A real-analytic proof of the Simon-Wolff theorem

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November 20, 2019

Abstract

We derive the Simon-Wolff localization criterion from the spectral averaging using an intuitive measure theoretic lemma.

1 Introduction

Let A be a cyclic self-adjoint operator in a (separable) Hilbert space H, φ $(\|\varphi\| = 1)$ its cyclic vector, and $P = P_\varphi$ the orthogonal projection onto the one-dimensional subspace $\mathbf{C}\varphi$ $(Pz = (z, \varphi)\varphi$ for all $z \in H)$. Define the operator family A_t by

$$A_t = A + tP, \;\; t \in \mathbb{R}.$$

Denote by $\mu_A^\varphi(d\lambda)$ the spectral measure of the vector φ for A (see [RS]).

The celebrated Simon-Wolff theorem says [SW]:

Theorem 1 *Let Δ be a Borel subset of $\mathbb{R}$. The operator A_t has only pure point spectrum on Δ for Lebesgue a.e. $t \in \mathbb{R}$ if and only if*

$$\int_{\mathbb{R}} \frac{\mu_A^\varphi(d\lambda)}{(\lambda - E)^2} < \infty \;\; \text{for Lebesgue a.e. } E \in \Delta. \tag{1}$$

This theorem plays a fundamental role in localization theory (for some of its applications see [SW], [CL], [PF]). Several proofs of this theorem and its generalizations are known [SW], [H], [CHM], [P], [S]. The proof in [S] is particulary short, but it uses the so called Aronszajn-Donoghue theory [Ar], [D] formulated in terms of boundary values of a certain analytic function. The goal of the present work is to give an intuitively clear proof the Simon-Wolff theorem which does not mention analytic functions at all.

The remaining part of the paper is organized as follows. Section 2 contains several simple auxiliary statements. The Simon-Wolff theorem is proved in Section 3. The proof uses a lemma that may be interesting in its own right. The proof of the lemma is contained in the last section.

2 A different formulation of the Simon-Wolff theorem and a definition of the function τ

Throughout the paper we will use the following notations

$$I(E) := \int_{\mathbb{R}} \frac{\mu(d\lambda)}{(\lambda - E)^2} \qquad \text{and} \qquad J(E) := \int_{\mathbb{R}} \frac{\mu(d\lambda)}{\lambda - E},$$

where $\mu = \mu_A^\varphi$ is the spectral measure of φ for A. Let also

$$S := \{E \in \Delta \,|\, I(E) < \infty\}. \tag{2}$$

The Simon-Wolff theorem states that

$$A_t \textit{ has only pure point spectrum in } \Delta \textit{ for a.e. } t \in \mathbb{R} \quad \textit{iff} \quad |\Delta| = |S|.$$

In this statement, we are going to replace S by the set

$$D := \{E \in \Delta \big| \; \exists\, t \in \mathbb{R} \text{ for which } \; E \in \sigma_p(A_t)\}, \tag{3}$$

where $\sigma_p(A_t)$ denotes the set of eigenvalues of the operator A_t.

Proposition 1 *The integral $J(E)$ is well-defined for any $E \in S$. If*

$$E \in S \quad \textit{and} \quad J(E) \neq 0, \tag{4}$$

then $E \in D$. Any point of $D \setminus S$ is an eigenvalue of A.

Proof. The Spectral Theorem (see [RS]) states that, for a cyclic self-adjoint operator A and its normalized cyclic vector φ, there exists a unitary operator $U\colon H \to L^2(\mathbb{R}, \mu)$ such that $A = U^{-1} M_\lambda U$ and $U\varphi = \mathbf{1}$. Here $\mathbf{1}(\lambda) \equiv 1$ and M_λ is the operator of multiplication by the identity function $m(\lambda) \equiv \lambda$ in $L^2(\mathbb{R}, \mu)$.

If, for some E,

$$I(E) < \infty,$$

then the function $z(\lambda) = (\lambda - E)^{-1}$ belongs to $L^2(\mathbb{R}, \mu)$. Hence the equation $(\lambda - E)z(\lambda) = \mathbf{1}$ has a solution in $L^2(\mathbb{R}, \mu)$; equivalently, the equation

$$(A - E)z = \varphi \tag{5}$$

has a solution in H. Observe now that $(z, \varphi) = J(E)$. Therefore, if $J(E) \neq 0$, then we can rewrite the equation (5) as $Az - \varphi = Ez$, or $Az - \frac{1}{(z,\varphi)}(z, \varphi)\varphi = Ez$. Since $(z, \varphi)\varphi = Pz$, the previous equation means that

$$A_t z = Ez, \tag{6}$$

where $t = -1/J(E)$. Therefore, conditions (4) imply that $E \in D$.

Let now $E \in D \setminus S$. According to the definition of the set D, there is a number $t \in \mathbb{R}$ such that the equation

$$(A - E)z = -tPz \tag{7}$$

has a non-trivial solution $z \in H$. Obviously, $-tPz = -t(z, \varphi)\varphi$ is a vector of the form $c\varphi$.

