DIFFUSION PROCESSES ON SOLVABLE GROUPS OF UPPER TRIANGULAR 3×3 MATRICES. APPLICATIONS IN ASIAN AND BASKET OPTIONS.

by

Julia S. Simonsen

A dissertation submitted to the faculty of
The University of North Carolina at Charlotte
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in
Applied Mathematics

Charlotte

2020

Approved by:

____
Dr. Stanislav Molchanov

____
Dr. Michael Grabchak

____
Dr. Susan Trammell

____
Dr. Boris Vainberg

ABSTRACT

JULIA S. SIMONSEN. Diffusion Processes on Solvable Groups of Upper Triangular 3×3 Matrices. Applications in Asian and Basket Options.. (Under the direction of DR. STANISLAV MOLCHANOV)

One of the general questions in algebraic groups is about the asymptotic behavior of the probability of return of a random walk defined on these groups. An upper-triangularity of a matrix is preserved by a sum, product, inverse, thus they form a group. Growth rate of a group and the asymptotic behavior of the probability of return of a random walk are closely related. Solvable groups have an exponential growth rate and in well-established literature, it was shown that the asymptotic behavior of the probability of return on these groups has a fractional-exponential decal. The results in S. Molchanov, V. Konakov and S. Menozzi paper, are different from the previous finding. They showed that in the case of solvable groups of upper-triangular 2x2 matrices the return probability of the Brownian motions has a polynomial decay. In this dissertation, we extended this research to the case of solvable groups of upper-triangular 3x3 matrices. The elements in the 3x3 matrices that define a Brownian motion on these groups contain integrals of geometric Brownian motions. These integrals have an important role in mathematical finance in particular, in Asian and Asian-Basket options. We proved some properties of these integrals and showed that certain cases of geometric Asian-basket call options with two assets have a higher risk that the same type of put options. Which implies that some trading strategies might benefit from a reevaluation using a new risk assessment of geometric Asian-Basket.

ACKNOWLEDGMENTS

Firstly, I would like to express my sincere gratitude to my advisor Dr. Stanislav Molchanov for the continuous support of my PhD study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. His insightful feedback pushed me to sharpen my thinking and brought my work to a higher level. I could not have imagined having a better advisor and mentor for my PhD study.

Besides my advisor, I would like to thank the rest of my thesis committee: Both Dr. Michael Grabchak and Dr. Boris Vainberg for their insightful comments and encouragement, but also for the hard questions which incented me to widen my research from various perspectives.

Lastly, I would like to thank my family for all their love and encouragement. For my parents who raised me with a love of science. My dad, Dr. Lev Steinberg, helped me tremendously rising up my confidence and helping me getting over my insecurities in academic world. To children, Stella and Samuel, have continually provided the requisite breaks from mathematics and the motivation to finish my degree with expediency. My mom, Galia Galimzianova, thank you for helping out with the children and allowing me to spend more time on this PhD thesis.

And most of all, I owe my deepest gratitude to my husband, Dr. Jeremy Simonsen, for his affection, encouragement, understanding and patience. He supported me without any complaint or regret that enabled me to complete my PhD thesis.

Contents

List of Figures

List of Tables

CHAPTER 1: INTRODUCTION TO GROUP THEORY

A *group* is a nonempty set G, called the underlying set of the group, together with a binary operation on G with the following properties:

- Associativity: $(ab)c = a(bc)$
- Identity: $1a = a1 = a$
- Inverses: $aa^{-1} = a^{-1}a = 1$

Let's list few definitions that are going to be used thru out this dissertation:

Definition 1. If we have a commutative property: $ab = ba$, then the group is *abelian*.

Definition 2. A group is *finite*, if the underlying set G is a finite set; otherwise, it is *countable*.

Definition 3. *Lie product* is given by a communicator: $[X, Y] = XY - YX$

Definition 4. *Lie algebra* $\mathrm{Lie}(Y, X_2, ..., X_m)$ is the smallest vector space of smooth vector fields which contains $\{Y, X_2, ..., X_m\}$ closed under the Lie product.

Definition 5. A *matrix Lie group* (roughly speaking - a continuous group) is a subgroup G of $GL(n, R)$ with the following property: if any sequence of matrices in G converges to some matrix A, then either A is in G or A is not invertible.

