alpha,\alpha)$. In response, Attacker, having for example, one bomb, will use probabilities $(1-2\beta,\beta,\beta)$ to plant a bomb. The unique Nash equilibrium point in this and the more general model can be found in an explicit form.

Remark 3 Game LBT Model is difficult and not solved completely. There is a completely solved case - Symmetric LBT (S-LBT), which consists of two parts: $A(n,k)$ ([SonSon][12]) and $B(n,\lambda)$ in this paper. For General LBT (G-LBT), when model parameters are increasing, the model becomes rather difficult, here we just discuss it under some special settings.

1.2 Symbols and outline

We consider random variables T_i, S_i, C_i, $i=1,2,...,n$ taking two values 0 and 1;

$$T_i = \begin{cases} 1 & \text{when the } i^{th} \text{ box contains a lock} \\ 0 & \text{when the } i^{th} \text{ box contains no lock} \end{cases}$$

$$S_i = \begin{cases} 1(\text{or } +) & \text{when the } i^{th} \text{ box is tested as positive, indicating lock is in present} \\ 0(\text{or } -) & \text{when the } i^{th} \text{ box is tested as negative, indicating lock is not in present} \end{cases}$$

$$C_i = \begin{cases} 1 & \text{when the } i^{th} \text{ box is destroyed} \\ 0 & \text{when the } i^{th} \text{ box is not destroyed} \end{cases}$$

n: Number of boxes

m: Number of bombs

x: Number of boxes with a minus signal

t: Number of boxes containing a lock with minus signal p: Probability of explosion

$a = P(S = 1|T = 1)$: Sensitivity

$b = P(S = 0|T = 0)$: Specificity

Sometimes, the complement of an event D is denoted as D'.

Dissertation Outline

In this dissertation, Chapter 2-3 consider optimal strategy of Attackers under different settings of LBT model.

In Chapter 4, in the general setting, the Nash Equilibrium point is discussed.

CHAPTER 2: INDEPENDENT IDENTICAL LOCKS ALLOCATION UNDER SYMMERTIC LBT MODEL

2.1 Parameter notation and model building

1 $B(n, \lambda)$ model

Under Symmetric LBT (**S-LBT**) model setting, where allocation of locks and testing has strictly symmetrical structure, we will discuss posterior distribution of locks given signal, and optimal strategy of attackers.

Define signal vector $s = (s_1, s_2, ...s_n)$, with s_i either be $-$ or $+$. And r.v. N is number of boxes with minus signals among all n boxes.

The symmetry in S-LBT model implies two useful formulas:

$$P(s_1, s_2, ..., s_n) = P(N = x)/\binom{n}{x} \tag{1}$$

$$P(T_i = 0|s_1, s_2, ..., s_n) = P(T_i = 0|s_i, N = x) \tag{2}$$

$B(n, \lambda)$ model assumes that the chance that a randomly selected box containing a lock is the same (λ). Thus locks are identically and independently distributed in any boxes. To be noticed, when number of locks (k) is fixed, we would have $A(n, k)$ model. We will compare results under these two models.

2 Probability Space

We have probability space $\{(\gamma,\, s)\}$, where γ is a vector of distribution of locks. In $B(n, \lambda)$ model, number of Locks K is a random variable. Suppose $K = k$ and there are

n boxes in total, then locks' position vector $\gamma = (i_1, i_2, ...i_k)$ with $1 \leq i_1 < i_2 < ... < i_k \leq n$, where i_k stands the k^{th} lock's position among n boxes. And $s = (s_1, s_2, ..., s_n)$ is a vector of signals. The probability of each outcome $p(\gamma, s) = b_0(\gamma)P(s|\gamma)$, where $b_0(\gamma)$ is prior distribution of locks, and $P(s|\gamma) = P(S_1 = s_1, ...S_n = s_n|\gamma)$

2.2 Lock's distribution given signal in model $B(n, \lambda)$

2.2.1 Conditional probability of signal given lock's position

Let us introduce r.v.s N_1, the number of minuses in locked boxes. N_2, the number of minuses in unlocked boxes. And $N = N_1 + N_2$ is the total number of minuses after testing. Number of Locks K is a random variable. Suppose we have $K = k$ locks in total, so probability of having k locks is $p(k) = \binom{n}{k}\lambda^k(1-\lambda)^{(n-k)}$. Then r.v.$N_1$ (number of false minus) is from binomial distribution with k trials, and probability of success $1-a$, $N_1 \sim Bin(k, 1-a)$

r.v. N_2 (number of true minus) is from binomial distribution with $(n-k)$ trials, and probability of success b. $N_2 \sim Bin(n-k, b)$.

N_1 and N_2 are independent. Thus distribution of N is $P(N = x) = g_B(x) = \sum_k p(k)g_{n,k}(x)$, where $g_{n,k}(x)$ is calculated for a fixed k.

When $K = k$, r.v. N has distribution $g_{n,k}(x) \equiv g(x)$, obtained by convolution formula. And then $g_B(x) \equiv P(N = x)$ can be calculated by the second formula below

$$\begin{aligned} g(x) \equiv g_{n,k}(x) &= \sum_j p_1(j)p_2(x-j) && (3) \\ &= \sum_t p_1(x-t)p_2(t) && (4) \\ &= \sum_{i=0}^{min(k,x)} \binom{k}{i}(1-a)^i a^{k-i}\binom{n-k}{x-i} b^{x-i}(1-b)^{n-k-x+i}. && (5) \end{aligned}$$

Thus $g_B(x) = \sum_k p(k)g_{n,k}(x)$

.

And we use notation $t = N_1(\gamma, s), x = N(s)$.

Proposition 1. For $B(n, \lambda)$ model,

(a). When r.v. $K = k$, and locks' distribution vector is $\gamma(k)$, for all signal vector s *with* $N_1(\gamma, s) = t$, *and* $N(s) = x$,, probability of signal vector s is

$$\begin{aligned} P(s|\gamma(k)) &= P(s|N_1 = t, N = x, K = k) = p(t, x|k) \\ &= (1-a)^t a^{(k-t)} b^{(x-t)} (1-b)^{(n-k-(x-t))} && (6) \end{aligned}$$

(b). When r.v. $K = k$, locks' joint distribution

$$\begin{aligned} s(t, x|k) &= P(N_1 = t, N_1 + N_2 = x|k) = P(N_1 = t, N_2 = x - t|k) \\ &= p_1(t)p_2(x-t) \\ &= \binom{k}{t}(1-a)^t a^{k-t}\binom{n-k}{x-t} b^{x-t}(1-b)^{n-k-x+t} && (7) \end{aligned}$$

(c). Unconditional locks' joint distribution for $B(n, \lambda)$ is

$$\begin{aligned} s_B(t, x) &= \sum_k s(t, x|k)p(k) \\ &= \sum_k \binom{k}{t}(1-a)^t a^{k-t}\binom{n-k}{x-t} b^{x-t}(1-b)^{n-k-x+t} p(k) && (8) \end{aligned}$$

(d). Conditional locks' distribution for $B(n, \lambda)$ is

$$\begin{aligned} s_B(t|x) &= \frac{s_B(t,x)}{g_B(x)} \\ &= \frac{\sum_k p(k) s(t,x|k)}{\sum_k p(k) g_{n,k}(x)} \\ &= \frac{\sum_k \binom{k}{t}(1-a)^t a^{k-t}\binom{n-k}{x-t} b^{x-t}(1-b)^{n-k-x+t} p(k)}{\sum_k p(k) \sum_{i=0}^{min(k,x)} \binom{k}{i}(1-a)^i a^{k-i}\binom{n-k}{x-i} b^{x-i}(1-b)^{n-k-x+i}} \end{aligned} \tag{9}$$

Example 1, $B(n, \lambda)$**.** With $\lambda = .5$, $n = 7$, $a = 7/12$, $b = 9/12$, find conditional distribution $s_B(t|x)$

```
> s_c
  0     1     2      3      4       5       6        7
0 1 0.643 0.413 0.2657 0.1708 0.10979 0.07058 0.045373
1 0 0.357 0.459 0.4428 0.3795 0.30498 0.23527 0.176452
2 0 0.000 0.128 0.2460 0.3163 0.33887 0.32676 0.294087
3 0 0.000 0.000 0.0456 0.1171 0.18826 0.24205 0.272303
4 0 0.000 0.000 0.0000 0.0163 0.05229 0.10085 0.151279
5 0 0.000 0.000 0.0000 0.0000 0.00581 0.02241 0.050426
6 0 0.000 0.000 0.0000 0.0000 0.00000 0.00208 0.009338
7 0 0.000 0.000 0.0000 0.0000 0.00000 0.00000 0.000741
> apply(s_c, 2, sum)
0 1 2 3 4 5 6 7
1 1 1 1 1 1 1 1
```

Figure 1: $s_B(t|x)$ for $\lambda = 0.5$, row is x, column is t

Example 2, $B(n, \lambda)$**.** For $\lambda = .7$, $n = 7$, $a = 7/12$, $b = 9/12$

```
> s_c
  0     1     2      3     4      5       6       7
0 1 0.435 0.190 0.0826 0.036 0.0157 0.00682 0.00297
1 0 0.565 0.492 0.3212 0.186 0.1015 0.05305 0.02695
2 0 0.000 0.319 0.4163 0.363 0.2632 0.17192 0.10482
3 0 0.000 0.000 0.1799 0.313 0.3412 0.29715 0.22646
4 0 0.000 0.000 0.0000 0.102 0.2211 0.28890 0.29356
5 0 0.000 0.000 0.0000 0.000 0.0573 0.14980 0.22832
6 0 0.000 0.000 0.0000 0.000 0.0000 0.03236 0.09866
7 0 0.000 0.000 0.0000 0.000 0.0000 0.00000 0.01827
> apply(s_c, 2, sum)
0 1 2 3 4 5 6 7
1 1 1 1 1 1 1 1
```

Figure 2: $s_B(t|x)$ for $\lambda = 0.7$, row is x, column is t

2.2.2 Posterior distribution for $B(n, \lambda)$

Given a signal vector s, and suppose number of locks $K = \kappa$ is fixed, if prior distribution $b_0(\gamma(\kappa))$ is uniform, then $b_0(\gamma(\kappa)) = p(\kappa)/\binom{n}{\kappa}$. Then what is the distribution of κ locks' position $\gamma = \gamma(\kappa)$?

In order to solve this problem, we need to introduce ADL (aposterior distribution of locks) first. For both S- and G-LBT models we describe above, our notation imply the following basic equalities:

$$P(S_i = 1|T_i = 1) = a, \quad P(S_i = 0|T_i = 0) = b, \quad P(C_i = 1|T_i = 1) = 0, \quad P(C_i = 1|T_i = 0, u_i) = p(u_i)$$

where u_i is the number of bombs in box i, and $p(u)$ is the probability of distribution of an unlocked box with u bombs. The independece of explosions implies that $p(u) = 1 - q^u, q = 1 - p$. Note that function $p(u)$ is increasing and concave upward, and $\Delta p(u) \equiv p(u+1) - p(u)$ is decreasing. This property of diminishing utility of each extra bomb plays an important role in the structure of optimal strategy.

One interest in all models is posterior probabilities $P(T_i = 0|s), s = (s_1, ..., s_n)$ and a more general *aposterior distribution of locks* (ADL) with

$$b(\gamma|s) = P(T_i = 1, i \in \gamma, T_i = 0, i \notin \gamma|S_i = s_i, i = 1, ..., n) \tag{11}$$

The following theorem (theorem 1) describes ADL (posterior distribution of locks) $b(\gamma(\kappa)|s)$ for an arbitrary and uniform $b_0(\gamma(\kappa))$. With uniform prior distribution all signals with the same values $N_1 = t, N = x$ have the same probability and as a result $b(\gamma(\kappa)|s) = b(\gamma(\kappa)|t, x)$. For all possible allocation of κ locks, the ADL $b(\gamma(\kappa)|s)$ is presented on element of an upper triangular $\binom{n}{\kappa} \times 2^n$-dimensional array $B(\gamma(\kappa)|s)$, where $\gamma(\kappa)$ takes all $\binom{n}{\kappa}$ possible values.