Let us show that $c = 0$. For that purpose, assume the opposite, i.e. that $c \neq 0$. In this case, (7) can be re-written in the equivalent form

$$(\lambda - E)z(\lambda) = c\mathbf{1},$$

where $z(\lambda) := Uz$. Hence, $z(\lambda) = c/(\lambda - E)$, which tells us that

$$|c|^2 \int_{\mathbb{R}} \frac{\mu(d\lambda)}{(\lambda - E)^2} = ||z||^2 < \infty.$$

Therefore, if $c \neq 0$, then $E \in S$, which contradicts our assumptions. Thus, $c = 0$ and the right hand side of (7) is zero, which means that E is an eigenvalue of A. $\square$

Corollary 1 *Let S and D be the sets defined in* (2) *and* (3). *Then*

$$S \setminus D = Z \cap \Delta, \tag{8}$$

where

$$Z := \{E \in \mathbb{R} \mid I(E) < \infty \text{ and } J(E) = 0\}. \tag{9}$$

Proof. It follows from Proposition 1 that

$$S \setminus D \subset Z \cap \Delta.$$

It remains to prove that if $E \in S$ and $J(E) = 0$, then $E \notin D$. Assume the opposite, i.e. that $E \in S \cap D$ and $J(E) = 0$. Since a point of the set S can not be an atom of the measure μ, we conclude that E is not an eigenvalue of the operator A. We also conclude from the relation $E \in D$, that there is a number $t \neq 0$ and a non-zero vector $z \in H$, obeying the condition

$$A_t z = Ez. \tag{10}$$

Observe that tPz can not be zero, otherwise, (10) would mean that E is an eigenvalue of A. So,

$$-tPz = c\varphi, \qquad \text{with } c \neq 0. \tag{11}$$

Therefore, (10) and (11) imply that the function $z(\lambda) = Uz$ coincides with $c/(\lambda - E)$, because

$$(\lambda - E)z(\lambda) = c\mathbf{1}.$$

In this case, $(z, \varphi) = cJ(E) = 0$. Consequently, $Pz = 0$, which contradicts (11). $\square$

Proposition 2 *The set Z is countable.*

Proof. Proposition 2 follows from the observation that if E', $E \in Z$ are distinct, then the two functions $1/(\lambda - E')$ and $1/(\lambda - E)$ are orthogonal to each other in the separable space $L^2(\mathbb{R},\, \mu)$:

$$(E - E') \int_{\mathbb{R}} \frac{\mu(d\lambda)}{(\lambda - E)(\lambda - E')} = \int_{\mathbb{R}} \frac{\mu(d\lambda)}{(\lambda - E')} - \int_{\mathbb{R}} \frac{\mu(d\lambda)}{(\lambda - E)} = 0.$$

The statement of Proposition 2 follows also from Proposition 8 stating the fact that Z coincides with the set of eigenvalues of a selfadjoint operator A_∞ acting in the orthogonal compliment to the vector φ. A description of this operator is given in the Appendix of the present paper and in Section I.5 of [S]. □

According to our observations, $S \setminus D = Z \cap \Delta$ is countable. We also see that $D \setminus S = \sigma_p(A) \cap \Delta$ is countable as well. Therefore, $|D| = |S|$ and the Simon-Wolff theorem is equivalent to the following statement:

The spectrum of A_t is pure point in Δ for a.e. $t \in \mathbb{R}$ iff $|\Delta| = |D|$.

Lemma 1 [D] *If $A_t y = Ey$ and $y \neq 0$, then $(y, \varphi) \neq 0$.*

Proof [D]. Assume the converse. Then $Py = 0$, hence $Ay = Ey$. Since y is cyclic for A, the linear span of the vectors $(A - \lambda)^{-1}\varphi$ $(\lambda \in \mathbf{C} \setminus \mathbb{R})$ is dense in H. But

$$(y, (A - \lambda)^{-1}\varphi) = ((A - \bar{\lambda})^{-1} y, \varphi) = (E - \bar{\lambda})^{-1}(y, \varphi) = 0,$$

and the above linear span cannot be dense, which is a contradiction. ■

Consequently, we can normalize eigenvectors y of the operators A_t by

$$(y, \varphi) = 1. \tag{12}$$

In what follows, we always assume that eigenvectors are normalized according to (12).

Lemma 2 *If $A_{t_1} y_1 = E_1 y_1$, $A_{t_2} y_2 = E_2 y_2$, and $(y_1, \varphi) = (y_2, \varphi) = 1$, then*

$$t_1 - t_2 = (E_1 - E_2)(y_1, y_2). \tag{13}$$

Proof.

$$(A_{t_1} y_1,\, y_2) - (y_1,\, A_{t_2} y_2) = (E_1 y_1, y_2) - (y_1, E_2 y_2) = (E_1 - E_2)(y_1, y_2).$$

On the other hand,

$$((A + t_1 P) y_1, y_2) - (y_1, (A + t_2 P) y_2) = t_1 (P y_1, y_2) - t_2 (y_1, P y_2)$$

$$= t_1((y_1, \varphi)\varphi, y_2) - t_2(y_1, (y_2, \varphi)\varphi) = t_1 (y_1, \varphi)(\varphi, y_2) - t_2 \overline{(y_2, \varphi)}(y_1, \varphi) = t_1 - t_2.$$

□

Now, we define the function τ on the set D as follows:

Definition. For any $E \in D$ the value $\tau(E)$ equals the number $t \in \mathbb{R}$ for which $E \in \sigma_p(A_t)$. According to (13) this t is uniquely defined.

