THE TYPE B PERMUTOHEDRON AND THE POSET OF INTERVALS AS A TCHEBYSHEV TRANSFORM

GÁBOR HETYEI

ABSTRACT. We show that the order complex of intervals of a poset, ordered by inclusion, is a Tchebyshev triangulation of the order complex of the original poset. Besides studying the properties of this transformation, we show that the dual of the type B permutohedron is combinatorially equivalent to the order complex of the poset of intervals of a Boolean algebra (with the minimum and maximum elements removed).

INTRODUCTION

Inspired by Postnikov's seminal work [14], we have seen a surge in the study of root polytopes in recent years. A basic object in these investigations is the permutohedron. Less widely known are the results on *Tchebyshev transform* of a poset, introduced by the present author [6, 7] and studied by Ehrenborg and Readdy [4], respectively the (generalized) Tchebyshev triangulations of a simplicial complex, first introduced by the present author in [8] and studied in collaboration with Nevo in [9]. The key idea of a Tchebyshev triangulation may be summarized as follows: we add the midpoint to each edge of a simplicial complex, and perform a sequence of stellar subdivisions, until we obtain a triangulation containing all the newly added vertices. Regardless of the order chosen, the face numbers of the triangulation will be the same, and may be obtained from the face numbers f_j of the original complex by replacing the powers of x with Tchebyshev polynomials of the first kind if we work with the appropriate generating function. The appropriate generating function in this setting is the polynomial $\sum_j f_{j-1}((x-1)/2)^j$.

It is easy to verify that the face numbers of the type A and type B permutohedra are connected by a similar formula. These permutohedra are simple polytopes and their duals are simplicial polytopes. The suspicion arises that the dual of the type B permutohedron is a Tchebyshev triangulation of the dual of its type A cousin.

The present work contains the verification of this conjecture. The dual of the type A permutohedron is known to be the order complex of the Boolean algebra, and the dual of the type B permutohedron turns out to be an order complex as well, namely of the partially ordered set of intervals of the Boolean algebra, ordered by inclusion. We show that the operation of associating the poset of intervals to a partially ordered sets turns always induces a Tchebyshev triangulation at the level of order complexes. This observation may be helpful in constructing "type B analogues" of other polytopes and partially ordered sets.

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This paper is structured as follows. After the Preliminaries, we introduce the poset of intervals in Section 2 and show that the order complex of the poset of intervals is always a Tchebyshev transform of the order complex of the original poset. We also introduce a graded variant of this operation that takes a graded poset into a graded poset. In Section 2 we show that the dual of the type B permutohedron is the order complex of the graded poset of intervals of the Boolean algebra. In Section 4 we show how to compute the flag f-vector of graded a poset of intervals. The operation is recursive, unfortunately. Finally, in Section 5 we make the first steps towards computing the effect of taking the graded poset of intervals on the *cd*-index of an Eulerian poset.

1. Preliminaries

1.1. **Graded Eulerian posets.** A partially poset is *graded* if it contains a unique minimum element $\widehat{0}$, a unique maximum element $\widehat{1}$ and a rank function ρ satisfying $\rho(\widehat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ for each x and y such that y covers x. The numbers of chains containing elements of fixed sets of ranks in a graded poset P of rank $n+1$ is encoded by the *flag f-vector* $(f_S(P) \ : \ S \subseteq \{1, \ldots, n\})$. The entry f_S in the flag f-vector is the number of chains $x_1 < x_2 < \cdots < x_{|S|}$ such that their set of ranks $\{\rho(x_i) \ : \ i \in \{1, \ldots, |S|\}\}$ is S. Inspired by Stanley [16] we introduce the *upsilon-invariant* of a graded poset P of rank $n+1$ by

$$\Upsilon_P(a, b) = \sum_{S \subseteq \{1,\ldots,n\}} f_S u_S$$

where $u_S = u_1 \cdots u_n$ is a monomial in noncommuting variables a and b such that $u_i = b$ for all $i \in S$ and $u_i = a$ for all $i \notin S$. It should be noted that the term upsilon invariant is *not used elsewhere* in the literature, most sources switch to the *ab-index* $\Psi_P(a, b)$ defined to be equal to $\Upsilon_P(a - b, b)$. The *ab*-index may be also written as a linear combination of monomials in a and b, the coefficients of these monomials form the *flag h-vector*. A graded poset P is *Eulerian* if every nontrivial interval of P has the same number of elements of even rank as of odd rank. All linear relations satisfied by the flag f-vectors of Eulerian posets were found by Bayer and Billera [1]. A very useful and compact rephrasing of the Bayer-Billera relations was given by Bayer and Klapper in [2]: they proved that satisfying the Bayer-Billera relations is equivalent to stating that the *ab*-index may be rewritten as a polynomial of $c = a + b$ and $d = ab + ba$. The resulting polynomial in noncommuting variables c and d is called the *cd-index*.

As an immediate consequence of the above cited results we obtain the following.

Corollary 1.1. *The cd-index of a graded Eulerian poset P may be obtained by rewriting $\Upsilon_P(a, b)$ as a polynomial of $c = a + 2b$ and $d = ab + ba + 2b^2$.*

Note that this statement is a direct consequence of $\Upsilon_P(a - b, b) = \Psi_P(a, b)$ which is equivalent to $\Upsilon_P(a, b) = \Psi_P(a + b, b)$.

1.2. **Tchebyshev triangulations and Tchebyshev transforms.** A finite simplicial complex $\triangle$ is a family of subsets of a finite vertex set V. The elements of $\triangle$ are called *faces*, subject to the following rules: a subset of any face is a face and every singleton is a face. The *dimension* of a face is one less than the number of its elements, the dimension $d - 1$ of the complex $\triangle$ is the maximum of the dimension of its faces. The number of j-dimensional faces is denoted by $f_j(\triangle)$ and the vector $(f_{-1}, f_0, \dots, f_{d-1})$ is the *f-vector* of the simplicial complex. We define the *F-polynomial* $F_\triangle(x)$ of a finite simplicial complex $\triangle$ as

$$F_\triangle(x) = \sum_{j=0}^{d} f_{j-1}(\triangle) \cdot \left(\frac{x-1}{2}\right)^j. \tag{1.1}$$

The *join* $\triangle_1 * \triangle_2$ of two simplicial complexes $\triangle_1$ and $\triangle_2$ is the simplicial complex $\triangle_1 * \triangle_2 = \{\sigma \cup \tau \ : \ \sigma \in \triangle_1, \tau \in \triangle_2\}$. It is easy to show that the F-polynomials satisfy $F_{\triangle_1 * \triangle_2}(x) = F_{\triangle_1}(x) \cdot F_{\triangle_2}(x)$. A special instance of the join operation is the *suspension* operation: the suspension $\triangle * \partial(\triangle^1)$ of a simplicial complex $\triangle$ is the join of $\triangle$ with the boundary complex of the one dimensional simplex. (A $(d-1)$-dimensional simplex is the family of all subsets of a d-element set, its boundary is obtained by removing its only facet from the list of faces.) The *link* of a face σ is the subcomplex $\mathrm{link}_\triangle(\sigma) = \{\tau \in \triangle : \ \sigma \cap \tau = \emptyset, \ \sigma \cup \tau \in K\}$. A special type of simplicial complex we will focus on is the *order complex* $\triangle(P)$ of a finite partially ordered set P: its vertices are the elements of P and its faces are the increasing chains. The order complex of a finite poset is a *flag complex*: its minimal non-faces are all two-element sets (these are the pairs of incomparable elements).