In this background setting, the number of locks is rv K with Binomial distribution with n trials and probability of success λ. Thus, rv K has distribution $p(k) = p(k|n,\lambda), k = 0, 1, ..., n$. When $K = k$, rv N has conditional distribution $g_{n,k}(x)$, and then $g_B(x) \equiv P(N = x)$ can be calculated by the second formula below

$$g_A(x) \equiv g_{n,k}(x) = \sum_j p_1(j)p_2(x-j) \equiv \sum_t p_1(x-t)p_2(t), \qquad g_B(x) = \sum_{k=0}^{n} p(k)g_{n,k}(x). \quad (12)$$

Summation over j in the convolution formula above is taken over values j such that $0 \leq j \leq k,\ 0 \leq x - j \leq n - k$. Similar holds for summation over t, where $0 \leq x - t \leq k,\ 0 \leq t \leq n - k$. Further, in all convolution formulas we may not specify the exact range of summation assuming that all probabilities involved in sums are well defined.

Theorem 1. ADL in case $B(n,\lambda)$.

a) *For a prior $b_0(\gamma(\kappa))$, and any position γ and signal s, according to definition of ADL (formula 11), the ADL $b(\gamma|s)$ is given by Bayes' formula*

$$b(\gamma(\kappa)|s) = \frac{b_0(\gamma(\kappa))P(s|\gamma(\kappa))}{\sum_k \sum_\sigma b_0(\sigma(k))P(s|\sigma(k))} \tag{13}$$

Where $P(s|\gamma(\kappa)) = P(s|N_1 = t, N = x, K = k) \equiv p(t,x|k)$ is given by formula (6) with $t = t(\gamma, s), x = N(s)$

b) *For the* uniform *distribution $b_0(\gamma(k)) = p(k)/\binom{n}{k}$: for any signal s holds formula* (2); *for any position γ and signal s, with $t(\gamma,s) = t$,*

$$b(\gamma(\kappa)|s) = b(\gamma(\kappa)|t,x) \equiv \frac{p(\kappa)s_B(t,x|\kappa)}{g_B(x)\binom{x}{t}\binom{n-x}{\kappa-t}}, \tag{14}$$

Proof. **of Theorem 1.** The first equality in point a) represents Bayes' formula. The equality $b(\gamma|s) = p(t,x)$ and formula (6) were proved in Introduction.

To prove b), note that

$s_B(t,x) = \sum_k \binom{k}{t}(1-a)^t a^{k-t}\binom{n-k}{x-t} b^{x-t}(1-b)^{n-k-(x-t)}p(k)$. Using the following equality,

$$\binom{n}{k}\binom{k}{t}\binom{n-k}{x-t} = \binom{n}{x}\binom{x}{t}\binom{n-x}{k-t}, \tag{15}$$

and the uniform prior $b_0(\gamma(\kappa)) = p(\kappa)/\binom{n}{\kappa}$. Hence $b(\gamma(\kappa)|s)$ takes form $b(\gamma(\kappa)|s) = \frac{p(\kappa)P(s|\gamma(\kappa))/\binom{n}{\kappa}}{\sum_k \sum_\sigma p(k)P(s|\sigma(k))/\binom{n}{k}}$. To estimate the sum in the denominator, let us prove the following equalities:

$$
\begin{aligned}
\sum_k \sum_\sigma b_0(\sigma) P(s|\sigma) &= \sum_k \frac{p(k)}{\binom{n}{k}} \sum_t \sum_{\sigma: t(\sigma,s)=t} p(t,x|k) \\
&= \sum_k \frac{p(k)}{\binom{n}{k}} \sum_t p(t,x|k) |\sigma : t(\sigma,s) = t| \\
&= \sum_k \frac{p(k)}{\binom{n}{k}} \sum_t p(t,x|k) \binom{x}{t} \binom{n-x}{k-t}
\end{aligned}
$$

By equality 15

$$
\begin{aligned}
&= \sum_k \frac{p(k)}{\binom{n}{k}} \sum_t \frac{s(t,x|k)\binom{n}{k}}{\binom{n}{x}} \\
&= \sum_k \frac{p(k)}{\binom{n}{x}} \sum_t s(t,x|k) \\
&= \sum_k \frac{p(k)}{\binom{n}{x}} g_{n,k}(x) \\
&= \frac{g_B(x)}{\binom{n}{x}}.
\end{aligned}
$$

Where $g_B(x) = \sum_k p(k) g_{n,k}(x)$

Thus

$$
\begin{aligned}
b(\gamma(\kappa)|s) &= \frac{p(\kappa) P(s|\gamma(\kappa))/\binom{n}{\kappa}}{\sum_k \sum_\sigma p(k) P(s|\sigma(k))/\binom{n}{k}} \\
&= \frac{\frac{p(\kappa)}{\binom{n}{\kappa}} \frac{s(t,x|\kappa)}{\binom{\kappa}{t}\binom{n-\kappa}{x-t}}}{\frac{g_B(x)}{\binom{n}{x}}}
\end{aligned}
$$

By equality 15

$$
\begin{aligned}
&= \frac{\frac{p(\kappa)}{\binom{n}{x}} \frac{s(t,x|\kappa)}{\binom{x}{t}\binom{n-x}{\kappa-t}}}{\frac{g_B(x)}{\binom{n}{x}}} \\
&= \frac{p(\kappa) s(t,x|\kappa)}{g_B(x)\binom{x}{t}\binom{n-x}{\kappa-t}}
\end{aligned}
$$

$\square$

Example 3, $B(n, \lambda)$**.** For $a = 7/12$, $b = 9/12$, $n = 7$, $\lambda = 0.7$, number of locks $\kappa = 2$, $p = 0.6$. Column is x (number of minus), and row is t (number of minus in locks)

```
> bb
            0            1            2            3            4            5           6           7
0 6.42109e-05 0.0001802048 5.057360e-04 1.419324e-03 0.0039832630 0.0111788350 0.000000000 0.000000000
1 0.00000e+00 0.0000429059 1.204133e-04 3.379342e-04 0.0009483960 0.0026616274 0.007469728 0.000000000
2 0.00000e+00 0.0000000000 2.866984e-05 8.046052e-05 0.0002258086 0.0006337208 0.001778507 0.004991293
3 0.00000e+00 0.0000000000 0.000000e+00 0.000000e+00 0.0000000000 0.0000000000 0.000000000 0.000000000
4 0.00000e+00 0.0000000000 0.000000e+00 0.000000e+00 0.0000000000 0.0000000000 0.000000000 0.000000000
5 0.00000e+00 0.0000000000 0.000000e+00 0.000000e+00 0.0000000000 0.0000000000 0.000000000 0.000000000
6 0.00000e+00 0.0000000000 0.000000e+00 0.000000e+00 0.0000000000 0.0000000000 0.000000000 0.000000000
7 0.00000e+00 0.0000000000 0.000000e+00 0.000000e+00 0.0000000000 0.0000000000 0.000000000 0.000000000
```

Figure 3: $b(\gamma(\kappa)|s)$

Note, when $N = x = 5$, $N_1 = t = 1$, $b(\gamma(2)|x = 5, t = 1) = \frac{p(2)s(t=1,x=5|2)}{g_B(5)\binom{5}{1}\binom{2}{1}} = 0.00266$

Given a fixed x, κ ($x = 5$,$\kappa = 2$), and suppose these 5 minus boxes are arranged in the first 5 places. Since $B(\gamma(2)|x = 5)$ is the probability of locks' position among n boxes.

i	1	2	3	4	5	6	7
$s, x = 5$	$-$	$-$	$-$	$-$	$-$	$+$	$+$
$\gamma, t = 0$						$\otimes$	$\otimes$

$$\begin{aligned} B(\gamma(2)|x = 5) &= P(L_1 = i_1, L2 = i_2|x = 5), i_1 < i_2 \\ &= b(\gamma(2)|x = 5, t = i) \\ &= \begin{cases} b(\gamma(2)|x = 5, t = 0) & i_1 = 6, i_2 = 7 \\ b(\gamma(2)|x = 5, t = 1) & i_2 \text{ is selected from box } 6, 7 \\ b(\gamma(2)|x = 5, t = 2) & \text{None of } i_1, i_2 \text{ are selected from box } 6, 7 \end{cases} \end{aligned}$$

We get table and histogram for $B(\gamma(2)|x = 5)$ (possibility that lock is in position i_1, i_2)

```
> B
  1            2            3            4            5           6           7
1 0 0.0006337208 0.0006337208 0.0006337208 0.0006337208 0.002661627 0.002661627
2 0 0.0000000000 0.0006337208 0.0006337208 0.0006337208 0.002661627 0.002661627
3 0 0.0000000000 0.0000000000 0.0006337208 0.0006337208 0.002661627 0.002661627
4 0 0.0000000000 0.0000000000 0.0000000000 0.0006337208 0.002661627 0.002661627
5 0 0.0000000000 0.0000000000 0.0000000000 0.0000000000 0.002661627 0.002661627
6 0 0.0000000000 0.0000000000 0.0000000000 0.0000000000 0.000000000 0.011178835
7 0 0.0000000000 0.0000000000 0.0000000000 0.0000000000 0.000000000 0.000000000
```

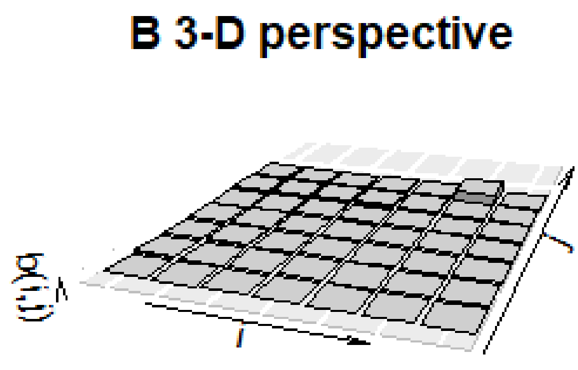


Figure 4: B table and histogram

While in case $A(n,k)$, when all settings keeps the same as Example 3, ($a = 7/12$, $b = 9/12$, $n = 7$, $k = 2$, $p = 0.6$) When $N = x = 5$, $N_1 = t = 1$, we have

```
> bb #b(i,j|x)
           0          1           2           3           4          5          6          7
0 0.04761905 0.06086957 0.080401094 0.110776187 0.161361142 0.25330270 0.00000000 0.00000000
1 0.00000000 0.01449275 0.019143118 0.026375283 0.038419319 0.06031017 0.10447761 0.00000000
2 0.00000000 0.00000000 0.004557885 0.006279829 0.009147457 0.01435956 0.02487562 0.04761905
3 0.00000000 0.00000000 0.000000000 0.000000000 0.000000000 0.00000000 0.00000000 0.00000000
4 0.00000000 0.00000000 0.000000000 0.000000000 0.000000000 0.00000000 0.00000000 0.00000000
5 0.00000000 0.00000000 0.000000000 0.000000000 0.000000000 0.00000000 0.00000000 0.00000000
6 0.00000000 0.00000000 0.000000000 0.000000000 0.000000000 0.00000000 0.00000000 0.00000000
7 0.00000000 0.00000000 0.000000000 0.000000000 0.000000000 0.00000000 0.00000000 0.00000000
```

Figure 5: $b(\gamma|s)$

Note, when $N = x = 5$, $N_1 = t = 1$, $b(\gamma|x = 5, t = 1) = P(L_1 = i_1, L_2 = i_2|x = 5, t = 1) = \frac{s(t=1|x=5)}{\binom{5}{1}\binom{2}{1}} = 0.06031017$,

Given a fixed x ($x = 5$), and suppose these 5 minus boxes are arranged in the first 5 places. Since $B(\gamma|x = 5)$ is the probability of locks' position among n boxes.

i	1	2	3	4	5	6	7
$s, x = 5$	$-$	$-$	$-$	$-$	$-$	$+$	$+$
$\gamma, t = 0$						$\otimes$	$\otimes$

$$
\begin{aligned}
B(\gamma|x=5) &= P(L_1 = i_1, L2 = i_2|x = 5), i_1 < i_2 \\
&= b(\gamma(2)|x = 5, t = i) \\
&= \begin{cases} b(\gamma|x = 5, t = 0) & i_1 = 6, i_2 = 7 \\ b(\gamma|x = 5, t = 1) & i_2 \text{ is selected from box } 6, 7 \\ b(\gamma|x = 5, t = 2) & \text{None of } i_1, i_2 \text{ are selected from box } 6, 7 \end{cases}
\end{aligned}
$$

We get table and histogram for $B(\gamma|x=5)$ (possibility that lock is in position i_1, i_2). After comparing the histogram for model $A(n,k)$ and $B(n,\lambda)$, we find that $A(n,k)$ model has a more obvious difference for those two locks' position between pair $(1,2),(1,3),(1,4),(1,5)$ and $(6,7)$

> B

	1	2	3	4	5	6	7
1	0	0.01435956	0.01435956	0.01435956	0.01435956	0.06031017	0.06031017
2	0	0.00000000	0.01435956	0.01435956	0.01435956	0.06031017	0.06031017
3	0	0.00000000	0.00000000	0.01435956	0.01435956	0.06031017	0.06031017
4	0	0.00000000	0.00000000	0.00000000	0.01435956	0.06031017	0.06031017
5	0	0.00000000	0.00000000	0.00000000	0.00000000	0.06031017	0.06031017
6	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.25330270
7	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

Figure 6: when $k=2$, likelihood of lock's position in 7 boxes

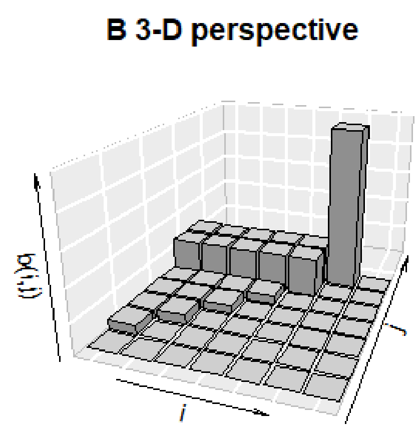


Figure 7: when $k=2$, histogram of lock's position

2.3 Optimal strategy for attackers

Now we know the posterior distribution of locks given information of signal. The next thing to consider is to realize Goal F1: How to maximize expected number of destroyed boxes?