Note that Proposition 1 tells us that τ vanishes on $D \setminus S$, because all points of $D \setminus S$ are eigenvalues of A. Observe also that the multiplicity of any eigenvalue E of A_t is 1. If it was larger than 1, then one would be able to find an eigenvector orthogonal to φ. Thus, for each $E \in D$, there is a unique vector $y_E \in H$, such that $(y_E, \varphi) = 1$ and $A_{\tau(E)} y_E = E y_E$. The equation (13) can be now written in the form

$$\tau(E_1) - \tau(E_2) = (E_1 - E_2)(y_{E_1}, y_{E_2}) \quad \text{for all} \quad E_1, E_2 \in D. \tag{14}$$

3 The proof of Theorem 1

To prove the Simon-Wolff theorem, we use the following lemma.

Lemma 3 *Let X be a Borel set in $\mathbb{R}$ and $\tau \colon X \to \mathbb{R}$ be a function such that for any non-isolated point $E \in X$ there exists a finite non-zero limit*

$$\tau^*(E) := \lim_{X \ni E' \to E} \frac{\tau(E') - \tau(E)}{E' - E} \neq 0. \tag{15}$$

Define the function $\tau^*(\cdot)$ at isolated points of X arbitrarily (so that $\tau^*(\cdot) \neq 0$). Then

$$|X| = \int_{\mathbb{R}} dt \sum_{E \in X \colon \tau(E) = t} \frac{1}{|\tau^*(E)|}, \tag{16}$$

where the integrand is a Borel function on $\mathbb{R}$ with values in $[0, \infty]$.

Although Lemma 3 is intuitively clear, a detailed proof seems appropriate. It is deferred until the last section.

Let N be a positive integer. Define the set D_N by

$$D_N := \{\, E \in D \mid \ \|y_E\|^2 \leq N\}.$$

Here y_E is the (unique) eigenvector of $A_{\tau(E)}$ corresponding to the eigenvalue E and normalized by $(y_E,\, \varphi) = 1$.

Proposition 3 *Let Δ be a Borel subset of $\mathbb{R}$. Then the set D_N is Borel for each N.*

Proof. Since the sets $D \setminus S$ and Z are countable, it is sufficient to prove that $D_N \cap (S \setminus Z)$ is Borel. For that purpose, we observe that

$$D_N \cap (S \setminus Z) = \left\{ E \in S \setminus Z \,\middle|\, \psi(E) \leq N \right\},$$

where the function $\psi \colon \mathbb{R} \to [0, \infty]$ is defined on $S \setminus Z$ by

$$\psi(E) = \frac{I(E)}{J^2(E)}.$$

The functions I and J are pointwise limits of Borel measurable functions

$$I_n(E) = \int_{\lambda \in \mathbb{R} \colon |\lambda - E| \geq \frac{1}{n}} \frac{\mu(d\lambda)}{(\lambda - E)^2}, \qquad J_n(E) = \int_{\lambda \in \mathbb{R} \colon |\lambda - E| \geq \frac{1}{n}} \frac{\mu(d\lambda)}{(\lambda - E)}.$$

Consequently, ψ is Borel measurable. Measurability of I_n and J_n follows from the fact that I_n and J_n are continuous on the complment of a countable set. $\square$

Lemma 4 *Suppose E is a non-isolated point of D_N. As $D_N \ni E' \to E \in D_N$, we have*

$$\frac{\tau(E') - \tau(E)}{E' - E} \to \|y_E\|^2. \tag{17}$$

Proof. According to (14), we only need to show that

$$(y_{E'}, y_E) \to \|y_E\|^2,$$

as $D_N \ni E' \to E \in D_N$. Assume the opposite, i.e. that there exists a positive number $\varepsilon > 0$ and a sequence $D_N \ni E_n \to E \in D_N$, such that

$$\big|(y_{E_n}, y_E) - \|y_E\|^2\big| \geq \varepsilon, \qquad \forall n. \tag{18}$$

It follows from (14) that the function τ is Lipschitz on D_N:

$$|\tau(E') - \tau(E)| \leq N|E' - E|, \qquad E', E \in D_N.$$

Consequently, the sequence of numbers $t_n := \tau(E_n)$ converges to $t := \tau(E)$.

For the corresponding eigenvector $y_n := y_{E_n}$, we have $A_{t_n} y_n = E_n y_n$; moreover, $\|y_n\|^2 \leq N$ and $(y_n, \varphi) = 1$.

Since any ball of radius N is weakly compact in H, we may assume without loss of generality that y_n converges weakly to some vector y; clearly, $\|y\|^2 \leq N$ and $(y, \varphi) = 1$.