Every finite simplicial complex $\triangle$ has a *standard geometric realization* in the vector space with a basis $\{e_v \ : \ v \in V\}$ indexed by the vertices, where each face σ is realized by the convex hull of the basis vectors e_v indexed by the elements of σ.

Definition 1.2. *We define a* Tchebyshev triangulation $T(\triangle)$ *of a finite simplicial complex $\triangle$ as follows. We number the edges $e_1, e_2, \dots, e_{f_1(\triangle)}$ in some order, and we associate to each edge $e_i = \{u_i, v_i\}$ a midpoint w_i. We associate a sequence $\triangle_0 := \triangle, \triangle_1, \triangle_2 \dots, \triangle_{f_1(\triangle)}$ of simplicial complexes to this numbering of edges, as follows. For each $i \geq 1$, the complex $\triangle_i$ is obtained from $\triangle_{i-1}$ by replacing the edge e_i and the faces contained therein with the one-dimensional simplicial complex L_i, consisting of the vertex set $\{u_i, v_i, w_i\}$ and edge set $\{\{u_i, w_i\}, \{w_i, v_i\}\}$, and by replacing the family of faces $\{e_i \cup \tau \ : \tau \in \mathrm{link}(e_i)\}$ containing e_i with the family of faces $\{\sigma' \cup \tau \ : \sigma' \in L_i\}$. In other words, we subdivide the edge e_i into a path of length 2 by adding the midpoint w_i and we also subdivide all faces containing e_i, by performing a stellar subdivision.*

It has been shown in [9] in a more general setting that a Tchebyshev triangulation of $\triangle$ as defined above is indeed a triangulation of $\triangle$ in the following sense: if we consider the standard geometric realization of $\triangle$ and associate to each midpoint w the midpoint of the line segment realizing the corresponding edge $\{u, v\}$ then the convex hulls of the vertex sets representing the faces of $T(\triangle)$ represent a triangulation of the geometric realization of $\triangle$. Furthermore, a direct consequence of [9, Theorem 3.3] is the following theorem.

Theorem 1.3 (Hetyei and Nevo)**.** *All Tchebyshev triangulations of a simplicial complex have the same f-vector.*

The following result has been shown in [8, Proposition 3.3] for a specific Tchebyshev triangulation. By the preceding theorem it holds for all Tchebyshev triangulations and motivates the choice of the terminology. The *Tchebyshev transform of the first kind of* *polynomials* used in the next result is the linear map $T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ sending x^n into the Tchebyshev polynomial of the first kind $T_n(x)$.

Theorem 1.4. *For any finite simplicial complex $\triangle$, the F-polynomial of any Tchebyshev triangulation $T(\triangle)$ is the Tchebyshev transform of the first kind of the F-polynomial of $\triangle$:*

$$F_{T(\triangle)}(x) = T(F_\triangle(x)).$$

The notion of the Tchebyshev triangulation of a simplicial complex was motivated by a poset operation, first considered in [6] and formally introduced in [7] .

Definition 1.5. *Given a locally finite poset P, its* Tchebyshev transform of the first kind *$T(P)$ is the poset whose elements are the intervals $[x, y] \subset P$ satisfying $x \neq y$, ordered by the following relation: $[x_1, y_1] \leq [x_2, y_2]$ if either $y_1 \leq x_2$ or both $x_1 = x_2$ and $y_1 \leq y_2$ hold.*

A geometric interpretation of this operation may be found in [7, Theorem 1.10]. The graded variant of this poset operation is defined in [4]. Given a graded poset P with minimum element $\widehat{0}$ and maximum element $\widehat{1}$, we introduce a new minimum element $\widehat{-1} < \widehat{0}$ and a new maximum element $\widehat{2}$. The *graded Tchebyshev transform of the first kind* of graded poset P is then the interval $[(\widehat{-1}, \widehat{0}), (\widehat{1}, \widehat{2})]$ in $T(P \cup \{\widehat{-1}, \widehat{2}\})$. By abuse of notation we also denote the graded Tchebyshev transform of a graded poset P by $T(P)$. It is easy to show that $T(P)$ is also a graded poset, whose rank is one more than that of P. The following result may be found in [8, Theorem 1.5].

Theorem 1.6. *Let P be a graded poset and $T(P)$ its graded Tchebyshev transform. Then the order complex $\triangle(T(P) \setminus \{(\widehat{-1}, \widehat{0}), (\widehat{1}, \widehat{2})\})$ is a Tchebyshev triangulation of the suspension of $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$.*

As a consequence of Theorem 1.6, we have

$$F_{\triangle(T(P)\setminus\{(\widehat{-1},\widehat{0}),(\widehat{1},\widehat{2})\})} = T(x \cdot F_{\triangle(P\setminus\{\widehat{0},\widehat{1}\})}). \tag{1.2}$$

It has been shown by Ehrenborg and Readdy [4] that there is a linear transformation assigning to the flag f-vectors of each graded poset $P$of rank $n + 1$ the flag f-vector of its Tchebyshev transform of the first kind $T(P)$. For Eulerian posets, they also compute the effect on the *cd*-index of taking the Tchebyshev transform of the first kind.

1.3. **Permutohedra of type A and B.** Permutohedra of type A and B have a vast literature, the results cited here may be found in [5] and in [17].

The type A permutohedron $\mathrm{Perm}(A_{n-1})$ is the convex hull of the $n!$ vertices $(\pi(1), \ldots, \pi(n)) \in \mathbb{R}^n$, where π is any permutation of the set $[1, n] := \{1, 2, \ldots, n\}$. The type B permutohedron $\mathrm{Perm}(B_n)$ is the convex hull of all points of the form $(\pm\pi(1), \pm\pi(2) \ldots, \pm\pi(n)) \in \mathbb{R}^n$. Combinatorially equivalent polytopes may be obtained by taking the A_{n-1}-orbit, respectively B_n orbit, of any sufficiently generic point in an $(n-1)$-dimensional (respectively n-dimensional) space, and the convex hull of the points in the orbit. [5, Section 2].