2.3.1 Ratio for signal in $B(n, \lambda)$

Since signal test is not perfect (sensitivity and specificity are less than 1), we introduce ratio $r(\lambda)$ here, to obtain efficient information of signals by comparing probability of no lock in this position given negative signal with positive signal.

Proposition 2. a) The ratio $r \equiv r_B(\lambda)$ is given by formula

$$r_B(\lambda) = \frac{P(T=0|S=0)}{P(T=0|S=1)} \equiv \frac{p^-(\lambda)}{p^+(\lambda)} = \frac{b}{(1-b)} \frac{\lambda a + (1-\lambda)(1-b)}{\lambda(1-a) + (1-\lambda)b}, \tag{16}$$

b) the probabilities used in (16) are given by formulas

$$p^-(\lambda) = \frac{(1-\lambda)b}{\lambda(1-a) + (1-\lambda)b}, \qquad p^+(\lambda) = \frac{(1-\lambda)(1-b)}{\lambda a + (1-\lambda)(1-b)}, \tag{17}$$

c) if $a + b > 1$, then function $r_B(\lambda)$ is increasing from 1 to $\frac{a}{1-a}\frac{b}{1-b} = c_1 c_2 = c > 1$, when λ is increasing from 0 to 1, otherwise, it is decreasing from 1 to $c < 1$.

Here $c_1 = \frac{a}{1-a}$ and $c_2 = \frac{b}{1-b}$ represent the quality of sensitivity and specificity, and $c = c_1 c_2$ represents the combined quality of testing.

Remark 1. Note that parameters a and b in function $r_B(\lambda)$ are not symmetrical, i.e., though $r(.5|a,b)r(.5|b,a) = c$ and $r_B(\lambda|a,b) \approx r_B(\lambda|b,a) \approx c$ for λ close to 1, generally $r_B(\lambda|a,b) \neq r_B(\lambda|b,a)$ for all $\lambda < 1$. This asymmetry property is in contrast to the symmetry of a and b for $r_A(x)$ in Problem $A(n,k)$. see of Proposition 2(b).

The visual plot of $\{a, b : r_A(\lambda|a, b) = constant\}$ is given in (Figure 9).

Example 1, $B(n, \lambda)$**.** For $\lambda = .5$ with $a = \frac{7}{12}, b = \frac{9}{12}$ by formula (30) we obtain $r = \frac{15}{7} \approx 2.143$, and with $a = \frac{9}{12}, b = \frac{7}{12}$, we obtain $r = \frac{49}{25} = 1.96$, and hence $r(.5|a, b)r(.5|b, a) = \frac{21}{5} = 4.2 = c$.

For $\lambda = .7$ with $a = \frac{7}{12}, b = \frac{9}{12}$, we obtain $r = \frac{87}{31} \approx 2.806$, and with $a = \frac{9}{12}, b = \frac{7}{12}$, we obtain $r = \frac{13}{5} = 2.6$, and $r(.7|a, b)r(.7|b, a) \approx 7.297$.

Example 2, $B(n, \lambda)$**.** Let $a = \frac{7}{12}, b = \frac{9}{12}$, probability of explosion $p = 0.6$ by formula (30), we obtain graph of ratio r w.r.t. λ as follows:

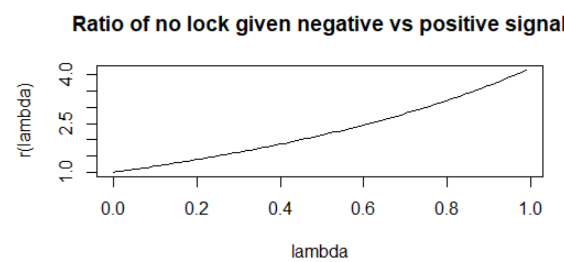


Figure 8: ratio for a=7/12,b=9/12, p=0.6

Example 3, $B(n, \lambda)$**.** find a and b for a fixed value r, such as $\{(a, b) : r(a, b) = c\}$

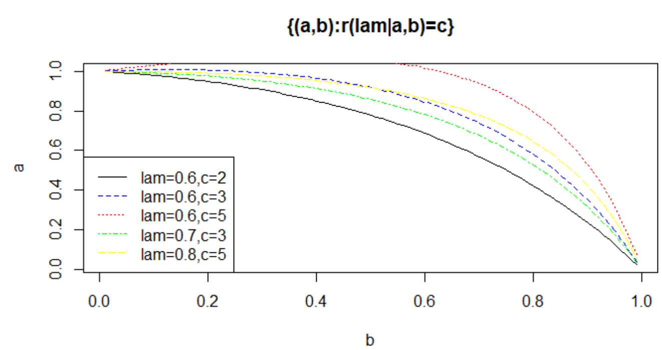


Figure 9: $r = c$

For $A(n,k)$, we have the following conclusion that the symmetry of Defense strategy implies that k locks are allocated at random between n boxes. Let us show that the probability that a particular box has a lock is $\lambda = k/n$. The number of combinations of k locks having one lock on a fixed position and the other $k-1$ locks having any of remaining $n-1$ positions is $\binom{n-1}{k-1}$. Then $\lambda = \binom{n-1}{k-1}/\binom{n}{k}$. The first of the two trivial equalities for binomial coefficients below with $m = n$ yields $\lambda = k/n$. The second equality in (18) will be used later.

$$\binom{m-1}{k-1}\Big/\binom{m}{k} = \frac{k}{m}; \qquad \binom{m}{k-1}\Big/\binom{m}{k} = \frac{k}{m+1-k}. \tag{18}$$

For the case A we obtain two different representations for $r_{n,k}(x)$ using total probability formula for *different* partitions.

Theorem 2. a) The crucial ratio $r_{n,k}(x|a,b) \equiv r_{n,k}(x),\ 0 < x < n$, is given by formula

$$r_{n,k}(x) \equiv \frac{P(T=0|S=0,x)}{P(T=0|S=1,x)} \equiv \frac{p^-(x)}{p^+(x)} = \frac{b}{(1-b)}\,\frac{(n-x)}{x}\,\frac{g_{n-1,k}(x-1)}{g_{n-1,k}(x)}, \tag{19}$$

b) the probabilities used in (19) $p^-(x) \equiv P(T=0|S=0,x)$ and $p^+(x) \equiv P(T=0|S=1,x))$ for $0 < x < n$ are given by formulas

$$p^-(x) = \frac{n-k}{x} * b * \frac{g_{n-1,k}(x-1)}{g_{n,k}(x)}, \quad p^+(x) = \frac{n-k}{n-x} * (1-b) * \frac{g_{n-1,k}(x)}{g_{n,k}(x)}. \tag{20}$$

c) functions $r_{n,k}(x) \equiv r_{n,k}(x|a,b)$ as functions of parameters a, b for all $n, k, 0 < x < n$ depend only on parameter $c = \frac{a}{1-a}\frac{b}{1-b}$, (see Remark 1), and hence satisfy the equality $r_{n,k}(x|a,b) = r_{n,k}(x|b,a) = r_{n,k}(x|\theta,\theta)$, where $\theta = \frac{\sqrt{c}}{1+\sqrt{c}}$.

d) functions $r_{n,n-1}(x) = c$ for all x, functions $r_{n,k}(x)$ for $k < n-1$ are monotonically increasing in x for $0 < x < n$, and $r(x) > 1$ when $a + b > 1$, and < 1 when $a + b \leq 1$.

e) functions $r_{n,k}(x)$ are monotonically decreasing for all fixed $k, 0 < x < n$ when n is increasing.

Proof. **of Theorem 2(c). Proof that $r(x)$ depends on a, b through c,** First, we can represent $g_{n,k}(x)$ as

$$g_{n,k}(x) = \sum_i \binom{k}{i} a^{k-i} \binom{n-k}{x-i} b^{x-i}(1-b)^{n-k-x+i} = a^k b^x (1-b)^{n-k-x} \sum_i \binom{k}{i}\binom{n-k}{x-i}\frac{1}{c^i},(21)$$

where $d_1(x) = max(0, x-n+k) \leq i \leq min(k, x) = d_2(x)$. Then we can represent $r_{n,k}(x)$ as

$$r_{n,k}(x) = \frac{n-x}{x}\frac{\sum_{d_1(x-1)\leq i\leq d_2(x-1)} \binom{k}{i}\binom{n-k-1}{x-i-1}c^{-i}}{\sum_{d_1(x)\leq i\leq d_2(x)} \binom{k}{i}\binom{n-k-1}{x-i}c^{-i}}.$$

Therefore $r(x$ depends only on c. We also have $\binom{n-k-1}{x-i-1} = \binom{n-k-1}{x-i}\frac{n-(x-i)}{x}$. Using these equalities and formula (22), we can show that $r_{n,k}(x$ grows in c as a function of c.

□

To compare with Example 1 in page 18, we can look into $B(n = 10, k = 5)$ model, with the same value $a = 7/12$, $b = 9/12$, and number of minus box $x = 0, 1, 2, ...10$. $r(x)$ is given by formula 19

	1	2	3	4	5	6	7	8	9	10
x	1.000000	2.000000	3.0000	4.000000	5.000000	6.000000	7.000000	8.000	9.000000	10
r(x)	1.733945	1.832666	1.9515	2.090505	2.243105	2.394689	2.536165	2.664	2.777778	0

Figure 10: $r(x)$ for $A(10, 5)$

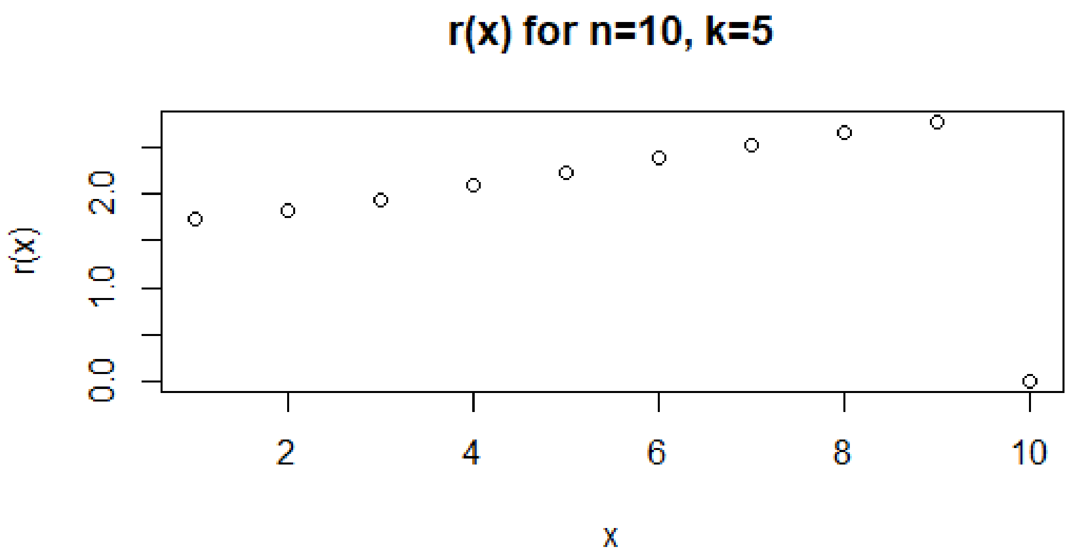


Figure 11: graph of $r(x)$ for $A(10, 5)$

2.3.2 Threshold for number of bombs

Again, for S–LBT models the problem description above and our notation imply the following basic equalities:

$$P(S_i = 1|T_i = 1) = a, \quad P(S_i = 0|T_i = 0) = b,$$

$$P(C_i = 1|T_i = 1) = 0, \quad P(C_i = 1|T_i = 0, u_i) = p(u_i), \tag{22}$$

The independence of explosions implies that $p(u) = 1 - q^u, q = 1 - p$. Note that function $p(u)$ is increasing and concave upward, and $\Delta p(u) \equiv p(u+1) - p(u)$ is decreasing. This property of diminishing utility of each extra bomb plays an important role in the structure of optimal strategy.