In the equality

$$Ay_n + t_n(y_n,\, \varphi)\varphi = E_n y_n$$

we have $t_n \to t,\ (y_n,\, \varphi) = (y,\, \varphi) = 1,\ E_n \to E$, and $y_n \overset{w}{\to} y$. This implies the weak convergence of Ay_n to some vector z. Moreover,

$$z + t(y,\, \varphi)\varphi = Ey. \tag{19}$$

On the other hand, $(Ay_n, h) = (y_n, Ah)$ for any $h \in Dom(A)$. Passing to the limit as $n \to \infty$ in this relation, we obtain the relation

$$(z, h) = (y, Ah), \qquad \forall h \in Dom(A),$$

which means that $y \in Dom(A)$ and $Ay = z$. This, along with (19), implies the equality

$$A_t y = Ey.$$

Put differently, the sequence of vectors y_n converges weakly to y_E. The latter contradicts (18). $\square$

If E is an eigenvalue of A_t, then E is an atom of μ_t, the spectral measure of A_t, corresponding to the vector φ. By the Spectral Theorem [RS], the mass of this atom equals

$$\mu_t(\{E\}) = \|\mathcal{E}_{A_t}(\{E\})\varphi\|^2 = \left|\left(\varphi,\, \frac{y_E}{\|y_E\|}\right)\right|^2 = \frac{1}{\|y_E\|^2} \tag{20}$$

(here $\mathcal{E}_{A_t}(Y)$, for any Borel set Y, denotes the spectral projection for A_t associated with Y; we use the fact that, by Lemma 5, φ is cyclic for A_t).

Let us apply Lemma 3 to $X = D_N$ and $\tau(E)$ defined in this section. Therefore,

$$|D_N| = \int_{-\infty}^{\infty} dt \sum_{E \in D_N :\, \tau(E)=t} \frac{1}{\tau^*(E)} = \int_{-\infty}^{\infty} dt \sum_{E \in D_N :\, \tau(E)=t} \frac{1}{||y_E||^2} = \int_{-\infty}^{\infty} dt \sum_{E \in D_N} \mu_t(\{E\}).$$

(The second equality makes use of (17). The third equality follows from (20).) Passing to the limit as $N \to \infty$, we obtain:

$$|D| = \int_{\mathbb{R}} dt \sum_{E \in \Delta \cap \sigma_p(A_t)} \mu_t(\{E\}) dt$$

or, equivalently,

$$|D| = \int_{\mathbb{R}} \mu_t^p(\Delta) dt. \tag{21}$$

Here the symbol μ_t^p denotes the pure point component in the standard decomposition of μ_t into its pure point and continuous components: $\mu_t = \mu_t^p + \mu_t^c$.

On the other hand, there is a fundamental identity due to Atkinson [At] (which was later rediscovered and/or cleverly used by Javrjan [J], Wegner [W], Carmona [C], Kotani [K], Delyon-Lévy-Souillard [DLS], and Simon-Wolff [SW]):

$$|\Delta| = \int_{\mathbb{R}} \mu_t(\Delta) dt \tag{22}$$

Subtracting (21) from (22), we obtain:

$$|\Delta \setminus D| = \int_{-\infty}^{\infty} \mu_t^c(\Delta) dt.$$

It follows that *Lebesgue a.e. point of Δ belongs to D if and only if*

$$\mu_t^c(\Delta) = 0 \tag{23}$$

for Lebesgue a.e. $t \in \mathbb{R}$.

Since, φ is cyclic for A_t for all t (see the lemma below), the equation (23) – the absence of the continuous component of μ_t on Δ – is equivalent to the fact that the operator A_t has in Δ only pure point spectrum. This completes the proof of the Simon-Wolff theorem up to the following statement:

Lemma 5 *For any $t \in \mathbb{R}$, the vector φ is cyclic for A_t.*

Proof. Assume the converse. Then there is a nonzero $y \in H$ such that $((A_t - \lambda I)^{-1}\varphi, y) = 0$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By the resolvent identity,

$$(A - \lambda I)^{-1} - (A_t - \lambda I)^{-1} = (A_t - \lambda I)^{-1}(tP)(A - \lambda I)^{-1},$$

so that

$$(A - \lambda I)^{-1}\varphi = (A_t - \lambda I)^{-1}\varphi + c_t(A_t - \lambda I)^{-1}\varphi$$

with some $c_t \in \mathbb{C}$. By the assumption, this vector is orthogonal to y for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, which contradicts the cyclicity of φ for A. $\square$

4 Proof of Lemma 3

Before we prove Lemma 3, we make several remarks.

Remark 1. The existence of the finite limit (15) for all non-isolated points E of X implies that the function $\tau\colon X \to \mathbb{R}$ is continuous.

Remark 2. Let us denote the integrand in (16) by $g(t)$. The statement that $g(\cdot)$ is a Borel function implies that $\tau(X) = \{t \in \mathbb{R}\colon\ g(t) > 0\}$ is a Borel set. Note that, in general, a set of the form $f(Y)$, where the set Y is Borel and the function f is continuous, is not necessarily Borel [Ke, Theorem 14.2].

The ***proof*** of Lemma 3 consists of justification of several statements.

Lemma 6 $\tau^* : X \to \mathbb{R}$ *is a Borel function.*

Proof. Denote the graph of $\tau(\cdot)$ by G. Let $\widehat{G}$ be a countable dense subset of G, and let $\widehat{X} := \mathrm{pr}_E(\widehat{G})$ be the projection of $\widehat{G}$ onto the E-axis.

For any non-isolated point $E \in X$, we have

$$\tau^*(E) = \limsup_{X \ni E' \to E} f_{E'}(E) = \limsup_{\widehat{X} \ni E' \to E} f_{E'}(E),$$

where

$$f_{E'}(E) := \frac{\tau(E') - \tau(E)}{E' - E}$$

is a Borel function on $X \setminus \{E'\}$.