The type A and B permutohedra are simple polytopes, their duals are simplicial polytopes. The boundary complexes of these duals are combinatorially equivalent to the *Coxeter complexes* of the respective Coxeter groups. The Coxeter complex of the symmetric group A_{n-1} on $[1,n]$ is the order complex of $P([1,n]) - \{\emptyset, [1,n]\})$, where $P([1,n])$ is the Boolean algebra of rank n. The Coxeter complex of the Coxeter group B_n is the order complex of the face lattice of the n-dimensional crosspolytope [17, Lecture 1]. In either case we consider the order complexes of the respective graded posets without their unique minimum and maximum elements: adding these would make the order complex contractible, whereas the boundary complexes of simplicial polytopes are homeomorphic to spheres. The standard n-dimensional crosspolytope is the convex hull of the vertices $\{\pm e_i \ : \ i \in [1,n]\}$, where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. Each nontrivial face of the crosspolytope is the convex hull of a set of vertices of the form $\{e_i, i \in K^+\} \cup \{-e_i, i \in K^-\}$, where K^+ and K^- is are disjoint subsets of $[1,n]$ and their union is not empty. Keeping in mind that each face of a polytope is the intersection of all the facets containing it, we have the following consequence.

Corollary 1.7. *Each facet of* $\mathrm{Perm}(B_n)$ *is uniquely labeled with a pair of sets* (K^+, K^-) *where* K^+ *and* K^- *is are subsets of* $[1,n]$*, satisfying* $K^+ \subseteq [1,n] - K^-$ *and* $(K^+, [1,n] - K^-) \neq (\emptyset, [1,n])$. *For a set of valid labels*

$$\{(K_1^+, K_1^-), (K_2^+, K_2^-), \ldots, (K_m^+, K_m^-)\}$$

the intersection of the corresponding set of facets is a nonempty face of $\mathrm{Perm}(B_n)$ *if and only if*

$$K_1^+ \subseteq K_2^+ \subseteq \cdots \subseteq K_m^+ \subseteq [1,n] - K_m^- \subseteq [1,n] - K_{m-1}^- \subseteq \cdots \subseteq [1,n] - K_1^- \quad \textit{holds.}$$

The triangle of f-vectors of the duals to the type B permutohedra is given in sequence A145901 in [13].

2. The poset of intervals as a Tchebyshev transform

Corollary 1.7 inspires considering the following operation on partially ordered sets.

Definition 2.1. *An* interval $[u,v]$ *in a partially ordered set* P *is the set of all elements* $w \in P$ *satisfying* $u \leq w \leq v$. *For a finite partially ordered set* P *we define the* poset $I(P)$ of the intervals of P *as the set of all intervals* $[u,v] \subseteq P$*, ordered by inclusion.*

We may identify the singleton intervals $[u,u]$ in $I(P)$ with the elements of P. This subset of elements forms an antichain in $I(P)$, however, under this identification, the order complex of $I(P)$ looks like a triangulation of the order complex of P, see Figures 1 and 2. Figure 1 shows a partially ordered set and its order complex. The poset of its intervals and the order complex thereof may be seen in Figure 2.

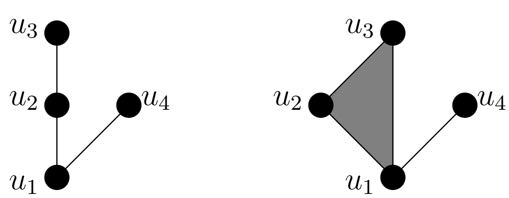

Figure 1. A partially ordered set P and its order complex $\triangle(P)$

In Figure 2 we marked the vertices of the order complex associated to non-singleton intervals with white circles.

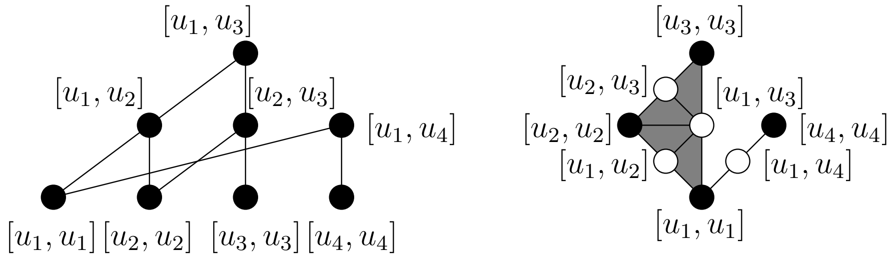

FIGURE 2. The poset $I(P)$ of intervals of P and its order complex

Theorem 2.2. *For any finite partially ordered set P the order complex $\triangle(I(P))$ of its poset of intervals is isomorphic to a Tchebyshev triangulation of $\triangle(P)$ as follows. For each $u \in P$ we identify the vertex $[u, u] \in \triangle(I(P))$ with the vertex $u \in \triangle(P)$ and for each nonsingleton interval $[u, v] \in I(P)$ we identify the vertex $[u, v] \in \triangle(I(P))$ with the midpoint of the edge $\{[u, u], [v, v]\}$. We number the midpoints $[u_1, v_1], [u_2, v_2], \ldots$ in such an order that $i < j$ holds whenever the interval $[u_i, v_i]$ contains the interval $[u_j, v_j]$.*

Proof. We illustrate the Tchebyshev triangulation process with the poset shown in Figure 2. We list its nonsingleton intervals in the following order: $[u_1, u_3]$, $[u_1, u_2]$, $[u_2, u_3]$, $[u_1, u_4]$. Figure 3 shows the stage of the process when we already added $[u_1, u_3]$ and $[u_1, u_2]$ but none of the remaining nonsingleton intervals. The following statement

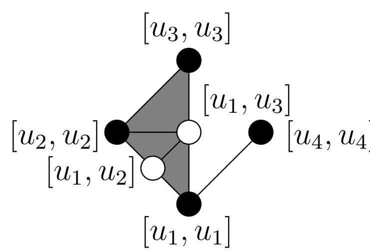

FIGURE 3. The second step of the Tchebyshev triangulation process

may be shown by induction on the number of stages in the process: in each stage, the resulting complex is a flag complex, whose minimal nonfaces are the following:

1. Pairs of singletons $\{[u, u], [v, v]\}$ such that u and v are not comparable in P.
2. Pairs of singletons $\{[u, u], [v, v]\}$ such that $u < v$ holds in P, but the interval $[u, v]$ has already been added to the triangulation.
3. Pairs of intervals from $I(P)$ such that neither one contains the other.

In each stage of the process, the nonsingleton interval $[u, v]$ added is the first midpoint of any edge whose endpoints are contained in the interval $[u, v]$ of P. At the beginning of the stage the restriction of the current complex to intervals contained in $[u, v]$ only contains singleton intervals, and it is isomorphic to the order complex of $[u, v]$. Subdividing the edge $\{[u, u], [v, v]\}$ and all faces containing this edge results in a complex where both $[u, u]$ and $[v, v]$ can not appear in the same face any more, each such face is replaced with 2 faces: one containing $\{[u, u], [u, v]\}$ the other containing $\{[v, v], [u, v]\}$. All intervals $[u', v']$ containing $[u, v]$ have already been added in a previous stage, and now we add the edge $\{[u,v],[u',v']\}$. The cumulative effect of all these changes is that we obtain a new flag complex satisfying the listed criteria. □

When P is a graded poset then $[u',v']$ covers $[u,v]$ in $I(P)$ exactly when the rank function ρ of P satisfies $\rho(v') - \rho(u') = \rho(v) - \rho(u) + 1$. Hence we may define the following graded variant of the operation $P \mapsto I(P)$.