We consider strategy $\pi = (u_1, ..., u_n|s)$ as an allocation of m bombs between boxes, given signal $s = (s_1, ..., s_n)$, $\sum_{j=1}^{n} u_j = m$, and we introduced $U^-(\pi|s) = \{u_j, j \in B^-(s)\}$ and $U^+(\pi|s) = \{u_j \in B^+(s)\}$ as two possible sets of the values of u_j in minus $B^-(s)$ and plus $B^+(s)$ boxes. By symmetry of prior distribution of locks and testing, all strategies with the same pair of sets $U^-(\pi|s), U^+(\pi|s)$ can be obtained using permutations of these sets among corresponding boxes, and they all have the same value, denoted as $w^\pi(x, m)$ for a problem $N(s) = x$. We denoted also $v(x, m) = \sup_\pi v^\pi(x, m)$, the value function over all strategies, given m and x, and $v(m)$, the value function over all .

Let J be a subset of boxes and $C(J)$ an event that *all* boxes in J are destroyed, C_j box j is destroyed. Then, given strategy π, we have $w^\pi(m|x) = \sum_{i=1}^{n} P(C_i|u_i, s_i, x)$. The conditional independece of testing and explosions, formula (22), and total probability formula imply the following formula for the conditional probability of the destruction of a particular box with u bombs, and for any event F generated by testing (signals), $P(C|u,F) = P(C|u,T=0)P(T=0|F) = p(u)P(T=0|F),\ u \geq 1$. Using this formula and the definitions of $r(\lambda)$, $p^-(\lambda)$ and $p^+(\lambda)$, we have:

$$
\begin{aligned}
P(C|u,S=1,\lambda) &= P(T=0|S=1,\lambda)P(C|u,T=0) = p^+(\lambda)p(u), \quad p(u) = 1-q^u, \\
P(C|u,S=0,\lambda) &= P(T=0|S=0,\lambda)P(C|u,T=0) = p^-(\lambda)p(u) = r(\lambda)p^+(\lambda)p(u).
\end{aligned}
\tag{23}
$$

The next Proposition justifies our claim that the optimal strategy in all problems are (separately) UAP in minus and plus boxes.

Theorem 3. *If $0 < \lambda < 1$, then the optimal strategy is to distribute all bombs between minus and plus boxes $d(\lambda)$-UAP, where $d(\lambda)$ is defined by formula*

$$d(\lambda) = min(i \geq 1 : r(\lambda)q^i < 1), \tag{24}$$

Theorem 4. *Let $\pi(\lambda) = (u_l, l = 1,2,...,n)$ be an optimal strategy. Then $|u_s - u_t| \leq 1$ when the signals in boxes s,t have the same sign.*

Theorem 5. *Let $\pi(\lambda) = (u_l, l = 1,2,...,n)$ be a strategy, $0 < \lambda < 1$, $u^- = i$ be the number of bombs in some minus box, $u^+ = j$ be the number of bombs in some plus box, and $d = d(\lambda)$ is defined by formula (24). Then, if $i - j > d$ or, if $j \geq 1$ and $i - j < d - 1$, then strategy π is not optimal, or, equivalently, if π is optimal, and $j = 0$, then $1 \leq i \leq d$, and if $j \geq 1$, then $i - j = d$ or $d - 1$.*

Proof. **of Theorem 3.** Suppose that theorem 3 is not true and let us say $u_s = i, u_t = j, i - j \geq 2$ and $S_s = S_t = 1$. The concavity of function $p(\cdot)$ implies that $p(i+1) + p(j-1) > p(i) + p(j)$. Then, using the formulas in (23), we have

$$\begin{aligned} P(C=1|i+1, S=1, \lambda) + P(C|j-1, S=1, \lambda) &= p^+(\lambda)[p(i+1) + p(j-1)] \\ &> p^+(\lambda)[p(i) + p(j)] \\ &= P(C|i, S=1, \lambda) + P(C|j, S=1, \lambda). \end{aligned}$$

Thus our initial strategy is not optimal. The proof for $S_s = S_t = 0$ is similar with $p^+(\lambda)$ replaced by $p^-(\lambda) = r(\lambda)p^+(\lambda)$.

□

Proof. **of Theorem 4.** Suppose that Proposition 4 is not true and let us say $u_s = i, u_t = j, i - j \geq 2$ and $S_s = S_t = 1$. The concavity of function $p(\cdot)$ implies that $p(i+1) + p(j-1) > p(i) + p(j)$. Then, using the formulas in (23), we have

$$\begin{aligned} P(C=1|i+1, S=1, x) + P(C|j-1, S=1, x) &= p^+(x)[p(i+1) + p(j-1))] > \\ > p^+(x)[p(i) + p(j)] &= P(C|i, S=1, x) + P(C|j, S=1, x). \end{aligned} \tag{25}$$

Thus our initial strategy is not optimal. The proof for $S_s = S_t = 0$ is similar with $p^+(x)$ replaced by $p^-(\lambda) = r(\lambda)p^+(\lambda)$.

□

Proof. **of Theorem 5.** *Proof.* Let $d(\lambda) = d$. As always, we assume that $a + b > 1$ and then $r(\lambda) > 1$ for $0 < \lambda < 1$, and hence $u^- \geq u^+$. Let us denote $P(\cdot|\lambda) = P(\cdot|\lambda)$, and denote the incremental utilities for minus and plus boxes as $\Delta C^-(i|\lambda) = P(C|i+1, S = 0, \lambda) - P(C|i, S = 0, \lambda)$, $\Delta C^+(j|\lambda) = P(C|j + 1, S = 1, \lambda) - P(C|j, S = 1, \lambda)$. Using formulas in (23), it is easy to check that $\Delta C^+(j|\lambda) = pp^+(\lambda)q^j$ and $\Delta C^-(i|\lambda) = pr(\lambda)p^+(\lambda)q^i$, and then we have

$$\Delta(i - 1, j) = \Delta C^-(i - 1|\lambda) - \Delta C^+(j|\lambda) = pq^j p^+(\lambda)[r(\lambda)q^{i-j-1} - 1].$$

By definition of $d(x)$, we have $r(\lambda)q^{d(\lambda)-1} \geq 1$ and $r(\lambda)q^{d(\lambda)} < 1$. Then if $i - j > d$, then formula (26) implies that $\Delta(i - 1, j) < 0$, i.e., a transfer of one bomb from a minus box from this pair to a plus box will increase the value of a strategy. Similarly, if $j \geq 1$ and $i - j < d - 1$ for such pair, then using formula for $\Delta(i + 1, j - 1)$ similar to formula (26), we can show that the inverse transfer will increase the value.

Note also that if $r(\lambda)q^{d(\lambda)-1} = 1$, then $d(\lambda)$-UAP strategy remains optimal but not anymore unique since then in formula (26) gives zero for $i - j = d(\lambda)$. Note also, that if $p = 1$, i.e. $q = 0$, then $d(\lambda) = 1$ for all $0 < \lambda < 1$, and if p is decreasing to zero, then $d(\lambda)$ tends to infinity. $\square$

Example 1: $B(n, \lambda)$, with $a = 7/12$, $b = 9/12$, $n = 8$, number of bombs $m = 50$, $p = 0.6$

From the Figure 12, we can find that when $\lambda = 0.5$, $r(0.5) = 2.143$ and $d = 2$.

```
> data
           0     1     2     3     4     5     6     7      8      9
lambda     0 0.100 0.200 0.300 0.400 0.500 0.600 0.700 0.8000 0.9000
r(lambda)  1 1.186 1.390 1.615 1.865 2.143 2.455 2.806 3.2069 3.6667
p_minus    1 0.942 0.878 0.808 0.730 0.643 0.545 0.435 0.3103 0.1667
p_plus     1 0.794 0.632 0.500 0.391 0.300 0.222 0.155 0.0968 0.0455
d          1 1.000 1.000 1.000 1.000 1.000 2.000 2.000 2.0000 1.0000
```

Figure 12: $d(\lambda)$ with 50 bombs and prob of explosion is 0.6

Thus when $\lambda = 0.5$, if there are totally 50 bombs, attacker will place them into 8 boxes according the following strategy to maximize damage.

i	1	2	3	4	5	6	7	8
$s, x = 5$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
bombs	7	7	7	7	7	5	5	5

In model $A(n,k)$, with $a = 7/12$, $b = 9/12$, $n = 8$, $k = 4$ number of bombs $m = 50$, $p = 0.6$

From the Figure 13, we can find that when $x = 5$, $r(5) = 2.5066767$ and $d = 2$.

```
> data
                1         2         3         4         5         6         7   8
x       1.0000000 2.0000000 3.0000000 4.0000000 5.0000000 6.0000000 7.0000000 8.0
r(x)    1.7710843 1.9132821 2.0944363 2.3066439 2.5066767 2.6808081 2.8285714 0.0
p_minus 0.8076923 0.7788204 0.7424901 0.6975786 0.6454941 0.5929401 0.5439560 0.5
p_plus  0.4560440 0.4070599 0.3545059 0.3024214 0.2575099 0.2211796 0.1923077 NaN
d       1.0000000 1.0000000 1.0000000 1.0000000 2.0000000 2.0000000 2.0000000 1.0
```

Figure 13: threshold for $A(8,4)$

Thus when $x = 5$, if there are totally 50 bombs, attacker will place them into 8 boxes according the following strategy to maximize damage.

i	1	2	3	4	5	6	7	8
$s, x = 5$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
bombs	7	7	7	7	7	5	5	5

2.3.3 Value function

Theorem 6. *Let, given signal s, the total number of minuses $N = x, 0 \leq x \leq n$. Then*

a) *if $x = 0$ or n, then the optimal strategy is to distribute all bombs between boxes UAP and the value function $v(m|0) = v(m|n)$ for $m = n*i+e,\ i = 0, 1, ...,\ 0 \leq e < n$, is given by formula*

$$v(m|0) \quad = \quad v(m|n) = \sum_{k=0}^{n} \frac{n-k}{n}[ep(i+1) + (n-e)p(i)]P(k); \tag{26}$$

b) *If $0 < \lambda < 1$, then the optimal strategy is to distribute all bombs between minus and plus boxes $d(\lambda)$-UAP, where $d(\lambda)$ is defined by formula*

$$d(\lambda) = min(i \geq 1 : r(\lambda)q^i < 1), \tag{27}$$

$q = 1 - p$ and $r(\lambda) = r_A(\lambda)$ is defined by formula (16).