By the definition of $\limsup$,

$$\tau^*(E) = \lim_{k\to\infty} \left(\sup\{f_{E'}(E) \mid E' \in X \cap U^*_{1/k}(E)\right),$$

where $U^*_{1/k}(E) = [E - 1/k, E + 1/k] \setminus \{E\}$. Since $\widehat{G}$ is dense in G, we also have

$$\tau^*(E) = \lim_{k\to\infty} \Big(\sup\{f_{E'}(E) \mid E' \in \widehat{X} \cap U^*_{1/k}(E)\Big).$$

Therefore, $\tau^*(E) < a$ iff there are $m, k \in \mathbb{N}$ such that $f_{E'}(E) \le a - 1/m$ whenever $E' \in \widehat{X}$ and $0 < |E - E'| \le 1/k$. In other words, $\tau^*(E) < a$ iff there are $m, k \in \mathbb{N}$ such that for any $E' \in \widehat{X}$ we have either $f_{E'}(E) \le a - 1/m$ or $E \notin U^*_{1/k}(E')$. This means that the set $\{E \in X \mid \tau^*(E) < a\}$ equals

$$\bigcup_m \bigcup_k \bigcap_{E' \in \widehat{X}} \left(\left\{ E \in X \;\middle|\; f_{E'}(E) \le a - \frac{1}{m} \right\} \bigcup (X \setminus U^*_{1/k}(E') \right)$$

and hence is a Borel set. $\square$

Definition 1 We will say that a Borel subset Y of X is *good* if the following conditions are fulfilled:

(i) the function $g_Y\colon\ \mathbb{R} \to [0, \infty]$ defined by the equation

$$g_Y(t) := \sum_{E \in Y\colon\ \tau(E) = t} \frac{1}{|\tau^*(E)|}$$

is Borel;

(ii) the equality holds:

$$\int_{\mathbb{R}} dt \sum_{E \in Y:\ \tau(E)=t} \frac{1}{|\tau^*(E)|} = |Y|. \tag{24}$$

Lemma 3 states that the set X is good.

Lemma 7 *Given a Borel set Y $(Y \subset X)$, let*

$$\widetilde{Y} := \{E \in Y |\ \text{the set } Y \cap (E-\varepsilon, E+\varepsilon) \text{ is uncountable for all } \varepsilon > 0\}.$$

The set $\widetilde{Y}$ is Borel; it is good iff Y is good.

Proof. The set $Z_Y := Y \setminus \widetilde{Y}$ consists of all points $E \in Y$ such that E is contained in an interval whose intersection with Y is countable. These intervals can be chosen to have rational endpoints, which shows that Z_Y is countable. Consequently, the set $\widetilde{Y}$ is Borel. Since the functions

$$g_Y(t) = \sum_{E \in Y:\ \tau(E)=t} \frac{1}{|\tau^*(x)|} \quad \text{and} \quad g_{\widetilde{Y}}(t) = \sum_{E \in \widetilde{Y}:\ \tau(E)=t} \frac{1}{|\tau^*(x)|}$$

differ only on the countable set $\tau(Z_Y)$, they are both Borel or both non-Borel. In the former case they have the same integral over $\mathbb{R}$. It is also obvious that $|\widetilde{Y}| = |Y|$. Therefore, if one of the sets Y and $\widetilde{Y}$ is good, the other set is good as well. $\square$

Lemma 8

(a) If sets $Y_1, Y_2, \ldots$ are good and disjoint, then the set $Y: = Y_1 \sqcup Y_2 \sqcup \ldots$ is good.

(b) If sets $Y_1, Y_2, \ldots$ are good and $Y_1 \subset Y_2 \subset \ldots$, then the set $Y := Y_1 \cup Y_2 \cup \ldots$ is good.

(c) If sets $Y_1, Y_2, \ldots$ are good, $Y_1 \supset Y_2 \supset \ldots$ and $|Y_1| < \infty$, then the set $Y := Y_1 \cap Y_2 \cap \ldots$ is good.

(d) The empty set $\varnothing$ is good.

Proof. (a) For any $t \in \mathbb{R}$, we have $g_Y(t) = \sum_n g_{Y_n}(t)$, so $g_Y(\cdot)$ is Borel and

$$\int_{\mathbb{R}} g_Y(t)dt = \sum_n \int_{\mathbb{R}} g_{Y_n}(t)dt = \sum_n |Y_n| = |Y|.$$

(b) We have $g_{Y_n}(t) \nearrow g_Y(t)$ for all $t \in \mathbb{R}$; therefore, $g_Y(\cdot)$ is Borel and

$$\int_{\mathbb{R}} g_Y(t)dt = \lim_n \int_{\mathbb{R}} g_{Y_n}(t)dt = \lim_n |Y_n| = |Y|.$$

(c) We have $g_{Y_n}(t) \searrow g_Y(t)$ for all $t \in \mathbb{R}$; therefore, $g_Y(\cdot)$ is Borel and, since $\int_{\mathbb{R}} g_{Y_1}(t)dt = |Y_1| < \infty$,

$$\int_{\mathbb{R}} g_Y(t)dt = \lim_n \int_{\mathbb{R}} g_{Y_n}(t)dt = \lim_n |Y_n| = |Y|.$$

(d) Obvious. $\square$

Corollary 2 *Suppose a Borel subset Y of X is such that for any $n \in \mathbb{N}$ the set $Y \bigcap [-n, n]$ is good. Then the set Y is good too.*

Proof. We have $Y = \bigcup_n (Y \bigcap [-n, n])$, so the statement follows from Lemma 8(b). $\square$

The corollary can be applied to the set X. Therefore, in the rest of the proof we will assume that *the set X is bounded.*

In the remaining part of the proof, we will gradually expand the class of subsets of X known to be good until it includes the set X itself. Every time we state that all subsets of X having a certain property are good, it will be sufficient (due to Lemma 7) to consider only those subsets that have no isolated points.