Definition 2.3. *For a graded poset P we define its* graded poset of intervals $\widehat{I}(P)$ *as the poset of all intervals of P,* including the empty set, *ordered by inclusion.*

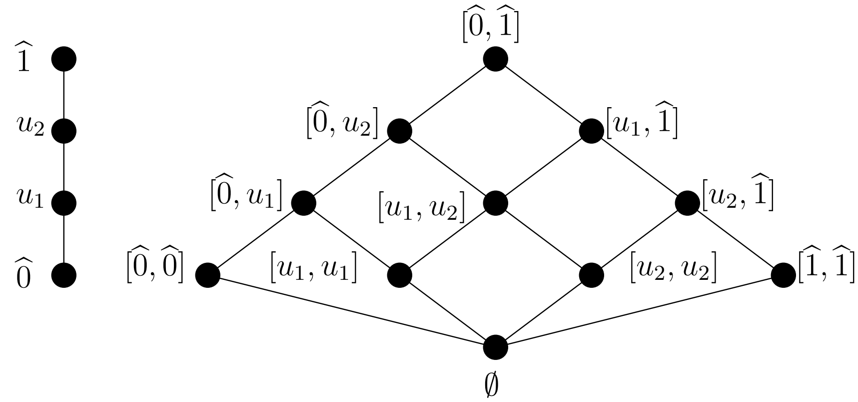

FIGURE 4. The graded poset of intervals of a chain

Remark 2.4. Figure 4 represents the graded poset of intervals of a chain of rank 3. It is worth comparing this illustration with [7, Figure 2] where the Tchebyshev transform of a chain of rank 3 is represented. The two posets are not isomorphic, not even after taking the dual of the Tchebyshev transform to make the number of elements at the same rank equal.

The following statement is straightforward.

Proposition 2.5. *If P is a graded poset of rank n with rank function ρ then $\widehat{I}(P)$ is a graded poset of rank $n+1$, in which the rank of a nonempty interval $[u,v]$ is $\rho(v) - \rho(u) + 1$.*

In analogy to Theorem 1.6 we have the following result.

Proposition 2.6. *Let P be a graded poset and $\widehat{I}(P)$ its graded poset of intervals. Then the order complex $\triangle(\widehat{I}(P) - \{\emptyset, [\widehat{0},\widehat{1}]\})$ is a Tchebyshev triangulation of the suspension of $\triangle(P - \{\widehat{0},\widehat{1}\})$.*

Proof. By Theorem 2.2, the order complex $\triangle(\widehat{I}(P) - \{\emptyset\})$ is a Tchebyshev triangulation of $\triangle(P)$. The order complex $\triangle(P)$ is the join of $\triangle(P - \{\widehat{0},\widehat{1}\})$ with the one-dimensional simplex on the vertex set $\{\widehat{0},\widehat{1}\}$. Performing the Tchebyshev triangulation results in subdividing every simplex containing the edge $\{\widehat{0},\widehat{1}\}$ into two simplices. The removal of the midpoint $[\widehat{0},\widehat{1}]$ leaves us exactly with those faces which are contained in a face of $\triangle(P)$ that does not contain the edge $\{\widehat{0},\widehat{1}\}$. Hence we obtain a Tchebyshev triangulation of a suspension of $\triangle(P - \{\widehat{0},\widehat{1}\})$: the suspending vertices are $\widehat{0}$ and $\widehat{1}$. □

We conclude this section with the following observation regarding the *direct product* of two graded posets. Recall that the direct product $P \times Q$ of two graded posets P and Q is defined as the set of all ordered pairs (u, v) where $u \in P$ and $v \in Q$, subject to the partial order $(u_1, v_1) \leq (u_2, v_2)$ holding exactly when $u_1 \leq u_2$ holds in P and $v_1 \leq v_2$ holds in Q.

Proposition 2.7. *If P and Q are graded posets then $\widehat{I}(P \times Q)$ is isomorphic to $\widehat{I}(P) \times \widehat{I}(Q)$.*

The straightforward verification is left to the reader.

Remark 2.8. It is worth comparing Proposition 2.7 above with [4, Theorem 9.1] where it is stated that the Tchebyshev transform of the Cartesian product of two posets is the *diamond product* of their Tchebyshev transforms. The diamond product is a different operation.

3. The dual of the type B permutohedron as a Tchebyshev triangulation

After introducing $X := K^+$ and $Y := [1, n] - K^-$, we may rephrase Corollary 1.7 as follows.

Corollary 3.1. *We may label each facet of the type B permutohedron $\text{Perm}(B_n)$ with a nonempty interval $[X, Y]$ of the Boolean algebra $P([1, n])$ that is different from $P([1, n]) = [\emptyset, [1, n]]$. The set $\{[X_1, Y_1], [X_2, Y_2], \ldots, [X_m, Y_m]\}$ labels a collection of facets with a nonempty intersection if and only if the intervals form an increasing chain in $\widehat{I}(P([1, n])) - \{\emptyset, [\emptyset, [1, n]]\}$.*

As we have seen, the representation of each face of $\text{Perm}(B_n)$ as an intersection of facets is unique. Hence we obtain the following result.

Proposition 3.2. *The dual of $\text{Perm}(B_n)$ is a simplicial polytope whose boundary complex is combinatorially equivalent to the order complex $\triangle(\widehat{I}(P([1, n])) - \{\emptyset, [\emptyset, [1, n]]\})$.*

As a consequence of this statement and of Proposition 2.6, we obtain the following result.

Corollary 3.3. *The dual of $\text{Perm}(B_n)$ is a simplicial polytope whose boundary complex is combinatorially equivalent to a Tchebyshev triangulation of the suspension of $\triangle(P([1, n]) - \{\emptyset, [1, n]\})$.*

It is worth noting that the order complex $\triangle(P([1, n]) - \{\emptyset, [1, n]\})$ is known to be combinatorially equivalent to the boundary complex of the permutohedron $\text{Perm}(A_{n-1})$. We may also think of this complex as the barycentric subdivision of the boundary of an $(n-1)$-dimensional simplex.