The value function $v(x, m)$ for $m = m^- + m^+ = i * x + e + j * (n - x) + e'$, where the tuple (i, e, j, e') is (uniquely) defined by value x and $d(x)$-UAP strategy, is given by formula

$$v(x, m) = p^+(\lambda)[r(\lambda)(ep(i+1) + (x-e)p(i)) + (e'p(j+1) + (n-x-e')p(j))]. \tag{28}$$

c) *The value function $v(m), m = 1, 2, ...$ is given by formula*

$$\begin{aligned} v(m) \quad &= \quad \sum_{x=0}^{n} P(N = x)v(x, m) \\ &= \quad \sum_{x=0}^{n} v(x, m)g_A(x). \end{aligned} \tag{29}$$

Example 2:$B(n, \lambda)$, $a = 7/12$, $b = 9/12$, $n = 8$, $\lambda = 0.5$, number of bombs $m = 50$, $p = 0.6$

Resulting the following table with first row to be the value of x (number of locks in minus boxes), value function is $\sum_{x=0}^{8} vg_x = 3.27$, where $vg_x = v(x, m)g_A(x)$

```
> data
```

	v0	1	2	3	4	5	6	7	8
x	0.00e+00	1.0000	2.0000	3.000	4.000	5.000	6.000	7.0000	8.0000
i	0.00e+00	7.0000	7.0000	6.000	6.000	6.000	6.000	6.0000	6.0000
e_minus	0.00e+00	0.0000	0.0000	2.000	2.000	2.000	2.000	2.0000	2.0000
m_minus	0.00e+00	7.0000	14.0000	20.000	26.000	32.000	38.000	44.0000	50.0000
j	6.00e+00	6.0000	6.0000	6.000	6.000	6.000	6.000	6.0000	0.0000
e_plus	2.00e+00	1.0000	0.0000	0.000	0.000	0.000	0.000	0.0000	0.0000
m_plus	5.00e+01	43.0000	36.0000	30.000	24.000	18.000	12.000	6.0000	0.0000
v_m	3.99e+00	2.0943	2.4170	2.739	3.061	3.383	3.705	4.0265	3.9861
g_B	9.08e-04	0.0102	0.0499	0.140	0.244	0.274	0.192	0.0766	0.0134
vg	3.62e-03	0.0213	0.1205	0.382	0.748	0.926	0.710	0.3085	0.0534

Figure 14: $B(8, 0.5)$ with 50 bombs and prob of explosion $p = 0.6$

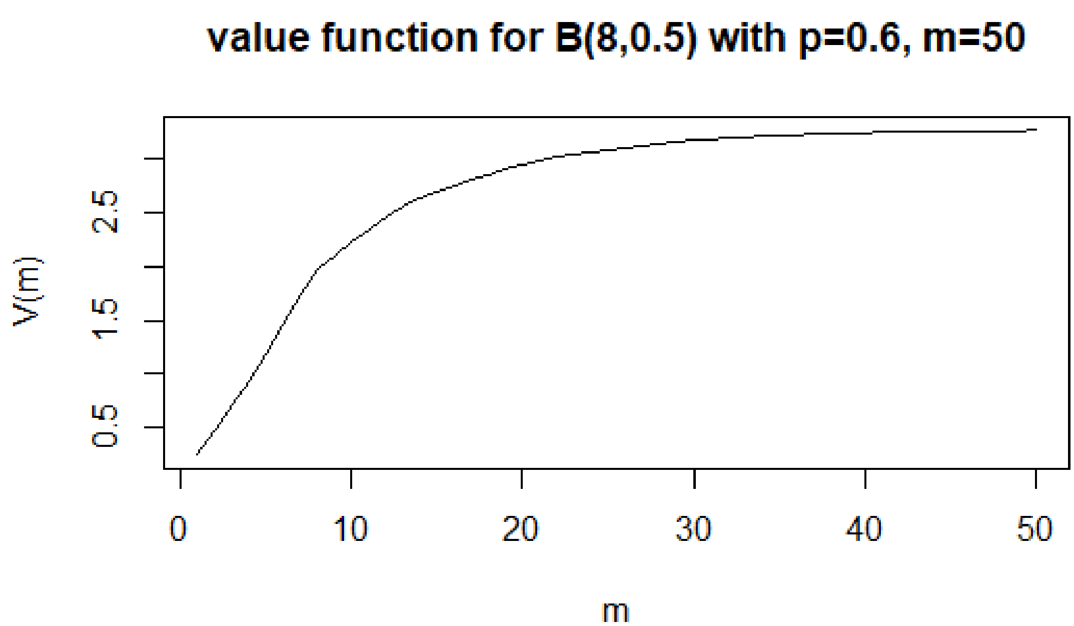


Figure 15: $B(8, 0.5)$ value function w.r.t number of bombs when prob of explosion $p = 0.6$

For $A(n,k)$ with $n = 8$, $k = 4$ Resulting the following table with first row to be the value of x (number of locks in minus boxes), value function is $\sum vg = 3.99$

> data

	v0	1	2	3	4	5	6	7	8
x	0.00e+00	1.00000	2.0000	3.000	4.000	5.000	6.000	7.0000	8.00000
r(x)	NaN	1.77108	1.9133	2.094	2.307	2.507	2.681	2.8286	0.00000
p_minus	NaN	0.80769	0.7788	0.742	0.698	0.645	0.593	0.5440	0.50000
p_plus	5.00e-01	0.45604	0.4071	0.355	0.302	0.258	0.221	0.1923	NaN
d	1.00e+00	1.00000	1.0000	1.000	1.000	2.000	2.000	2.0000	1.00000
i	0.00e+00	7.00000	7.0000	6.000	6.000	7.000	6.000	6.0000	6.00000
e_minus	0.00e+00	0.00000	0.0000	2.000	2.000	0.000	4.000	3.0000	2.00000
m_minus	0.00e+00	7.00000	14.0000	20.000	26.000	35.000	40.000	45.0000	50.00000
j	6.00e+00	6.00000	6.0000	6.000	6.000	5.000	5.000	5.0000	0.00000
e_plus	2.00e+00	1.00000	0.0000	0.000	0.000	0.000	0.000	0.0000	0.00000
m_plus	5.00e+01	43.00000	36.0000	30.000	24.000	15.000	10.000	5.0000	0.00000
v_m	3.99e+00	3.98672	3.9874	3.987	3.987	3.987	3.987	3.9864	3.98607
g_B	4.52e-04	0.00672	0.0413	0.136	0.259	0.291	0.190	0.0661	0.00954
vg	1.80e-03	0.02679	0.1647	0.542	1.033	1.161	0.756	0.2636	0.03801

Figure 16: $A(8,4)$ with 50 bombs and prob of explosion $p = 0.6$

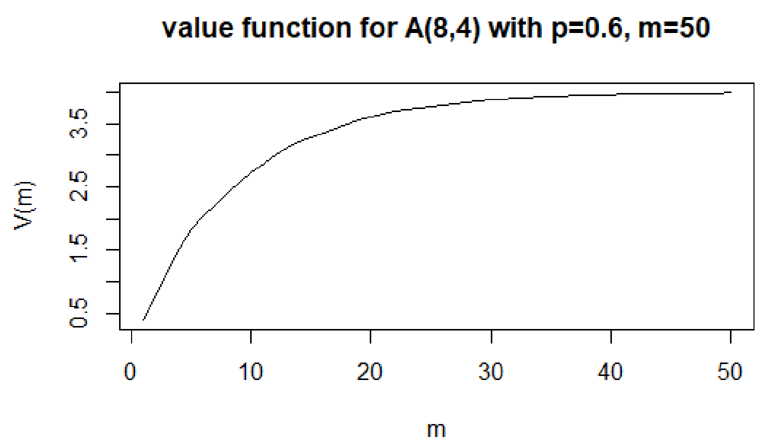


Figure 17: $A(8,4)$ value function w.r.t number of bombs when prob of explosion $p = 0.6$

CHAPTER 3: INDEPENDENT LOCKS ALLOCATION UNDER GENERAL LBT MODEL

3.1 Notations And Conditions

Under the setting of General **G-LBT** model when there are different kinds of boxes, locks and bombs, with possibly different values of benefits and costs for Defender and/or Attacker, and testing is not uniform in respect to different boxes, e.g., when Def can test only a subset of all boxes, or parameters of testing a and b depend on the box number, we can construct posterior distribution of locks and obtain optimal strategy of attackers.

G-LBT model $G_B(n, \Lambda, m)$:

Defender: There are totally n boxes and each box has a value c_i, such that $V = (c_1, c_2, c_3, ...c_n)$. Defenders allocate locks independently with non-identical probability in different boxes, i.e. $\Lambda = (\lambda_1, \lambda_2, ...\lambda_n)$, where λ_i indicates probability of containing a lock in i^{th} box, thus with restriction $\sum_i \lambda_i = k$ and $0 < \lambda_i < 1$ for $i = 1, 2, ...n$. So there are totally $\kappa = 2^n$ allocation of locks, let's redefine locks' allocation vector $\gamma = (\gamma_1, \gamma_2, ...\gamma_\kappa)$.

Attacker: there are m bombs, among which u bombs are placed into boxes with $u = (u_1, u_2, ...u_n)$, and $\sum u_i = m$, and they test each box to obtain signal vector $s = (s_1, s_2, ...s_n)$, with s_i either be $-$ or $+$

Assume boxes themselves have different values ($V = c_1, c_2, c_3, ...c_n$), i.e. $c_1 \geq c_2 \geq$

$c_3 > ... \geq c_n$, Attacker's goal is to maximize destruction (i.e. valued boxes)

Remark: Here we have two cases:

Case 1: Attackers know each λ_i, $i = 1, 2, ...n$

Case 2: Attackers only know k

Due to the complexity of this model, here we just discuss a little about Case 1 and leave Case 2 for future work

3.2 Parameter r in model $G_B(n, \Lambda, m)$

Thus instead of computing for $r(\lambda)$, we calculate for $r_i(s)$,

$$r_i(s) = v_i P(T_i = 0|s)p, \tag{30}$$

Where $P(T_i = 0|s)$ is given by Posteriori Distribution, such that $P(T_i = 1|s) = \sum_{\gamma_i : i \in \gamma} b(\gamma_i|s)$, thus $P(T_i = 0|s) = 1 - P(T_i = 1|s)$

Example 1: $G_B(n, \Lambda, m)$ Assume $n = 4$, $V = \{7, 5, 3, 1\}$, $\Lambda = (0.2, 0.1, 0.2, 0.5)$, $a = 7/12$, $b = 9/12$, probability of explosion $p = 0.8$

and let parameter k here to be 1

When $S = \{-, -, +, +\}$, $m = 1$, from table 1, Attacker should place bomb into box 1

Table 1: A summary of destruction value for $G_B(n, \Lambda, m)$.

Boxes	signal s	$r(s)$
1	$-$	4.9170732
2	$-$	2.1073171
3	$+$	0.2325581
4	$+$	0.2857143

CHAPTER 4: NASH EQUILIBRIUM POINTS

In game theory, the Nash equilibrium, named after the mathematician John Forbes Nash Jr., is a proposed solution of a non-cooperative game involving two or more players in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy. It is a concept within game theory where the optimal outcome of a game is where there is no incentive to deviate from their initial strategy. More specifically, the Nash equilibrium is a concept of game theory where the optimal outcome of a game is one where no player has an incentive to deviate from his chosen strategy after considering an opponent's choice. Overall, an individual can receive no incremental benefit from changing actions, assuming other players remain constant in their strategies. A game may have multiple Nash equilibria or none at all.

In our case, even when Attackers have already selected the optimal strategy of bombs placement for any signal received, Defenders can still minimize their potential loss by choosing optimal strategy of locks allocation

This is the case of Nash Equilibrium

4.1 Notations And Conditions

Under the setting of General **G-LBT** model when there are different kinds of boxes, locks and bombs, with possibly different values of benefits and costs for D and/or R, and testing is not uniform in respect to different boxes, e.g., when Def can test only a subset of all boxes, or parameters of testing a and b depend on the box number, we can construct posterior distribution of locks and obtain optimal strategy of attackers.

G-LBT model $G_B(n, \Lambda, m)$:

Defender: There are totally n boxes and each box has a value c_i, such that $V = (c_1, c_2, c_3, ...c_n)$. Defenders allocate locks independently with non-identical probability in different boxes, i.e. $\Lambda = (\lambda_1, \lambda_2, ...\lambda_n)$, where λ_i indicates probability of containing a lock in i^{th} box, thus with restriction $\sum_i \lambda_i = k$ and $0 < \lambda_i < 1$ for $i = 1, 2, ...n$. So there are totally $\kappa = 2^n$ allocation of locks, let's redefine locks' allocation vector $\gamma = (\gamma_1, \gamma_2, ...\gamma_\kappa)$.