Lemma 9 *Suppose that, under the assumptions of Lemma 3, a Borel subset Y of X has the property that there are two constants A, B $(0 < A \leq B < \infty)$ such that*

$$A(E' - E) \leq \tau(E') - \tau(E) \leq B(E' - E) \tag{25}$$

for all $E, E' \in Y$ $(E < E')$. Then the set Y is good.

Proof. Let $I = [\inf Y, \sup Y] \equiv [\alpha, \beta]$ $(\alpha < \beta)$ and $J = [\tau(\alpha), \tau(\beta)]$. By (25), $\tau(\cdot)$ can be continued (uniquely) to a continuous function $\tau \colon I \to J$ that is linear on each connected component of the open set $I \setminus \overline{Y}$, where $\overline{Y}$ is the closure of Y. Then (25) still holds for all $E, E' \in I$ $(E < E')$. Hence $\tau \colon I \to J$ is a homeomorphism, and $\tau(Y)$, like Y, is a Borel set. In addition, $t \mapsto \tau^*(\tau^{-1}(t))$ is a Borel function on $\tau(Y)$ as a composition of two Borel functions.

Since (25) holds for all $E, E' \in Y$ $(E < E')$, the function $\tau^{-1} \colon J \to I$ is absolutely continuous and has, for a.e. $t \in J$, a derivative $(\tau^{-1})'(t) \in [B^{-1}, A^{-1}]$. For any $u, v \in J$ $(u < v)$, the equation holds:

$$\tau^{-1}(v) - \tau^{-1}(u) = \int_u^v (\tau^{-1})'(t)dt.$$

In other words, if W is a subinterval of J, then

$$|\tau^{-1}(W)| = \int_W (\tau^{-1})'(t)dt.$$

It follows (by summation) that the same is true if W is any relatively open subset of the closed interval J and, by approximation from outside, if W is any Borel subset of J. In particular, this is true for $W = \tau(Y)$:

$$|Y| = \int_{\tau(Y)} (\tau^{-1})'(t)dt.$$

Since $(\tau^{-1})'(t) = \dfrac{1}{\tau'(\tau^{-1}(t))}$ and $\tau'(E) = \tau^*(E)$ for $E \in Y$, we get

$$|Y| = \int_{\tau(Y)} \frac{dt}{\tau^*(\tau^{-1}(t))}.$$

As $\tau \colon Y \to \tau(Y)$ is a bijection, this is equivalent to (24). $\square$

Lemma 10 *Suppose that, under the assumptions of Lemma 3, a Borel subset Y of X is such that for some constants A, B $(0 < A \leq B < \infty)$ and $\delta > 0$ the double inequality*

$$A(E' - E) \leq \tau(E') - \tau(E) \leq B(E' - E) \tag{26}$$

holds for all $E, E' \in Y$ with $0 < E' - E < \delta$. Then the set Y is good.

Proof. Partition Y into a countable set of Borel sets of diameter $< \delta$ each. All of them are good by Lemma 9, so Y is good by Lemma 8(a). $\Box$

Lemma 11 *Under the assumptions of Lemma 3, for any $a, b \in \mathbb{R}$ such that $0 < a \leq b < \infty$, the set $X_{[a,b]}$ of all $E \in X$ for which*

$$a \leq \tau^*(E) \leq b \tag{27}$$

is good.

Proof. As we did before, denote by $\widehat{X}$ a countable subset of X such that the set $\{(E, \tau(E)) \mid E \in \widehat{X}\}$ is dense in the graph of τ, and denote by $g_{E'}(\cdot)$ the function

$$g_{E'}(E) := \frac{\tau(E') - \tau(E)}{E' - E}, \qquad E \in X \setminus \{E'\}.$$

Fix $m \in \mathbb{N}$ such that $1/m < a$ and define a sequence of sets X_k^m $(k \in \mathbb{N})$ as follows: the set X_k^m consists of all $E \in X$ such that

$$a - \frac{1}{m} \leq g_{E'}(E) \leq b + \frac{1}{m} \tag{28}$$

for all $E' \in \widehat{X}$ with $0 < |E' - E| \leq 1/k$.

The set X_k^m is Borel. To show this, let us verify that the set $(X_k^m)^-$ of all $E \in X$ satisfying the first inequality in (28) for all $E' \in \widehat{X}$ with $0 < |E' - E| \leq 1/k$ is Borel. The arguments are similar to those in the proof of Lemma 6. A point $E \in X$ belongs to $(X_k^m)^-$ iff $g_{E'}(E) \geq a - \frac{1}{m}$ for all $E' \in \widehat{X} \cap U^*_{1/k}(E)$. This is equivalent to the fact that for any $E' \in \widehat{X}$, E satisfies at least one of the two conditions: either $g_{E'}(E) \geq a - 1/m$ or $E \notin U^*_{1/k}(E')$. Since the functions $g_{E'}(\cdot)$ $(E' \in \widehat{X})$ are Borel and the set $\widehat{X}$ is countable, this shows that the set $(X_k^m)^-$ is Borel. Similarly, the set $(X_k^m)^+$ (defined in an obvious way) is Borel, so the set $X_k^m = (X_k^m)^- \cap (X_k^m)^+$ is Borel too.