Figure 5 represents "half" of the dual of $\text{Perm}(B_3)$. The boundary of the triangle whose vertices are labeled with singleton intervals $[\{i\}, \{i\}]$ is shown in bold. (In general, the reader should imagine the boundary of a simplex, whose vertices are labeled with $[\{i\}, \{i\}]$.) The vertices of the barycentric subdivision of the boundary are marked with black circles. These correspond to singleton intervals of the form $[X, X]$, where X is a subset of $[1, 3]$. (In general, X is a subset of $[1, n]$.) The suspending vertex $\emptyset$ is marked

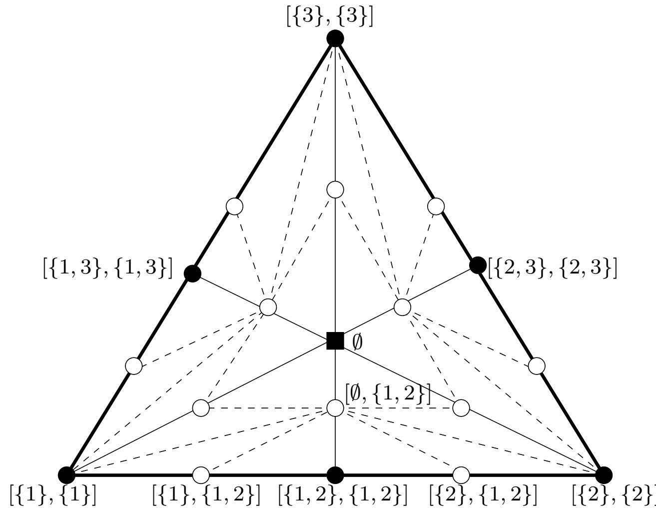

FIGURE 5. Half of the dual of $\mathrm{Perm}(B_3)$

with a black square. The other suspending vertex $[1, 3]$ (in general: $[1, n]$) is not shown in the picture. One would need to make another picture showing the boundary of the triangle with the suspending vertex, and "glue" the two pictures along the boundary of the triangle. The midpoints of the edges are marked with white circles. These are labeled with intervals $[X, Y]$ such that X is properly contained in Y. The edges arising when we take the appropriate Tchebyshev triangulation are indicated with dashed lines. Note that this part of the picture is different on the "other side" of the dual of $\mathrm{Perm}(B_3)$: on the side shown the largest intervals labeling midpoints are of the form $[\emptyset, [1, 3] - \{i\}]$ (in general $[\emptyset, [1, n] - \{i\}]$) whereas on the other side the largest such intervals are of the form $[\{i\}, [1, 3]]$ (in general: $[\{i\}, [1, n]]$). We leave to the reader as a challenge to draw the other side of the dual of $\mathrm{Perm}(B_3)$.

Remark 3.4. As a consequence of Corollary 3.3, we may compute the F-polynomial of the dual of $\mathrm{Perm}(B_n)$ using (1.2), and obtain that these polynomials have the same coefficients (up to sign) as the derivative polynomials for secant. For the Tchebyshev transform of a Boolean algebra this was first observed in [7, Corollary 9.3], and at the level of counting faces in the order complex of a graded poset there is no difference between considering the operator $P \mapsto T(P)$ and the operator $P \mapsto \widehat{I}(P)$. However, this observation is not new. The interest of the present section is to provide more detailed information regarding the face structure of the dual of the type B permutohedron, which may allow a more refined face count in the future.

4. Computing the flag f-vector of the graded poset of intervals

In this section we show that for any graded poset P, the flag f-vector of its graded poset of intervals $\widehat{I}(P)$ may be obtained from the flag f-vector of P by a linear transformation. By "chain" in this section we always mean a chain containing the unique minimum element and the unique maximum element. This treatment is equivalent to excluding both of these elements from all chains.

Definition 4.1. *Given a chain* $\emptyset \subset [u_1, v_1] \subset [u_2, v_2] \subset \cdots \subset [u_k, v_k] \subset [u_{k+1}, v_{k+1}][\widehat{0}, \widehat{1}]$ *in the graded poset of intervals* $\widehat{I}(P)$ *of a graded poset* P*, we call the set*

$$\{u_1, v_1, u_2, v_2, \ldots, u_{k+1}, v_{k+1}\}$$

the support *of the chain.*

Obviously the support of a chain in $\widehat{I}(P)$ is a chain in P containing the minimum element $\widehat{0}$ and the maximum element $\widehat{1}$.

The next statement expresses the number of chains in $\widehat{I}(P)$ having the same support in terms of the *Pell numbers* $P(n)$. These numbers are given by the initial conditions $P(1) = 1$ and $P(2) = 2$ and by the recurrence $P(n) = 2 \cdot P(n-1) + P(n-2)$ for $n \geq 3$. A detailed bibliography on the Pell numbers may be found at sequence A000129 of [13].

Proposition 4.2. *Let* P *be a graded poset and let* $c : \widehat{0} = z_0 < z_1 < \cdots < z_{m-1} < z_m = \widehat{1}$ *be a chain in it. Then the number of chains* $\emptyset \subset [u_1, v_1] \subset [u_2, v_2] \subset \cdots \subset [u_k, v_k] \subset [u_{k+1}, v_{k+1}] = [\widehat{0}, \widehat{1}]$ *whose support is* c *is the sum* $P(m)+P(m+1)$ *of two adjacent Pell numbers.*

Proof. We proceed by induction on m. For $m = 1$ there are three chains: $\emptyset \subset [\widehat{0}, \widehat{1}]$, $\emptyset \subset [\widehat{0}, \widehat{0}] \subset [\widehat{0}, \widehat{1}]$ and $\emptyset \subset [\widehat{1}, \widehat{1}] \subset [\widehat{0}, \widehat{1}]$. For $m = 3$, there are the following seven chains with support $\widehat{0} < z_1 < \widehat{1}$:

(1) $\emptyset \subset [\widehat{0}, z_1] \subset [\widehat{0}, \widehat{1}]$,
(2) $\emptyset \subset [\widehat{0}, \widehat{0}] \subset [\widehat{0}, z_1] \subset [\widehat{0}, \widehat{1}]$,
(3) $\emptyset \subset [z_1, z_1] \subset [\widehat{0}, z_1] \subset [\widehat{0}, \widehat{1}]$,
(4) $\emptyset \subset [z_1, \widehat{1}] \subset [\widehat{0}, \widehat{1}]$,
(5) $\emptyset \subset [\widehat{1}, \widehat{1}] \subset [z_1, \widehat{1}] \subset [\widehat{0}, \widehat{1}]$,
(6) $\emptyset \subset [z_1, z_1] \subset [z_1, \widehat{1}] \subset [\widehat{0}, \widehat{1}]$, and
(7) $\emptyset \subset [z_1, z_1] \subset [\widehat{0}, \widehat{1}]$.

Let us list the elements of the chain in $\widehat{I}(P)$ in decreasing order. The largest element of the chain must be $[\widehat{0}, \widehat{1}]$, the unique maximum element. The next element is either the interval $[z_1, \widehat{1}]$ or the interval $[\widehat{0}, z_m]$ or the interval $[z_1, z_m]$. We can not make the minimum of this next interval larger than z_1 because that would force skipping z_1 in the support, similarly the maximum of this next interval is at least z_m. Applying the induction hypothesis to the intervals $[z_1, \widehat{1}]$, $[\widehat{0}, z_m]$ and $[z_1, z_m]$, respectively, we obtain that the number of chains is

$$2 \cdot (P(m) + P(m+1)) + (P(m-1) + P(m)) = P(m+1) + P(m+2).$$

□

Remark 4.3. The numbers $P(n) + P(n+1)$ are listed as sequence A001333 in [13]. They are known as the numerators of the continued fraction convergents to $\sqrt{2}$, and have many combinatorial interpretations. The even, respectively odd indexed entries in this sequence may also be obtained by substitutions into the Tchebyshev polynomials of the first, respectively second kind.