Let G be a matrix Lie group with Lie algebra g. If X and Y are elements of g, then the following results hold:

- $AXA^{-1} \in g$ for all $A \in g$
- $sX \in g$ for all real numbers s
- $X + Y \in g$
- $XY - YX \in g$

Definition 6. Let G be a group and let $g_0 = g$ and $g_k = [g_{k-1}, g_{k-1}]$ where $g_k \in G$ for $k = 0, 1, 2, \ldots$.. The group G is called a *solvable Lie group* if $g_k = e_G$ for some k, where e_G an identity element of group G.

Definition 7. Let G be a group and $g^0 = g$ and $g^k = [g, g^{k-1}]$ where $g^k \in G$ for $k = 0, 1, 2, ...$ The group G is called a *nilpotent Lie group* if $g^k = e_G$ for some k, where e_G an identity element of group G.

If the group is nilpotent, then it is solvable. For general groups, one of the most basic and natural questions about random walks concerns the asymptotic behavior of the probability of return to the starting point [13]. An important observation that an upper-triangularity of a matrix is preserved by a sum, product, inverse, thus they form a group. We will be focusing on solvable groups, which can be realized as subgroups of invertible upper triangular matrices.

1.1 Growth Rate in Groups

Suppose G is a finitely generated group; and T s a finite symmetric set of generators (symmetric means that if $x \in T$, then $x^{-1} \in T$. Any element $x \in G : x = a_1 \cdot a_2 \cdots a_k$ where $a_i \in T$.

Definition 8. The *closed ball* of radius n is $B_n(G,T) = \{x \in G | x = a_1 \cdot a_2 \cdots a_k$ where $a_i \in T$ and $k \leq n\}$.

Definition 9. The *growth rate* of the group G is $\#(n) = |B_n(G,T)|$, which is the number of elements in this closed ball.

Can we relate the growth rate of a group to the asymptotic behavior of the probability of return of a random walk defined on that group?

Definition 10. The growth rate in a group is called

- Exponential, if $\#(n) \geq a^n$, $a > 1$.
- Sub-Exponential, if $\#(n)$ growth slower than an any exponential
- Polynomial, if $\#(n) \leq C\left(n^k + 1\right)$, $k \leq \infty$

Consider a countable group G with finitely many generators a_1, a_2, $\ldots$, a_m. Let $x(t)$, where $t = 1, 2, ...$, be a left-invariant symmetric random walk on G: $p(g_1, g_2) = p(g_2, g_1)$ without any compactness constraints on the transition probabilities. According to Kesten [17], all such groups are divided into two classes: amenable groups, for which

$$\limsup_{n\to\infty} \frac{\ln p(2n,e,e)}{2n} = 0$$

and nonamenable groups, for which $p(2n,e,e) \leq Ce^{-\nu n}$, where $\nu > 0$.

An example of a nonamenable group is the free group. See details about group theory in Hall [13].

1.2 Random Walks on Countable Groups

First significant results in the direction of study of random walks in groups were two papers [18] and [17] that Harry Kesten published in 1959, in which he showed that for a symmetric random walk on a group, the return probability decays exponentially if and only if the group is non-amenable.

Definition 11. A group G is *amenable* if one can say what proportion of G any given subset takes up.

In this way, Kesten related the behavior of the random walk to the geometric structure of the group.

1.3 Connecting Volume Growth with Probability of Return

The idea of using volume growth to study random walks on groups was introduced by Varopoulos [27] in the early 1980's. In the case of groups with polynomial growth, the volume growth completely determines the behavior of the return probability.

Theorem 1 (Varopoulos's theorem). Let G be a group of polynomial growth of degree d. Then for a finitely supported symmetric random walk ξ_n on G, $P(\xi_{2n} = e)) \simeq n^{-\frac{d}{2}}$. Moreover, if G is any group such that the volume growth satisfies the lower bound Cn^d for all n, then ξ_n on G, $P(\xi_{2n} = e)) = O\left(n^{-\frac{d}{2}}\right)$.