Now note that in the definition of the set X_k^m the set $\widehat{X}$ can be replaced by X:

$$X_k^m = \left\{ E \in X \;\middle|\; g_{E'}(E) \in \left[a - \frac{1}{m}, b + \frac{1}{m}\right] \text{ if } E' \in X \text{ and } 0 < |E' - E| \leq \frac{1}{k} \right\}. \tag{29}$$

The sets X_k^m $(k \in \mathbb{N})$ are nested: $X_k^m \subset X_{k+1}^m$ for all $k \in \mathbb{N}$. The Borel set

$$X^m := \bigcup_{k \in \mathbb{N}} X_k^m \tag{30}$$

consists of all points $E \in X$ such that $g_{E'}(E) \in [a - 1/m, b + 1/m]$ for all $E' \in X$ $(E' \neq E)$ close enough to E. Consequently, the Borel set

$$X^\infty := \bigcap_m X^m \tag{31}$$

coincides with the set $X_{[a,b]}$ of all $E \in X$ such that $\tau^*(E) \in [a,b]$.

The set $X_{[a,b]}$ is good. To see why, we first note that for any $m, k \in \mathbb{N}$ the set X_k^m defined by (29) is good: this follows from Lemma 10, which should be applied with $A = a - 1/m,\ B = b + 1/m$ and $\delta = 1/k$. Second, the set X^m defined by (30) is good by Lemma 8(b). Finally, the set $X_{[a,b]} = X^\infty$ defined by (31) is good by Lemma 8(c) (we use the assumption that the set X is bounded and hence $|X| < \infty$). $\square$

Corollary 3 *Under the assumptions of Lemma 3, let $a, b \in \mathbb{R}$ be two constants such that $0 < a < b < \infty$. The set*

$$X_{(a,b]} := \{E \in X \mid a < \tau^*(E) \le b\}$$

is good.

Proof. We have

$$X_{(a,b]} = \bigcup_{n \in \mathbb{N}:\ n > 1/(b-a)} X_{[a+\frac{1}{n},b]},$$

so the statement follows from Lemma 8(b). $\square$

End of proof of Lemma 3.

We partition X into countably many disjoint Borel sets

$$X_k^+ := X_{(2^k,2^{k+1}]} \equiv \{E \in X \mid 2^k < \tau^*(E) \le 2^{k+1}\} \quad (k \in \mathbb{Z})$$

and

$$X_k^- := X_{[-2^{k+1},-2^k)} \equiv \{E \in X \mid -2^{k+1} \le \tau^*(E) < -2^k\} \quad (k \in \mathbb{Z}).$$

Each set X_k^+ is good by Corollary 3. Each set X_k^- is good by Corollary 3 applied to the function $-\tau(\cdot)$ instead of $\tau(\cdot)$. By Lemma 8(a), the set X is good. $\square$

5 Appendix: infinite coupling

In this section, we give a natural definition of a certain operator A_∞ playing the role of A_t for $t = \infty$. First, we extend the function $J(E)$ originally defined on the set S to a function on $(\mathbb{C} \setminus \mathbb{R}) \cup S$

$$J(z) := \int_{\mathbb{R}} \frac{\mu(d\lambda)}{\lambda - z}, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{32}$$

Proposition 4 *Let $J(z)$ be defined by* (32). *Then*

$$\pm \mathrm{Im}\, J(z) > 0 \qquad \textit{for} \qquad \pm\, \mathrm{Im}\, z > 0.$$

In particular, $J(z) \notin \mathbb{R}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Indeed,

$$\mathrm{Im}\, J(z) = \mathrm{Im}\, z \int_{\mathbb{R}} \frac{\mu(d\lambda)}{(\lambda - \mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2}.$$

$\square$

Proposition 5 *For any* $z \in \mathbb{C} \setminus \mathbb{R}$*,*

$$(A_t - z)^{-1} = (A-z)^{-1} - \frac{t}{1+tJ(z)}(A-z)^{-1}P(A-z)^{-1}. \tag{33}$$

Proof. Obviously, the range of the operator in the right hand side of (33) is contained in $Dom(A)$. Therefore, we can multiply this oparator by $A_t - z$ from the left. The product is equal to I, because

$$(A_t - z)(A-z)^{-1} = I + tP(A-z)^{-1}$$

and

$$P(A-z)^{-1}P = J(z)P.$$

□

Corollary 4 *For any* $z \in \mathbb{C} \setminus \mathbb{R}$*,*

$$(A_t - z)^{-1} \longrightarrow R(z) := (A-z)^{-1} - \frac{1}{J(z)}(A-z)^{-1}P(A-z)^{-1}, \qquad \textit{as } t \to \infty, \tag{34}$$

in the operator norm topology.