It is transparent in the proof of Proposition 4.2 that the contributions of chains of $\widehat{I}(P)$ with a fixed support to $\Upsilon_{\widehat{I}(P)}(a,b)$ depends only on the contribution of their support to $\Upsilon_P(a,b)$. This observation motivates the following definition.

Definition 4.4. *Given an ab-word w of degree n, we define $\iota(w)$ as the contribution of all chains of $\widehat{I}(P)$ with a fixed support to $\Upsilon_{\widehat{I}(P)}(a,b)$, whose support is the same chain of P, contributing the word w to $\Upsilon_P(a,b)$.*

Theorem 4.5. *The operator ι may be recursively computed using the following formulas.*

1. *$\iota(a^n) = (a+2b)a^n$ holds for $n \geq 0$. In particular, for the empty word ε we have $\iota(\varepsilon) = (a+2b)$.*
2. *$\iota(a^iba^j) = (a+2b)(a^iba^j + a^jba^i) + ba^{i+j+1}$ holds for $i,j \geq 0$.*
3. *$\iota(a^ibwba^j) = \iota(a^ibw)ba^j + \iota(wba^j)ba^i + \iota(w)ba^{i+j+1}$ holds for $i,j \geq 0$ and any ab-word w.*

Proof. The only chain that contributes a^n to the ab-index of a graded poset is the chain $\widehat{0} < \widehat{1}$ in a graded poset P of rank $n+1$. As seen in the proof of Proposition 4.2, there are 3 chains in $\widehat{I}(P)$ whose support is $\widehat{0} < \widehat{1}$, and their contribution is to the ab-index of $\widehat{I}(P)$ is $(a+2b)a^n$.

Similarly, the only chains that contribute a^iba^j to the ab-index of a graded poset are the chains $\widehat{0} < z_1 < \widehat{1}$ in a graded poset P of rank $i+j+2$, where the rank of z_1 is $i+1$. As seen in the proof of Proposition 4.2, there are 7 chains in $\widehat{I}(P)$ whose support is $\widehat{0} < z_1 < \widehat{1}$, and their contribution is to the ab-index of $\widehat{I}(P)$ is $(a+2b)(a^iba^j + a^jba^i) + ba^{i+j}$.

Finally, consider a chain $c : \widehat{0} < z_1 < z_2 < \cdots < z_k < z_{k+1} = \widehat{1}$ that contributes a^ibwba^j to the ab-index of a graded poset P of rank $n+1$. In such a chain the rank of z_1 is $i+1$ and the rank of of z_k is $n-j$. The largest element below $[\widehat{0},\widehat{1}]$ of any chain in $\widehat{I}(P)$ with support c is either $[\widehat{0}, z_k]$ (of rank $n-j+1$) or $[z_1,\widehat{1}]$ (of rank $n+1-i$) or $[z_1, z_k]$ (of rank $n-i-j+1$). The three terms correspond to the contributions of the chains of these three types. □

Corollary 4.6. *There is a linear map $I_n : \mathbb{R}^{2^n} \rightarrow \mathbb{R}^{2^{n+1}}$ sending the flag f-vector of each graded poset P of rank $n+1$ into the flag f-vector of its graded poset of intervals $\widehat{I}(P)$. This linear map may be obtained by encoding flag f-vectors with the corresponding upsilon-invariants, and extending the map ι by linearity.*

Example 4.7. Using Theorem 4.5 we obtain the following formulas.

$n=1$: $\iota(a) = a^2 + 2ba$, $\iota(b) = (a+2b)(b+b) + ba = 4b^2 + 2ab + ba$.
$n=2$: $\iota(a^2) = a^3 + 2ba^2$, $\iota(ab) = (a+2b)(ab+ba) + ba^2 = a^2b + aba + 2bab + 2b^2a + ba^2$, $\iota(ba) = a^2b + aba + 2bab + 2b^2a + ba^2 = \iota(ab)$, and

$$\begin{aligned}\iota(b^2) &= 2\iota(b)b + \iota(\varepsilon)ba = 2(4b^2 + 2ab + ba)b + (a+2b)ba \\ &= 8b^3 + 4ab^2 + 2bab + aba + 2b^2a.\end{aligned}$$

5. The graded poset of intervals of an Eulerian poset

Theorem 5.1. *If a graded poset P is Eulerian then the same holds for the graded poset of its intervals $\widehat{I}(P)$.*

Proof. It is well known consequence of Phillip Hall's theorem (see [15, Propositition 3.8.5]) that a graded poset is Eulerian if and only if the reduced characteristic of the order complex of each open interval (u,v) is $(-1)^{\rho(v)-\rho(u)}$ where ρ is the rank function. Since taking the graded poset of intervals results in taking a triangulation of the suspension of each such order complex, the reduced Euler characteristic remains unchanged. □

As a consequence of Theorem 5.1, the linear map I_n takes the flag f-vector of any graded Eulerian poset of rank $n+1$ into the flag f-vector of a graded Eulerian poset of rank $n+2$. It has been shown by Bayer and Billera [1] that for each n, one may make a list of F_{n+1} graded Eulerian partially ordered sets of rank $n+1$ whose flag f vectors are linearly independent, where F_{n+1} is the $(n+1)$st Fibonacci number ($F_1=1$, $F_2=2$). The upsilon invariants of such a basis span the vector space of upsilon invariants of all Eulerian posets of rank $n+1$, and the images under ι of these basis vectors have the property that the resulting upsilon invariants are also polynomials of $c=a+2b$ and $d=ab+ba+2b^2$. The same observation also holds for all linear combinations, hence we obtain the following result.

Theorem 5.2. *Extending the operator ι to linear combinations of ab-words by linearity, results in a linear operator that takes each polynomial of $c=a+2b$ and $d=ab+ba+2b^2$ into a polynomial of $c=a+2b$ and $d=ab+ba+2b^2$. This operator takes the cd-index of an Eulerian poset P into the cd-index of its graded poset of intervals $\widehat{I}(P)$.*

By abuse of notation, we will use the same symbol ι to denote the induced operator on cd words.