1.4 Growth rate and Asymptotic Behavior of Probability of Return

The class of amenable groups includes Abelian, nilpotent, and solvable groups. On such groups, a nontrivial ergodic theory can be developed See details in Tempelman [26]. Abelian and nilpotent groups have polynomial lower bound for the return probability. In the regards, of great interest are solvable groups, which can be realized as subgroup of groups of upper triangular matrices. It was believed that, for a Brownian motion on a solvable Lie group, the situation must be similar, that is, the exponential growth of the volume of a Riemann ball of radius r must imply the fractional-exponential decay of the transition density $p(t, e, e)$ as $t \to \infty$. It turned out that this is not so. According to Konakov-Menozzi-Molchanov [19] for classical solvable Lie groups of 2×2 matrices, we have $p(t, e, e) \sim \frac{c}{t^{\nu}}$.

1.5 Construction of Diffusion Processes on Lee Groups

Let's start from the functional central limit theorem (FCLT). Let $X_1, X_2, ..., X_n, ...$ are i.i.d random variables and $EX_i = 0$, $VarX_i = EX_i^2 = 1$. If $EX_i = a$ and $VarX_i = \sigma^2$, the the transformation $Y_i = \frac{X_i - a}{\sigma}$ for $i = 1, 2, ...$ leads to $EY_i = 0$ and $VarY_i = 1$.

Consider in $C([0, 1])$ the random element $x_n(t)$, $t \in [0, 1]$ given by formulas:

$$x(0) = 0$$
$$x\left(\frac{k}{n}\right) = \frac{x_1 + x_2 + ... + x_k}{\sqrt{n}}, \quad k = 1, 2, ...n$$

on the greed $\{\frac{k}{n}, \quad k = 0, 1, ..., n\}$ and by the linear interpolation between the points of the greed.

The usual CLT for fixed $0 < s \leq 1$ the distribution of $x_n(s)$ converges to $N(0, \sigma)$ if $n \to \infty$. We can show it by taking $k = [sn]$, and consider $\frac{s_k}{\sqrt{n}} = \frac{s_{[sn]}}{\sqrt{n}}$. Then $E\frac{s_k}{\sqrt{n}} = 0$, $var\frac{s_k}{\sqrt{n}} \to s$, $n \to \infty$ etc.

Similarly one can check that

$$(x_n(t_1), ..., x_n(t_m)) \xrightarrow[n\to\infty]{d} (b(t_1), ..., b(t_m))$$

for fixed m and $0 < t_1 < ... < t_m \leq 1$. Which means that the finite dimensional distributions of $x_n(t), \quad t \in [0,1]$ for $n \to \infty$ tends to the corresponding distributions of the Brownian motion (Wiener process).

1.6 Diffusion Processes on Solvable Groups of Upper-Triangular 2x2 matrices

Consider the solvable group T_2 of upper-triangular 2×2 matrices of the form [19]:

$$T_2 = \begin{bmatrix} e^{x_1} & y \\ 0 & e^{x_2} \end{bmatrix}, \quad x_1, x_2, y \in \mathbb{R}$$

as well as important subgroups of this group, such as the group

$$T_2 = \begin{bmatrix} e^{x} & y \\ 0 & 1 \end{bmatrix}, \quad x, y \in \mathbb{R}$$

Brownian motions on these groups can be constructed by using the multiplicative stochastic integral introduced by McKean in [23] and studied in detail by Ibero in [15]. The idea of this approach is to construct a matrix-valued stochastic integral in the Lie algebra of the group under consideration, project the increments of this integral onto the group itself by using an exponential mapping, and perform the multiplicative "gluing" of the resulting projections. Setting

$$g(0) = e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We see that, on each of the groups specified above, the Brownian motion $g(t)$ has the form

$$g_{T_2}(t) = \begin{bmatrix} e^{W_1(t)} & \int\limits_0^t e^{W_1(s)+W_2(t)-W_2(s)} dW_3(s) \\ 0 & e^{W_2(t)} \end{bmatrix}$$

$$p_{T_2}(t,e,e) \sim \frac{1}{4\sqrt{\pi}} t^{-2} \tag{1}$$

$$g_{\mathrm{Aff}(\mathbb{R})}(t) = \begin{bmatrix} e^{W_1(t)} & \int\limits_0^t e^{W_1(s)} dW_2(s) \\ 0 & 1 \end{bmatrix}$$

$$p_{\mathrm{Aff}(\mathbb{R})}(t,e,e) \sim \sqrt{\frac{\pi}{2}} t^{-\frac{3}{2}} \tag{2}$$

$$g_{T_{2,U}}(t) = \begin{bmatrix} e^{W_1(t)} & \int\limits_0^t e^{2W_1(s)-W_1(t)} dW_2(s) \\ 0 & e^{-W_1(t)} \end{bmatrix}$$

$$p_{T_{2,U}}(t,e,e) \sim \frac{1}{4} t^{-\frac{3}{2}} \tag{3}$$

The purpose of this dissertation is to establish this fact further and discuss diffusion processes on the solvable group T_3 of upper-triangular 3×3 matrices.