Attacker: there are m bombs, among which u bombs are placed into boxes with $u = (u_1, u_2, ...u_n)$, and $\sum u_i = m$, and they test each box to obtain signal vector $s = (s_1, s_2, ...s_n)$, with s_i either be $-$ or $+$

Assume boxes themselves have different values ($V = c_1, c_2, c_3, ...c_n$), i.e. $c_1 \geq c_2 \geq c_3 \geq ... \geq c_n$, Attacker's goal is to maximize destruction (i.e. valued boxes)

4.2 Nash Equilibrium Point for $G_B(n, \Lambda, m)$ model

Parameter k can be fixed or random. Fixed when we have dependent model $G_A(n, k, m)$, and the strategy of how to allocate locks is defined by probability vector $X = (x_1, x_2, ..., x_k)$. In our paper, the parameter k; $k < n$, is random, then the strategy of the Defender is a probability distribution $b(\gamma)$ on a set of all possible positions of locks. There are totally $\kappa = 2^n$ allocation of locks, thus locks' allocation vector $\gamma = (\gamma_1, \gamma_2, ...\gamma_\kappa)$.

In our Bayesian setting we assume that this prior distribution $b(\gamma)$ is known to attacker, though of course the real positions of the locks are not. After the locks are allocated, Attacker receives signal s and, having m bombs, distributes them among n sites deterministically or using some randomization trying to maximize the expected sum of values of all destroyed boxes. WLOG, we can assume that this distribution is deterministic and an optimal strategy of attacker $\pi(m|b(\gamma))$, with respect to a strategy of defender $b(\gamma)$, is a collection of her optimal responses $u(s|m, b(\gamma)) = u(s) = (u_1(s), u_2(s), ...u_n(s))$ to each signal s, where $u_i(s)$ is the number of bombs placed into site i; $i = 1, 2, ...n$. Using prior distribution $b(\gamma)$, the probabilities of signals $p(s)$, given this distribution, the posterior distribution of the positions of locks $b(\gamma|s)$ and the total expected damage (loss), $L(b(\gamma), \pi(m|b(\gamma))$ can be calculated. The goal of Defender is to select a prior distribution of locks $b_*(\gamma)$ to minimize this loss. Then the pair $(b_*(\gamma), \pi_*)$, where π_* is an optimal response of Attacker to strategy $b_*(\gamma)$ forms a classical Nash equilibrium (NE) point. The corresponding value of the game is $v_* = L(b_*(\gamma), \pi_*)$. As we will see, though $b_*(\gamma)$

are not unique, they all have common properties that result in a unique (up to some randomization) Attacker's strategy π_*, and thus a specific value of v_*.

We denote $G_B = G_B(n, \Lambda, m|a, b, V)$ this general Bayesian game, where n is the number of sites, n dimensional vectors a and b represent the quality of testing, (the sensitivities and the specificities), and vector $V = (c_1, ..., c_n)$ describes the values of each box.

With only one available bomb, $m = 1$, Attacker will place it into the next valuable site, and if $m > 1$ she should solve the problem of discrete optimization placing the next available bomb into the site with the maximal marginal utility.

The other extreme situation is when testing is not informative, i.e., when $a_i = b_i = 1/2$; $i = 1, 2, ..., n$, and then the posterior distribution $b(\gamma|s)$ coincides with the prior distribution $b(\gamma)$ for all signals s. Given prior distribution $b(\gamma)$, let us introduce probability $\alpha_i = P(T_i = 0)$.

For model $G_B(n = 2, \Lambda, m = 1)$, assume $p = 1$ and valued boxes has value $V = (c, 1)$, $\Lambda = (\lambda, 1-\lambda)$. At first glance, it seems that if c is much larger than 1, defender should place a unique lock into the most valuable site and then her loss is 1. But simple calculations show that the optimal distribution of locks is $(\frac{c}{c+1}, \frac{1}{c+1})$ and $v = \frac{c}{c+1} < 1$. Attacker can place her unique bomb into any site or place it at random.

Similarly, for the game $G_B(n = 3, \Lambda, m = 1)$ with vector of values $V = (4, 3, 2)$, and $\Lambda = (\lambda_1, \lambda_2, 1 - \lambda_1 - \lambda_2)$. We obtain that the optimal distribution of a unique lock is given by $b(\gamma) = (7/13, 5/13, 1/13)$ and $\alpha = (6/13, 8/13, 12/13)$, $c_i * \alpha_i = v_* = 24/13$ for $i = 1, 2, 3$. But if the vector of values is $V = (4, 3, 1)$, then $b(\gamma) = (4/7, 3/7, 0)$ and $\alpha = (3/7, 4/7, 1)$, $c_i * \alpha_i = v_* = 12/7$ for $i = 1, 2, 3$

Theorem 7. **(Non-Informative Case).** For $m = 1$; $V = (c_1, c_2, c_3, ...c_n)$ with $c_1 \geq c_2 \geq c_3 \geq ... \geq c_n$,

a) the class of optimal strategies $b_*(\gamma)$ has the following structure: there is $k_* = k_*(c)$; $k < k_* < n$ and constant $v_* = v(c)$ such that: $c_i\alpha_i = v_*$ for $1 \leq i \leq k_*$, and $v_* > c_i$; $\alpha_i = 1$ for $k_* < i \leq n$;

b) the optimal strategy for AT is to place a bomb at random between the sites with numbers $1, 2, 3...k_*$;

c) the value of the game is $v_* = (k_* - k)/C_{k_*}$, where $k_* = max\{j : j \geq k, c_j > (j-k)/C_j\}$ and $C_j = \sum_{i=1}^{j} 1/c_i$

Proof. **of Theorem 7(c).** When $1 \leq i \leq k_*$, we have $c_i\alpha_i = v_*$ for $i = 1, 2, ...k_*$.

When $k_* \leq i \leq n$, we have $\alpha_i = 1$

Hence in general we have

$$\begin{aligned}
E(\sum_{i=1}^{k_*} 1_{(T_i=0)}) = \sum_{i=1}^{k_*} \alpha_i &= k_* - k \\
\sum_{i=1}^{k_*} \frac{v_*}{v_i} &= k_* - k \\
\text{Let } C_* &= \sum_{i=1}^{k_*} \frac{1}{c_i} \\
\text{Thus } v_* &= \frac{k_* - k}{C_*}
\end{aligned}$$

Hence, when $i = 1, 2, ...k*$, $\alpha_i = \frac{v_*}{c_i} < 1$

Thus, we have $\frac{k_* - k}{C_* c_i} < 1$, or in other words, $c_i > \frac{k_* - k}{C_*}$.

So $k_* = max\{j : j \geq k, c_j > (j-k)/C_j\}$ $\square$

In other words, if $k_* < n$, then the sites with numbers greater than k_* should not be protected at all and the distribution of k locks in the first k_* sites should make all sites *equally desirable for attack*. In layman terms: if the strength of a chain is defined by the strength of the weakest link, and the resources to make links strong are limited, then make all links of equal strength. This is a special case of a more general "Chain-Link Optimization Principle".

Note also that for *any vector of values* V, the number $k < k_*$, the optimal strategy $b_*(\gamma)$ is always randomized and value $v_* > c_{k_*+1}$. As a result, attacker will allocate her m bombs among the first k_* sites if $m \leq k_*$.

Of course, the main interest in the Bayesian LBT game problem is the case of imperfect but informative testing. For simplicity we will assume that this means that $a_i > 1/2$, $b_i > 1/2$, $i = 1, 2, ...n$, though in the general case this property should be described using vectors a and b. To obtain the description of NE points, we have to solve three problems.

The first problem, is, given a strategy of defender $b(\gamma)$, to describe the optimal strategy (response) of attacker $\pi(m|b(\gamma))$, i.e., to describe the optimal allocation of bombs $u(s|m)$ given signal s and m available bombs. The full answer to this problem is given by a recursive procedure S described. The expected value of damage (loss) for the pair of strategies $(b(\gamma), \pi(m|b(\gamma)))$ can also be obtained.

The second, more difficult, problem is to find the optimal strategy or strategies $b_*(\gamma)$ of defender, minimizing this loss. So far, the proof of corresponding Theorem 2 is not 100% complete but its heuristic meaning is similar to the meaning of Theorem 1: these strategies have to make the potential expected losses in the sites, that are worth protecting, equal when attacker applies her optimal response to $b_*(\gamma)$. The difficulty here lies in the fact that in informative case the optimal response depends on signal s. Given a strategy of defender $b(\gamma)$, let us denote $L_i(m)$ the expected loss in site i when attacker applies her optimal strategy given signal s, $p(s)$ the probability of signal s and $L_i(m) = \sum_s p(s)L_i(s|m)$ the corresponding expected loss.

Theorem 8. **(Informative Case).** For $m = 1$; $c_1 \geq c_2 \geq ..._n$.

The class of optimal strategies $b_*(\gamma)$ has the following structure: the value of $L_i(m)$ must be equal for $1 \leq i \leq k_*$, where $k_*(m) : k < k_*(m) \leq n$ is the number of sites worth protecting

The third problem, to be solved, is to obtain the full description of all NE points, i.e. to describe all $b_*(\gamma)$ delivering the equality of $L_i(m)$ in Theorem 2.

Remark 1. The description of $b_*(\gamma)$ is based on the following interesting property of G game: to obtain the optimal response of attacker given any $b(\gamma)$ and signal s, i.e. to use procedure R, attacker need to know only the marginal probabilities $\alpha_i(s)$ for all s, but defender, trying to obtain $b_*(\gamma)$, need to know v(m), and to calculate the total expected loss, she needs to have $p(s)$ based on the whole distribution $b(\gamma)$. There is a simple example that shows that two distinct $b(\gamma)$ can have the same probabilities $\alpha_i(s)$. Thus, one of a side problems is to obtain the description of all $b(\gamma)$ having the same probabilities $\alpha_i(s)$.

Remark 2. The statements and interpretation of Theorems 1 and 2 can be expressed also using the concepts of *information* and *entropy*. Loosely speaking, the optimal DF strategy is to create the situation for AT with maximal possible entropy.

Remark 3.Here we define the function of real damage as $d(x) = \sum_s p(s|x)d(s|x))$, where $d(s|x)$ as real damage with optimal placement of a unique bomb is given by the maximum potential damage $d_i(s|x)$ for each signal, where $d_i(s|x) = c_i pP(T_i = 0|s)$ for $i = 1, 2, ...n$. And $p(s|x)$ is total probability of signal given by $p_0(s) = \sum_i p(s|\gamma_i)b_0(\gamma_i)$. and x as parameter of $\Lambda = (x, 1 - x)$

Example 1, Informative Case: $G_B(2, \Lambda, 1)$, where $\Lambda = (\lambda, 1-\lambda)$ with valued box $V = (2, 1)$, $a = 7/12$, $b = 9/12$. (This example will be compared to dependent case model $G_A(n, k, m)$ where $k = 1$).Here is the summary of prior probability for lock's location (Table 4) and distribution of signal vector (S) (Table 5)

Table 2: A summary of Locks' location and prior probability for $G_B(n, \Lambda, m)$.

Lock' Location (γ)	Probability of γ ($b_0(\gamma)$)
$\gamma_1 = (0, 0)$	$(1-\lambda)\lambda$
$\gamma_2 = (0, 1)$	$(1-\lambda)^2$
$\gamma_3 = (1, 0)$	λ^2
$\gamma_4 = (1, 1)$	$\lambda(1-\lambda)$

Table 3: A summary of signal vector and distribution for $G_B(n, \Lambda, m)$.

Signal (S)	$p(S\|\gamma_1)$	$p(S\|\gamma_2)$	$p(S\|\gamma_3)$	$p(S\|\gamma_4)$	$p_0(S) = \sum_i p(S\|\gamma_i)b_0(\gamma_i)$
$s_1 = (-, -)$	b^2	$b(1-a)$	$(1-a)b$	$(1-a)^2$	$-(4\lambda - 9)(4\lambda + 5)/144$
$s_2 = (-, +)$	$b(1-b)$	ab	$(1-a)(1-b)$	$(1-a)a$	$(4\lambda - 7)(4\lambda - 9)/144$
$s_3 = (+, -)$	$b(1-b)$	$(1-a)(1-b)$	ab	$a(1-a)$	$(4\lambda + 3)(4\lambda + 5)/144$
$s_4 = (+, +)$	$(1-b)^2$	$a(1-b)$	$a(1-b)$	a^2	$-(4\lambda + 3)(4\lambda - 7)/144$

For dependent model $G_A(n,k,m)$ with n=2,k=1,m=1 and with valued box $V = (2,1)$, $a = 7/12$, $b = 9/12$ The probability of allocating lock in box 1 is λ, the probability of allocating lock in box 2 is $1-\lambda$

Table 4: A summary of Locks' location and prior probability for $G_A(n,k,m)$.