Example 5.3. Using the formulas listed in Example 4.7 we obtain the following:
$n=1$: $\iota(c)=\iota(a+2b)=a^2+2ba+2(4b^2+2ab+ba)=c^2+2d$.
$n=2$: (we use $\iota(ab)=\iota(ba)$)

$$\begin{aligned}\iota(d)&=\iota(ab+ba+2b^2)\\&=2(a^2b+aba+2bab+2b^2a+ba^2)+2(8b^3+4ab^2+2bab+aba+2b^2a)\\&=2(ba^2+2aba+a^2b+4b^2a+4bab+4ab^2+8b^3)\\&=2(cd+dc).\end{aligned}$$

$$\begin{aligned}\iota(c^2)&=\iota((c^2-2d)+2d)=\iota(a^2)+2\iota(d)\\&=(a+2b)a^2+4(cd+dc)=c(c^2-2d)+4(cd+dc)=c^3+2cd+4dc.\end{aligned}$$

We conclude this section with explicitly computing $\iota(c^n)$ for all n. Note that c^n is the cd-index of the "ladder" poset L_n of rank $n+1$. This poset has exactly 2 elements: $-i$ and i for each rank i satisfying $0<i<n+1$, and any pair of elements at different ranks are comparable. The "ladder" poset L_2 of rank 3 is shown in Figure 6. To simplify our

notation in the proof of Theorem 5.4 below, we write the unique minimum element of L_n as 0 and the unique maximum element as $n+1$.

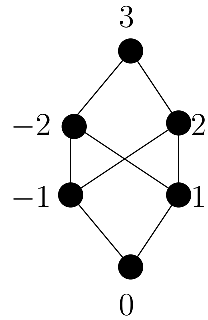

Figure 6. The "ladder" poset L_2 of rank 3

Theorem 5.4. *Assume that the finite vector* $(k_0, \ldots, k_r)$ *of nonnegative integers satisfies* $2r + k_0 + k_2 + \cdots + k_r = n$. *Then the coefficient of* $c^{k_0} d c^{k_1} d \cdots c_{k_r} d c^{k_r}$ *in* $\iota(c^n)$ *is* $2^r (k_1 + 1)(k_2 + 1) \cdots (k_r + 1)$.

Proof. The proof uses an *R-labeling* on the dual $\widehat{I}(L_n)^*$ of the graded poset of intervals $\widehat{I}(L_n)$ of the ladder poset L_n. A R-labeling is a labeling of the cover relations with an ordered set of labels, such that every interval has a unique maximal chain whose labels are rising. For more information on R-labelings see [15]. Since we work with the dual poset, we will show the dual statement, that is, the coefficient of $c^{k_1} d c_2^k d \cdots c_{k_r} d c^{k_{r+1}}$ in $\iota(c^n)^{\text{rev}}$, obtained from $\iota(c^n)$ by reversing all monomials, is $2^r (k_1+1)(k_2+1) \cdots (k_r+1)$.

All elements, except for the maximum element $\emptyset$ of $\widehat{I}(L_n)^*$ have the form $[\varepsilon \cdot i, \eta \cdot j]$ where $0 \leq i < j \leq n+1$ and $\varepsilon, \eta \in \{-1, 1\}$. If $i = 0$ then $\varepsilon = 1$ and if $j = n+1$ then $\eta = 1$.

We label the cover relations with the symbols U^-, U^+, L^- and L^+ as follows.

1. Cover relations of the form $[\varepsilon \cdot i, \eta \cdot j \prec (\varepsilon \cdot i, -(j-1)]$, $[\varepsilon \cdot i, \eta \cdot (i+1)] \prec [\varepsilon \cdot i, \varepsilon \cdot i]$, and $[\varepsilon \cdot i, \varepsilon \cdot i] \prec \emptyset$ receive the label U^-. In other words, if the upper end of the interval decreases and the upper end becomes a negative letter, or we create thereby a singleton or $\emptyset$, we record the label U^-.
2. Cover relations of the form $[\varepsilon \cdot i, \eta \cdot j \prec (\varepsilon \cdot i, (j-1)]$ for $j > i+1$ receive the label U^+. In other words, if the upper end of the interval decreases and becomes a positive letter, but the resulting interval is not a singleton, we record the label U^+.
3. Cover relations of the form $[\varepsilon \cdot i, \eta \cdot j] \prec [-(i+1), \eta \cdot j]$ and $[\varepsilon \cdot i, \eta \cdot (i+1)] \prec [\eta \cdot (i+1), \eta \cdot (i+1)]$ receive the label L^-. In other words, if the lower end of the interval increases and becomes a negative letter, or we create thereby singleton, we record the label L^-.
4. Cover relations of the form $[\varepsilon \cdot i, \eta \cdot j] \prec [(i+1), \eta \cdot j]$ for $j > i+1$ receive the label L^+. In other words, if the lower end of the interval increases and becomes a positive letter, but the resulting interval is not a singleton, we record the label L^+.

Next we replace each label U^-, U^+, L^-, and L^+ with an ordered pair $(U^-, r), (U^+, r), (L^-, r)$ and (L^+, r) where r is the rank of the top element of the cover relation labeled. For example, for $n = 5$ and the saturated chain

$$[0,6] \prec [0,-5] \prec [0,-4] \prec [1,-4] \prec [2,-4] \prec [2,3] \prec [3,3] \prec \emptyset$$

we record the sequence of edge labels

$$((U^-,1),(U^-,2),(L^+,3),(L^+,4),(U^+,5),(L^-,6),(U^-,7)).$$

We order the labels as follows:

$$(U^-,1) < (U^-,2) < \cdots < (U^-,n+1) < (U^+,n-1) < (U^+,n-2) < \cdots < (U^+,1) <$$
$$(L^-,1) < (L^-,2) < \cdots < (L^-,n) < (L^+,n-1) < (L^+,n-2) < \cdots < (L^+,1).$$

It is left to the reader to check that there is a unique rising chain in each interval $[[\varepsilon_1 \cdot i_1, \eta_1 \cdot j_1], [\varepsilon_2 \cdot i_2, \eta_2 \cdot j_2]]$ and in each interval $[[\varepsilon_1 \cdot i_1, \eta_1 \cdot j_1], \emptyset]$. For example, in the first case, beginning with $[\varepsilon_1 \cdot i_1, \eta_1 \cdot j_1]$ we want to keep decreasing the top element to a negative letter, as long as the new top element is larger than j_2, then we want to decrease the top element to $\eta_2 \cdot j_2$. From here on we want to keep increasing the bottom element to a negative letter, except possibly for the last step when we increase the bottom element to $\varepsilon_2 \cdot i_2$. Note that, following this pattern we will have at most one step with label (U^+, r) and at most one step of the form (L^+, s).