CHAPTER 2: TRANSITION DENSITY FUNCTION AND PARTIAL DIFFERENTIAL EQUATIONS

2.1 Partial Differential Equations

Let's describe the partial differential equation's (PDE) problem in the simplest case of the Laplacian:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}$$

acting in $C^2\left(\mathbb{R}^d\right)$. Define $\delta \in \mathbb{R}^d$ as a bounded domain with the regular boundary ∂D. Regularity means that for any point $x_0 \in \partial D$ the part of the boundary near x_0 in the appropriate coordinate system can be presented by the equation:

$$x_i = F_i\left(x_1, .., x_{d-1}, x_{d+1}, ...\right)$$

We will assume that $F_i(\cdot) \subset C^1$, though majority of the future results are applicable to picewise boundaries.

2.1.1 Dirichlet Problems

For given function $f(x) \in C\left(\partial D\right)$ find the solution of the equation:

$$\begin{cases} \Delta u(x) = 0, \quad x \in, \quad u(x) \in C(\bar{D}), \quad (\bar{D}) = D \cup \partial D \\ \\ \quad u(y) = \phi(y), \quad y \in \partial D \end{cases}$$

such functions are known as harmonic functions. They are not only of the class $C^2(D)$, but also analytic inside of the domain D. For some symmetric domains there are exact formulas for the solution of the Dirichlet problem.

2.1.2 Nonhomogeneous problem with Dirichlet boundary conditions

$$\begin{cases} \Delta u(x) + f(x) = 0, \quad x \in D \\ f(\cdot) \in C(D) \\ u(y) = 0, \quad y \in \partial D \end{cases} \tag{4}$$

This equation can be solved by the Fourier method. Consider the spectral problem:

$$-\delta\phi = \lambda\phi, \quad \phi|_{\partial D} = 0$$

The general results in the functional analysis give the existence of the complete orthonormal basis of the eigenfunctions $\{\phi_i(x), i \geq 1, x \in D\}$ in $L^2(D, dx)$:

$$\begin{cases} (\phi_i, \phi_j) = \int\limits_D \phi_i(x)\phi_j(x)dx = \delta_{i,j} \\ -\Delta\phi_i(x) = \lambda_i\phi_i(x) \\ \lambda_i > 0 \\ \phi_i|_{\partial D} = 0 \end{cases}$$

Using this basis we can solve Eq. (4). Put

$$u(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \quad f(x) = \sum_{n=1}^{\infty} a_n\phi_n(x)$$

then

$$\Delta u(x) = -\sum_{n=1}^{\infty} c_n\lambda_n\phi_n(x) + \sum_{n=1}^{\infty} a_n\phi_n(x)$$

Hence: $c_n = \frac{a_n}{\lambda_n}$ and

$$u(x) = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{\lambda_n} \phi_n(x)$$

This series converges at least in $L^2(D, dx)$.

2.1.3 Parabolic Problem

Let's consider the parabolic problem, which includes the times and space variables. Such problems appear in the description of the heat energy propagation and diffusion. The simplest equation here is the Cauchy problem in the full space:

$$\begin{cases} \dfrac{\partial u(t, \mathbf{x})}{\partial t} = \Delta u(f, x) \\ u(0, \mathbf{x}) = \phi(\mathbf{x}) \in C\left(\mathbb{R}^d\right) \end{cases} \tag{5}$$

Note that for $d = 1$, the solution of this equation is related to Brownian motion and is given by the formula:

$$\begin{cases} u(t, x) = \displaystyle\int_{\mathbb{R}^2} p(t, x, y)\phi(y)dy \\ p(t, x, y) = p(t, 0, y - x) = \dfrac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \end{cases}$$