Lock' Location (γ)	Probability of γ ($b_0(\gamma)$)
$\gamma_1 = (1,0)$	λ
$\gamma_2 = (0,1)$	$1-\lambda$

Table 5: A summary of signal vector and distribution for $G_A(n,k,m)$.

Signal (S)	$p(S\|\gamma_1)$	$p(S\|\gamma_2)$	$p_0(S) = \sum_i p(S\|\gamma_i)b_0(\gamma_i)$
$s_1 = (-,-)$	$b(1-a)$	$b(1-a)$	$45/144$
$s_2 = (-,+)$	$(1-a)(1-b)$	ab	$(63-48\lambda)/144$
$s_3 = (+,-)$	ab	$(1-a)(1-b)$	$(15+48\lambda)/144$
$s_4 = (+,+)$	$a(1-b)$	$a(1-b)$	$21/144$

Example 1a For $G_B(2, \Lambda, 1)$ model. Specially, let's take a look of posterior distribution and destruction at $\lambda = 0.7$. Posterior Distribution for locks are given by formula $b(\gamma_i|S) = P(S|\gamma_i)b_0(\gamma_i)/p_0(S)$, and shows in (Table 8).

Measure $r_i(S) = c_i pP(T_i = 0|S)$ and real damage $d(s|x) = p(s|x)d(s|x)) = p_0(s)c_i pP(T_i = 0|s)$, $i = 1, 2$.We have the following destruction table with (Table 9).

When $\lambda = 0.7$, destruction $d = d_1 + d_2 = 0.7883333$

Table 6: A summary of signal and Posterior Distribution when $\lambda = 0.7$.

Lock (γ)	Signal (S)	$b(\gamma_i\|S) = p(S\|\gamma_i)b_0(\gamma_i)/p_0(S)$
$\gamma_1 = (0, 0)$	$s_1 = (-, -)$	$-81\lambda(1-\lambda)/((4\lambda-9)(4\lambda+5)) = 0.35173697$
	$s_2 = (-, +)$	$27\lambda(1-\lambda)/((4\lambda-7)(4\lambda-9)) = 0.21774194$
	$s_3 = (+, -)$	$27\lambda(1-\lambda)/((4\lambda+3)(4\lambda+5)) = 0.12533156$
	$s_4 = (+, +)$	$-9\lambda(1-\lambda)/((4\lambda+3)(4\lambda-7)) = 0.07758621$
$\gamma_2 = (0, 1)$	$s_1 = (-, -)$	$-45(1-\lambda)^2/((4\lambda-9)(4\lambda+5)) = 0.08374690$
	$s_2 = (-, +)$	$63(1-\lambda)^2/((4\lambda-7)(4\lambda-9)) = 0.21774194$
	$s_3 = (+, -)$	$15(1-\lambda)^2/((4\lambda+3)(4\lambda+5)) = 0.02984085$
	$s_4 = (+, +)$	$-21(1-\lambda)^2/((4\lambda+3)(4\lambda-7)) = 0.07758621$
$\gamma_3 = (1, 0)$	$s_1 = (-, -)$	$-45\lambda^2/((4\lambda-9)(4\lambda+5)) = 0.45595533$
	$s_2 = (-, +)$	$15\lambda^2/((4\lambda-7)(4\lambda-9)) = 0.28225806$
	$s_3 = (+, -)$	$63\lambda^2/((4\lambda+3)(4\lambda+5)) = 0.68236074$
	$s_4 = (+, +)$	$-21\lambda^2/((4\lambda+3)(4\lambda-7)) = 0.42241379$
$\gamma_4 = (1, 1)$	$s_1 = (-, -)$	$-25\lambda(1-\lambda)/((4\lambda-9)(4\lambda+5)) = 0.10856079$
	$s_2 = (-, +)$	$35\lambda(1-\lambda)/((4\lambda-7)(4\lambda-9)) = 0.28225806$
	$s_3 = (+, -)$	$35\lambda(1-\lambda)/((4\lambda+3)(4\lambda+5)) = 0.16246684$
	$s_4 = (+, +)$	$-49\lambda(1-\lambda)/((4\lambda+3)(4\lambda-7)) = 0.42241379$

Table 7: A summary of destruction.
Notice: Box 1 has larger r for signal s_1 and $s2$. Box 2 has larger r for signal s_3 and s_4

Box	Signal (S)	$r_i(S) = c_iP(T_i = 0\|S)p$	$d(s\|\lambda = 0.7) = c_iP(T_i = 0\|s)p$
1	$s_1 = (-, -)$	$2(b(\gamma_1\|s_1) + b(\gamma_2\|s_1)) = 0.8709677$	$2p(\gamma_1)(p(s_1\|\gamma_1) + p(s_2\|\gamma_1))$
	$s_2 = (-, +)$	0.8709677	$+2p(\gamma_2)(p(s_1\|\gamma_2) + p(s_2\|\gamma_2)))$
	$s_3 = (+, -)$	0.3103448	$= 0.45$
	$s_4 = (+, +)$	0.3103448	
2	$s_1 = (-, -)$	$(b(\gamma_1\|s_1) + b(\gamma_3\|s_1)) = 0.8076923$	$p(\gamma_1)(p(s_3\|\gamma_1) + p(s_4\|\gamma_1))$
	$s_2 = (-, +)$	0.5000000	$+p(\gamma_3)(p(s_3\|\gamma_3) + p(s_4\|\gamma_3)))$
	$s_3 = (+, -)$	0.8076923	$= 0.3383333$
	$s_4 = (+, +)$	0.5000000	

For dependent case $G_A(n, k, m)$, we have destruction $d = d_1 + d_2 = 0.8895833$

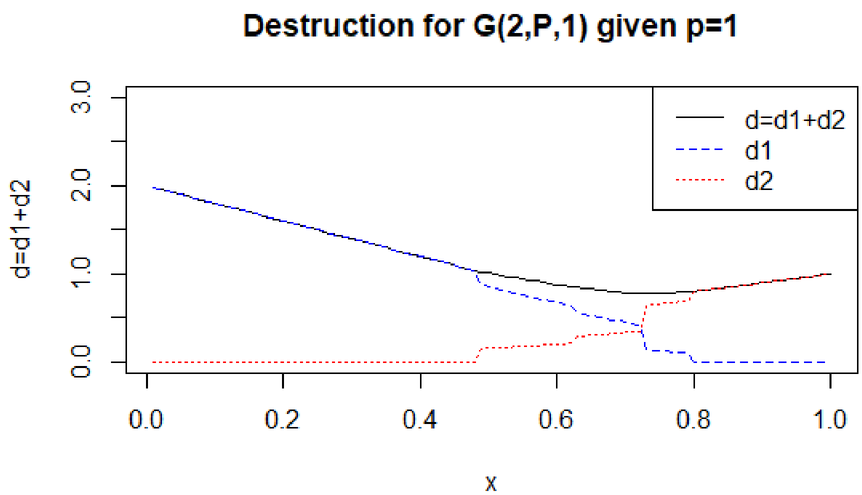
Table 8: A summary of signal and Posterior Distribution when $\lambda = 0.7$.

Lock (γ)	Signal (S)	$b(\gamma_i\|S) = p(S\|\gamma_i)b_0(\gamma_i)/p_0(S)$
$\gamma_1 = (1, 0)$	$s_1 = (-, -)$	$\lambda = 0.7$
	$s_2 = (-, +)$	$15\lambda/(63 - 48\lambda) = 0.35714286$
	$s_3 = (+, -)$	$63\lambda/(63\lambda + 15(1 - \lambda)) = 0.90740741$
	$s_4 = (+, +)$	$\lambda = 0.7$
$\gamma_2 = (0, 1)$	$s_1 = (-, -)$	$1 - \lambda = 0.3$
	$s_2 = (-, +)$	$63(1 - \lambda)/(63 - 48\lambda) = 0.64285714$
	$s_3 = (+, -)$	$(15 + 48\lambda) = 0.09259259$
	$s_4 = (+, +)$	$1 - \lambda = 0.3$

Table 9: A summary of destruction.
Notice: Box 1 has larger r for signal s_2. Box 2 has larger r for signal s_1,s_3 and s_4

Box	Signal (S)	$r_i(S) = c_i P(T_i = 0\|S)p$	$d(s\|\lambda = 0.7) = c_i P(T_i = 0\|s)p$
1	$s_1 = (-, -)$	$2b(\gamma_2\|s_1) = 0.6$	
	$s_2 = (-, +)$	1.2857143	$2p(\gamma_2)p(s_2\|\gamma_2)$
	$s_3 = (+, -)$	0.1851852	$= 0.2625$
	$s_4 = (+, +)$	0.6	
2	$s_1 = (-, -)$	$b(\gamma_1\|s_1) = 0.7$	
	$s_2 = (-, +)$	0.3571429	$p(\gamma_1)(p(s_1\|\gamma_1) + p(s_3\|\gamma_1) + p(s_4\|\gamma_1))$
	$s_3 = (+, -)$	0.9074074	$= 0.6270833$
	$s_4 = (+, +)$	0.7	

Example 1b Nash Equilibrium point. Continue on Example 1. $G_B(2, \Lambda, 1)$, where $\Lambda = (\lambda, 1 - \lambda)$ with valued box $V = (2, 1)$, $a = 7/12$, $b = 9/12$. From (Figure 19), we find at $\lambda = 0.72$, destruction will be minimized $d = 0.7728$.



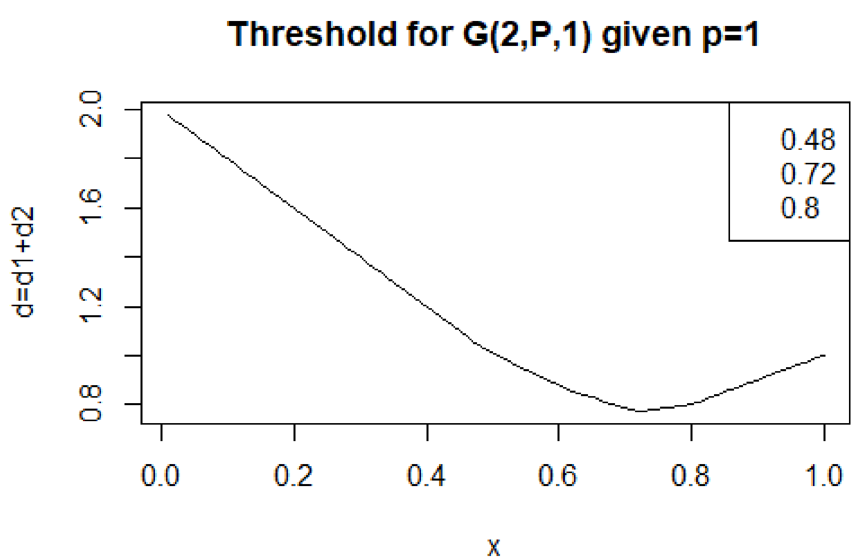
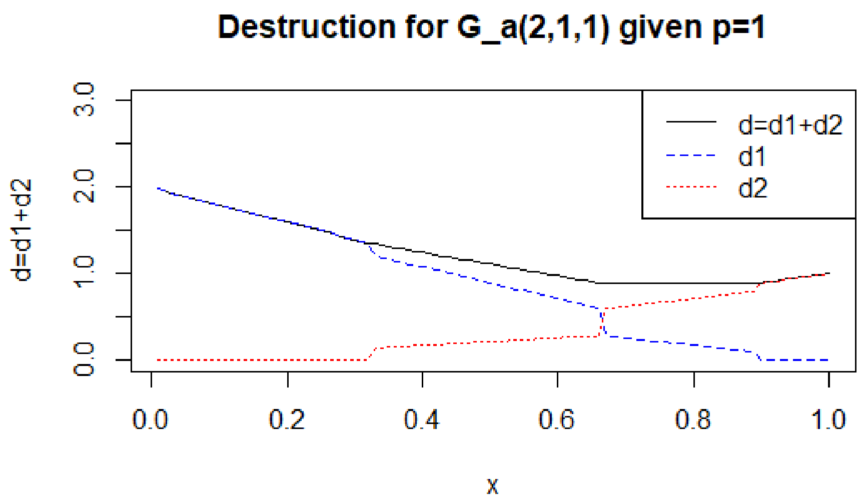