We may simplify our labeling to recording only the letters U^-, U^+, L^-, L^+ if we use the following convention: when a letter is followed by a different letter then the order $U^- < U^+ < L^- < L^+$ determines whether an ascent or a descent is created, for adjacent pairs of identical letters, the pairs U^-U^- and L^-L^- create an ascent, whereas the pairs U^+U^+ and L^+L^+ create a descent. The labeling of our saturated chains is such that the last letter is always U^- and the before last letter is U^- or U^+. For example, for $n = 2$, there are $4 \times 4 \times 2 = 32$ saturated chains, one for each label of the form $X_1^{\varepsilon_1} X_2^{\varepsilon_2} X_3^- U^-$ where X_1, X_2 and X_3 is any letter from the set $\{U, L\}$ and ε_1 and ε_2 are signs. To compute the ab-index $\Psi(n) := \Psi_{\widehat{I}(L_n)^*}(a, b)$, we record a letter a for each ascent and a letter b for each descent. Let us denote by $\Psi_0(n)$, $\Psi_1(n)$, $\Psi_2(n)$ and $\Psi_3(n)$, respectively the total weight of all saturated chains in $\widehat{I}(L_n)^*$ whose R-labeling begins with U^-, U^+, L^-, L^+ respectively. We obtain the following system of recurrences:

$$\begin{aligned}
\Psi_0(n+1) &= a(\Psi_0(n) + \Psi_1(n) + \Psi_2(n) + \Psi_3(n)) = a\Psi(n) \\
\Psi_1(n+1) &= b(\Psi_0(n) + \Psi_1(n)) + a(\Psi_2(n) + \Psi_3(n)) \\
\Psi_2(n+1) &= b(\Psi_0(n) + \Psi_1(n)) + a(\Psi_2(n) + \Psi_3(n)) \\
\Psi_3(n+1) &= b(\Psi_0(n) + \Psi_1(n) + \Psi_2(n) + \Psi_3(n)) = b\Psi(n)
\end{aligned} \tag{5.1}$$

subject to the initial conditions

$$\begin{aligned}
\Psi_0(1) &= a(a+b) \\
\Psi_1(1) &= ab + ba \\
\Psi_2(1) &= ab + ba \\
\Psi_3(1) &= b(a+b)
\end{aligned} \tag{5.2}$$

It is an immediate consequence of Equations (5.1) and (5.2) that $\Psi_1(n) = \Psi_2(n)$ holds for all $n \geq 1$. Introducing $c = a + b$ and $d = ab + ba$, as usual, (5.2) may be rewritten as

$$\begin{aligned} \Psi(1) &= c^2 + 2d \\ \Psi_1(1) &= d \end{aligned} \tag{5.3}$$

and repeated use of the recurrence (5.1) yields

$$\begin{aligned} \Psi_0(n+1) &= a\Psi(n) \\ \Psi_1(n+1) &= b(\Psi_0(n) + \Psi_1(n)) + a(\Psi_2(n) + \Psi_3(n)) \\ &= b(a\Psi(n-1) + \Psi_1(n)) + a(\Psi_1(n) + b\Psi(n-1)) \\ &= c\Psi_1(n) + d\Psi(n-1) \\ \Psi_2(n+1) &= c\Psi_1(n) + d\Psi(n-1) \quad \text{(since } \Psi_1(n+1) = \Psi_2(n+1)) \\ \Psi_3(n+1) &= b\Psi(n) \end{aligned}$$

Adding the last four equations and repeating the second, we obtain the recurrence formulas

$$\begin{aligned} \Psi(n+1) &= c\Psi(n) + 2c\Psi_1(n) + 2d\Psi(n-1) \\ \Psi_1(n+1) &= c\Psi_1(n) + d\Psi(n-1) \end{aligned} \tag{5.4}$$

Subtracting twice the second equation of (5.4) from the first we obtain

$$\Psi(n+1) - 2\Psi_1(n+1) = c\Psi(n), \quad \text{that is,}$$

$$\Psi_1(n+1) = \frac{1}{2} \cdot (\Psi(n+1) - c\Psi(n)).$$

Substituting this (with n and $n-1$ into the first equation of (5.4) we obtain

$$\Psi(n+1) = c\Psi(n) + c(\Psi(n) - c\Psi(n-1)) + 2d\Psi(n-1),$$

that is,

$$\Psi(n+1) = 2c\Psi(n) + (2d - c^2)\Psi(n-1). \tag{5.5}$$

The statement follows from the last equation by an easy induction on n. Indeed, for $k_1 = 0$, the coefficient of $c^{k_1} d c^{k_2} d \cdots c^{k_r} d c^{k_{r+1}}$ in $\Psi(n+1)$ is contributed by $2d\Psi(n-1)$ in (5.5) and we may apply the induction hypothesis to the word $c^{k_2} d \cdots c^{k_r} d c^{k_{r+1}}$, using the fact that the factor $(k_1 + 1)$ is 1. If $k_1 = 1$ then the coefficient of $c^{k_1} d c^{k_2} d \cdots c^{k_r} d c^{k_{r+1}}$ is contributed by $2c\Psi(n)$ and we may apply the induction hypothesis to the word $c^0 d c^{k_2} d \cdots c^{k_r} d c^{k_{r+1}}$, using the fact that the additional factor of 2 is equal to $(k_1 + 1)$. Finally, if $k_1 \geq 2$ then the coefficient of $c^{k_1} d c^{k_2} d \cdots c^{k_r} d c^{k_{r+1}}$ is contributed by $2c\Psi(n) - c^2\Psi(n-1)$. Applying the induction hypothesis to the words $c^{k_1-1} d c^{k_2} d \cdots c^{k_r} d c^{k_{r+1}}$ and $c^{k_1-2} d c^{k_2} d \cdots c^{k_r} d c^{k_{r+1}}$ we obtain that the coefficient of $c^{k_1} d c^{k_2} d \cdots c^{k_r} d c^{k_{r+1}}$ in $\Psi(n+1)$ is

$$(2k_1 - (k_1 - 1)) \cdot 2^r \cdot (k_2 + 1) \cdots (k_r + 1) = 2^r \cdot (k_1 + 1)(k_2 + 1) \cdots (k_r + 1).$$

□

Remark 5.5. Surprisingly this formula is the dual of the one obtained for the other Tchebyshev transform see [6, Theorem 7.1] and [4, Corollary 6.6] (see also [6, Table 1] and compare it with Example 5.3), although the two poset operations are very different. This observation also suggests that, when we comparing it to the Tchebyshev transform, one would want to consider the dual of the poset of intervals, ordered by reverse inclusion.

6. Concluding remarks

It would be interesting to find a similar interpretation of the type C permutohedron: the geometric modifications do not seem hard, associating a reasonably general poset operation seems difficult. Taking the graded poset of intervals seems to be a fairly straightforward operation, worthy of further study. Finding explicit formulas for cd-indices would be desirable but it seems harder than for the Tchebyshev transform studied in [6], [7] and [4]. The source of all difficulties seems that the operator ι recursively "rotates" the words involved: the recurrences call for cutting off certain initial segment of some words and placing their reverse at the end. That said, generalizations of permutohedra abound, and performing an analogous sequence of stellar subdivisions on their duals, respectively taking the graded poset of intervals for an associated poset may result in interesting geometric constructions, producing perhaps new type B analogues. Finally, applying the Tchebyshev transform studied in [6], [7] and [4] to a Boolean algebra creates a poset whose order complex has the same face numbers as the dual of a type B permutohedron. It may be interesting to find out whether the resulting polytope also has a nice geometric representation.

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Department of Mathematics and Statistics, UNC-Charlotte, Charlotte NC 28223-0001. WWW: `http://webpages.uncc.edu/ghetyei/`.