We will use Eq. (5) in a different form:

$$\begin{cases} \dfrac{\partial u(t, \mathbf{x})}{\partial t} = \dfrac{1}{2} \Delta u(f, \mathbf{x}) \\ u(0, \mathbf{x}) = \dfrac{1}{(2\pi t)^{\frac{d}{2}}} \displaystyle\int_{\mathbb{R}^d} e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{2t}} d\mathbf{y} \end{cases}$$

Let's consider the parabolic problems in the cylindrical domains $[0,T]\times D \subset \mathbb{R}^d$. For example, a nonhomogeneous Dirichlet problem:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + f(t,x), \quad f(t,\cdot)|_{\partial D} \; = 0 \\ u(t,y) = 0, \quad y \in \Delta D, \quad t \in [0,T] \\ u(t,x) = 0, \quad x \in D \end{cases}$$

A solution of this problem can be expressed in terms of eigenfunctions $(\phi_n(x), n \geq 1)$:

$$u(t,x) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x), \quad f(t,x) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$$

Then:

$$\sum_{n=1}^{\infty} \frac{\partial c_n(t)}{\partial t}\phi_n(x) = -\sum_{n=1}^{\infty} c_n(t)\lambda_n\phi_n(x) + \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$$

which means that

$$c_n(t) = \int_0^t e^{-\lambda_n(t-s)} a_n(s)ds$$

and

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} \int_0^t e^{-\lambda_n(t-s)} \left(f(s,\cdot), \phi_n\right) \phi_n(x)ds \\ &= \sum_{n=1}^{\infty} \int_0^t e^{-\lambda_n(t-s)} \left(\int_D f(s,y), \phi_n(y)\phi_n(x)dy \right) ds \\ &= \int_0^t \left[\int_D q(t-s,x,y)f(s,y)dy \right] ds \end{aligned}$$

where

$$q(t-s,x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n(t-s)}\phi_n(y)\phi_n(x)$$

The kernal $q(t-s,x,y)$ is the transition probability of the Brownian motion inside D with the annihilation on its boundary ∂D.

Consider the self-adjoint elliptic operator:

$$(\mathcal{A}f)(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial f}{\partial x_j}\right) \tag{6}$$

Definition 12. A matrix $[a_{i,j}(x)]$ is called positive-definite if $\forall \left(y \in \mathbb{R}^d\right) \neq 0$ we have

$$\sum a_{i,j}(x) y_i y_j > 0$$

Assume that the matrix $[a_{i,j}(x)]$ in Eq. (6) is symmetric and strictly positive-definite. If there exists a positive $\lambda_0 \leq 1$ such that $\forall \left(y \in \mathbb{R}^d\right)$ we have

$$\lambda_0 \sum_{i=0}^{d} y_i^2 \leq \sum_{i,j} y_i y_j \leq \lambda_0^{-1} \sum_{i=0}^{d} y_i^2$$

Then, we call the operator $\mathcal{A}$ uniformly elliptic and λ_0 the constant of ellipticity. If $a_{i,j}(x) = \delta_{i,j}$ and $\lambda_0 = 1$, then $\mathcal{A}$ is Laplacian. In the general theory elements $a_{i,j}(x)$ are only measurable (see Friedman [9]).

2.2 Infinitesimal Generator

The infinitesimal generator is a partial differential operator that encodes a lot of information about the stochastic process. Let $X_t : [0, \infty] \times \Omega \to \mathbb{R}^n$ defined on a probability space $(\Omega, \mathscr{F}, P)$ be an Itö diffusion satisfying a stochastic differential equation of the form:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

where W is an m-dimensional Brownian motion and $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$. The infinitesimal generator of X_t is the operator L, which is defined for

$f : \mathbb{R}^n \to \mathbb{R}$ by:

$$(\mathscr{L} f)(x) = \lim_{t \to 0} \frac{E_x[f(X_t)] - f(x)}{t} \tag{7}$$

Theorem 2. For any $f \in C^2\left(\mathbb{R}^d\right)$ (twice differentiable with continuous second derivative) such that the limit in Eq. (7) exists at a point $\mathbf{x} \in \mathbb{R}^d$, the infinitesimal generator of X can be presented in the following form:

$$(\mathscr{L} f)(\mathbf{x}) = \sum_{i=1}^{d} b_i(\mathbf{x}) \frac{\partial f(\mathbf{x})}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \left(\sigma(\mathbf{x})\sigma(\mathbf{x})^T\right)_{i,j} \frac{\partial^2 f(\mathbf{x})}{\partial x_i, \partial x_j}$$