Figure 18: Destruction and threshold for $G_B(n, \Lambda, m)$

for dependent cass $G_A(n,k,m)$, destruction is minimized at $\lambda = 0.67$ with value $d = 0.8895833$



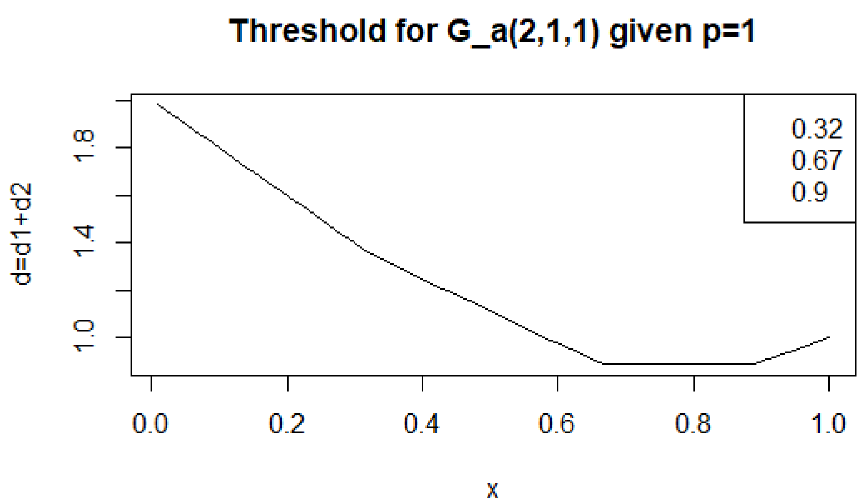


Figure 19: Destruction and threshold for $G_A(n,k,m)$

Example 2, Non-informative Case: $G_B(2, \Lambda, 1)$, where $\Lambda = (\lambda, 1-\lambda)$ with valued box $V = (2, 1)$, $a = 1/2$, $b = 1/2$. (This example will be compared to dependent case $G_A(n, k, m)$ where $k = 1$). See figure (21), and destruction is minimized at $\lambda = 2/3 = 0.67$

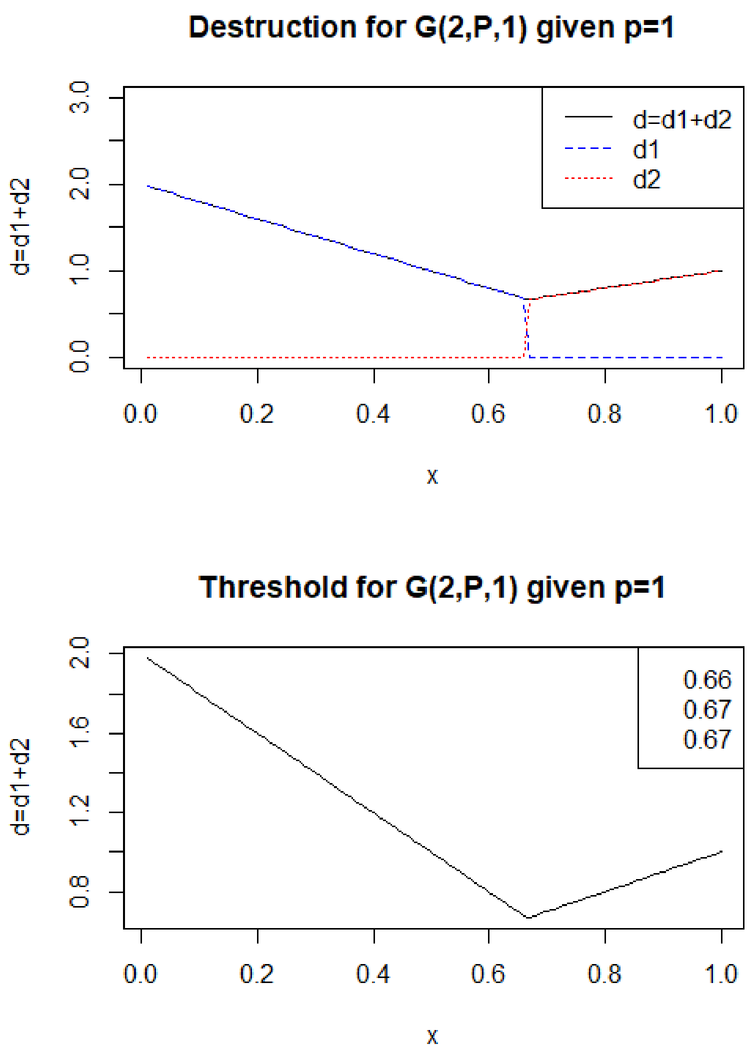


Figure 20: Destruction and Threshold for non-inform-general case in model $G_B(2, \Lambda, 1)$

For dependent case $G_A(2,1,1)$, we find the same conclusion as above. With destruction minimized at $\lambda = 2/3 = 0.67$

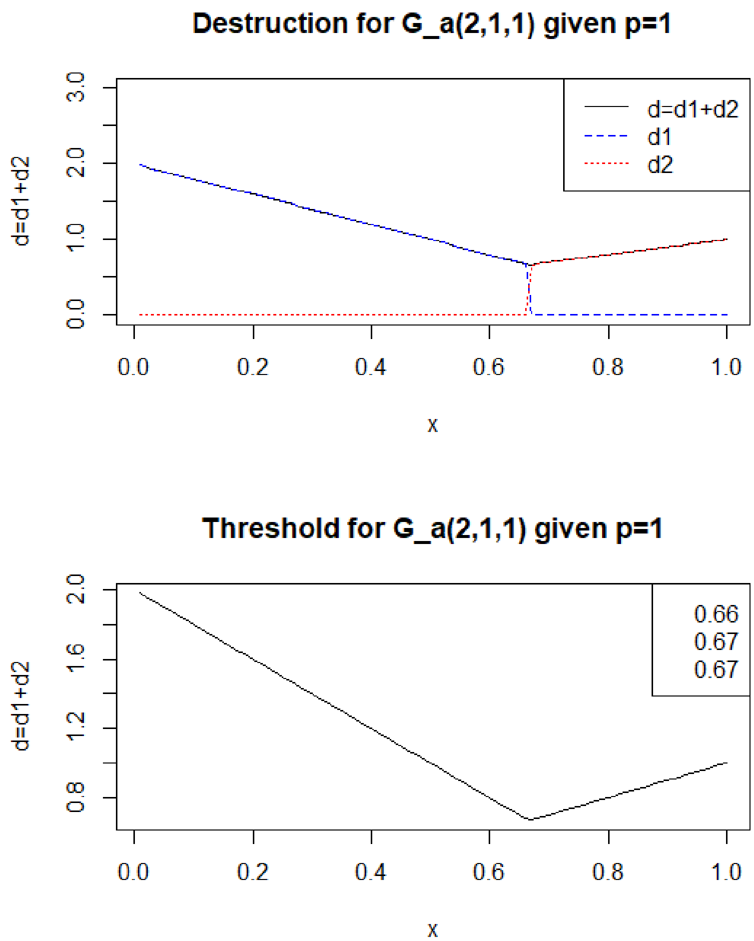


Figure 21: Destruction and Threshold for non-inform-general case in model $G_A(2,1,1)$

Example 3, Non-informative Case: $G_B(3, \Lambda, 1)$, where $\Lambda = (\lambda_1, \lambda_2, 1-\lambda_1-\lambda_2)$ with valued box $V = (4, 3, 1)$, $a = 1/2$, $b = 1/2$. See figure (23), and destruction is minimized at $\Lambda = (4/7, 3/7, 0) = (0.57, 0.43, 0)$ with value $12/7 = 1.72$,

```
> Defender_strategy(nn,P1,V)
$result
   box signal1 signal2 signal3    r joint_p    d.ps
1    1       0       0       0 1.72 0.05375 0.05375
2    1       1       0       0 1.72 0.05375 0.05375
3    1       0       1       0 1.72 0.05375 0.05375
4    1       1       1       0 1.72 0.05375 0.05375
5    1       0       0       1 1.72 0.05375 0.05375
6    1       1       0       1 1.72 0.05375 0.05375
7    1       0       1       1 1.72 0.05375 0.05375
8    1       1       1       1 1.72 0.05375 0.05375
9    2       0       0       0 1.71 0.07125 0.00000
10   2       1       0       0 1.71 0.07125 0.00000
11   2       0       1       0 1.71 0.07125 0.00000
12   2       1       1       0 1.71 0.07125 0.00000
13   2       0       0       1 1.71 0.07125 0.00000
14   2       1       0       1 1.71 0.07125 0.00000
15   2       0       1       1 1.71 0.07125 0.00000
16   2       1       1       1 1.71 0.07125 0.00000
17   3       0       0       0 1.00 0.12500 0.00000
18   3       1       0       0 1.00 0.12500 0.00000
19   3       0       1       0 1.00 0.12500 0.00000
20   3       1       1       0 1.00 0.12500 0.00000
21   3       0       0       1 1.00 0.12500 0.00000
22   3       1       0       1 1.00 0.12500 0.00000
23   3       0       1       1 1.00 0.12500 0.00000
24   3       1       1       1 1.00 0.12500 0.00000
```

Figure 22: r and destruction for signals in each box

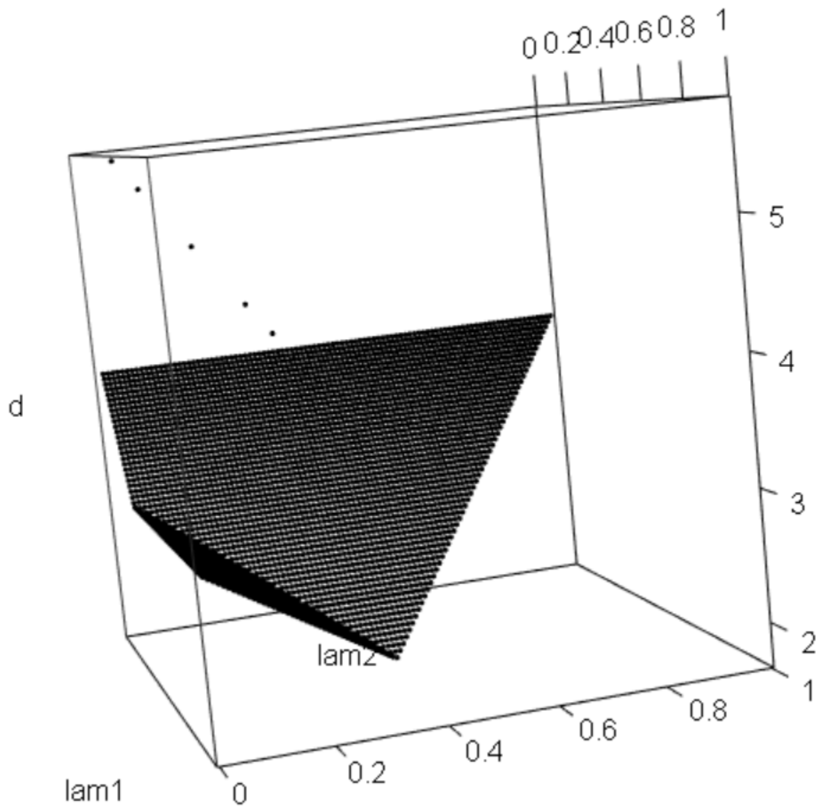


Figure 23: graph for destruction on different lambdas

CHAPTER 5: CONCLUSION AND FUTURE WORK

In Chapter 3, we developed General LBT model (**G-LBT**) under special condition of valued box and nonidentical probability of locks allocation when number of bombs is 1. In the future, we can also extend this model to a more general condition where number of bombs could be a random variable.

Moreover, **G-LBT** model could tolerate different kinds of bombs and locks. and testing is not uniform with respect to different boxes, in this case, Defender can test only a subset of all boxes, or parameters of testing a and b depend on the box number.

Note also that LBT models can be extended to the case when an integer number of bombs u is replaced by a nonnegative continuous variable and the destruction function $p(u)$ gives the probability of distribution with u resources allocated to an unlocked box. Similarly, we can convert integer locks to a continuous protection resource. We do not present any results about the G-LBT model in this paper.

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