Proposition 6 *The range of the bounded operator* $R(z)$ *is contained in the space* H_φ *of vectors orthogonal to* φ*. In particular,* H_φ *is invariant for* $R(z)$*.*

Proof. It is sufficient to show that $PR(z) = 0$. The latter follows once one combines the equality

$$PR(z) = P(A-z)^{-1} - \frac{1}{J(z)}P(A-z)^{-1}P(A-z)^{-1}$$

with the fact that $P(A-z)^{-1}P = J(z)P$. □

Proposition 7 *Let* A_∞ *be the operator in* H_φ *defined on* $Dom(A_\infty) = Dom(A) \cap H_\varphi$ *by*

$$A_\infty y = (I-P)Ay.$$

Then A_∞ *is densily defined and selfadjoint in* H_φ*. Moreover,*

$$(A_\infty - z)^{-1} = R(z)\big|_{H_\varphi} \qquad \forall z \in \mathbb{C} \setminus \mathbb{R}. \tag{35}$$

Proof. The operator A_∞ is symmetric, because

$$(A_\infty u, v) \in \mathbb{R}, \qquad \forall u, v \in Dom(A_\infty) \subset H_\varphi.$$

This opertor is densily defined. Indeed, let $y \in H_\varphi$ be given and let $h \in Dom(A)$ be a vector such that $(h, \varphi) = 1$ (such a vector exists, because A is densily defined). Suppose that $h_n \in Dom(A)$ is a sequence of vectors converging to y. Then

$$h_n - (h_n, \varphi)h \in Dom(A_\infty)$$

converges to y, as $n \to \infty$.

To establish that A_∞ is selfadjoint, we observe that the definition of $R(z)$ given by (34) leads to

$$(A_\infty - z)R(z)\big|_{H_\varphi} = (I-P)(I - \frac{1}{J(z)}P(A-z)^{-1})\big|_{H_\varphi} = I\big|_{H_\varphi}. \tag{36}$$

Consequently, the range of the operator $(A_\infty - z)$ is the whole space H_φ, which implies that A_∞ is selfadjoint. The relation (35) follows from (36). □

Proposition 8 *A real number E is an eigenvalue of A_∞ if and only if $E \in S$ and $J(E) = 0$.*

Proof. Again, we use the Spectral Theorem (see [RS]) which states that, for a cyclic self-adjoint operator A and its normalized cyclic vector φ, there exists a unitary operator $U\colon H \to L^2(\mathbb{R}, \mu)$ such that $A = U^{-1} M_\lambda U$ and $U\varphi = \mathbf{1}$. Here $\mathbf{1}(\lambda) \equiv 1$ and M_λ is the operator of multiplication by the identity function $m(\lambda) \equiv \lambda$ in $L^2(\mathbb{R}, \mu)$.

If, for some E,

$$I(E) < \infty \qquad \text{and} \qquad J(E) = 0, \tag{37}$$

then the function $z(\lambda) = (\lambda - E)^{-1}$ belongs to $L^2(\mathbb{R}, \mu)$ and is orthogonal to $\mathbf{1}$. Hence the equation $(\lambda - E)z(\lambda) = \mathbf{1}$ has a solution in $L^2(\mathbb{R}, \mu)$ orhtogonal to $\mathbf{1}$; equivalently, the equation

$$(A - E)z = \varphi \tag{38}$$

has a solution in H orthogonal to φ. Observe now that $(Az, \varphi) = ((A - E)z, \varphi) = (\varphi, \varphi) = 1$. Therefore, we can rewrite the equation (38) as $Az - (Az, \varphi)\varphi = Ez$. Since $(Az, \varphi)\varphi = PAz$, the previous equation means that

$$A_\infty z = Ez. \tag{39}$$

Therefore, conditions (37) imply that $E \in \sigma_p(A_\infty)$.

Let now $E \in \sigma_p(A_\infty)$. Then the equation

$$(A - E)z = PAz \tag{40}$$

has a non-trivial solution $z \in H_\varphi$. Obviously, $PAz = (Az, \varphi)\varphi$ is a vector of the form $c\varphi$.

Assume that $c \neq 0$. In this case, (40) can be re-written equivalently as

$$(\lambda - E)z(\lambda) = c\mathbf{1}, \tag{41}$$

where $z(\lambda) := Uz$. Hence, $z(\lambda) = c/(\lambda - E)$, which tells us that

$$I(E) = \int_{\mathbb{R}} \frac{\mu(d\lambda)}{(\lambda - E)^2} = |c|^{-2} ||z||^2 < \infty.$$

Therefore, if $c \neq 0$, then $E \in S$. On the other hand, since

$$0 = (z, \varphi) = c \int_{\mathbb{R}} \frac{\mu(d\lambda)}{\lambda - E} = cJ(E),$$

we obtain also that $J(E) = 0$.

It remains to consider the case $c = 0$. In this case, (41) tells us that the support of $z(\lambda)$ consists of the point $\lambda = E$. The latter implies that

$$(z, \varphi) = \int_{\mathbb{R}} z(\lambda)\, \mu(d\lambda) = z(E)\, \mu(\{E\}) \neq 0,$$

which contradicts the assumption $z \in H_\varphi$. $\square$

Corollary 5 *Let the sets S and D be defined by* (2) *and* (3) *correspondingly. Then*

$$D \setminus S = \sigma_p(A_0) \cap \Delta \qquad \textit{and} \qquad S \setminus D = \sigma_p(A_\infty) \cap \Delta.$$

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