It is known from the general theory of PDE's [9] that the solution to the following parabolic problem exists and unique:

$$\begin{cases} \dfrac{\partial u(t,\mathbf{x})}{\partial t} = \mathscr{L} u(t,\mathbf{x}) = \displaystyle\sum_{i=1}^{d} b_i(\mathbf{x}) \frac{\partial f(\mathbf{x})}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \left(\sigma(\mathbf{x})\sigma(\mathbf{x})^T\right)_{i,j} \frac{\partial^2 f(\mathbf{x})}{\partial x_i, \partial x_j} \\ u(0,\mathbf{x}) = \phi(\mathbf{x}) \in C\left(\mathbb{R}^d\right) \end{cases}$$

Moreover, the solution to the problem above can be presented in the following form:

$$u(t,\mathbf{x}) = \int_{\mathbb{R}^d} p(t,\mathbf{x},\mathbf{y})\phi(\mathbf{y})d\mathbf{y}$$

where $p(t,\mathbf{x},\mathbf{y})$ is the fundamental solution of the same problem:

$$\begin{cases} \dfrac{\partial p}{\partial t} = \mathscr{L}_{\mathbf{x}} p \\ p(0,\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) \end{cases}$$

where δ-function is the Dirac delta function and it is expressed as

$$\delta(\mathbf{x}-\mathbf{y}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip(\mathbf{x}-\mathbf{y})} dp$$

The Dirac delta can be loosely thought of as a function on the real line which is zero everywhere except at the origin [10], where it is infinite and it is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

Due to the max principle $p(t, x, y) > 0$, $t > 0$. Consider the case when $\phi(y) = 1$, then $u(t, x) = 1$; and one can conclude that $\int_{\mathbb{R}^d} p(t, x, y)dy = 1$, $\forall x \in \mathbb{R}^d$, $t > 0$. Finally, the solution at the moment $t + s$ can be constructed in two steps: solve the problem on $[0, s]$ and take $u(s, x)$ as the new initial function for the parabolic problem on $[s, s+t]$. It will lead to the fundamental relation:

$$p(t + s, \mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^d} p(s, \mathbf{x}, \mathbf{z})p(t, \mathbf{z}, \mathbf{y})d\mathbf{z}$$

The last formula gives the simplest manifestation of the Markov property for X_t, $t \geq 0$.

2.3 The Fundamental Solution of a Linear Parabolic Problem

Theorem 3. A solution of the parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathscr{L}u \\ u(0, x) = \psi(x) \in C\left(\mathbb{R}^d\right) \end{cases}$$

can be presented in terms of the fundamental kernal $p(t, x, y)$, which is the transition density of the diffusion process with the generator $\mathscr{L}$:

$$\begin{cases} u(t, x) = \displaystyle\int_{\mathbb{R}^d} p(t, x, y)\psi(y)dy \\ \dfrac{\partial p(t, x, y)}{\partial t} = \mathscr{L}_x p(t, x, y) \\ p(0, x, t) = \delta_y(x) \end{cases}$$

and for ant $t > 0$, $x, y \in \mathbb{R}^d$ we have upper and lower Gaussian estimates:

$$c_1^- e^{-c_0^- \frac{(x-y)^2}{t}} \leq p(t,x,y) \leq c_1^+ e^{-c_0^+ \frac{(x-y)^2}{t}} \tag{8}$$

where constants $c_{1,0}^{\pm}$ depend only on the dimension d and the ellipticity constant λ_0.

We can assume that $a_{i,j}(x) \in C^\infty\left(\mathbb{R}^d\right)$and approximate "bad" coefficients by C^∞ infinity coefficient. The central fact is that constants in Eq. (8) are independent on smoothness of coefficients.

The general (non-symmetric) elliptic operator has the Fokker-Planck form:

$$\begin{aligned}\mathscr{L}f(x) &= \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} a_{i,j}(x) \frac{\partial f}{\partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial f}{\partial x_i} \\ &= (\Box, a\nabla f) + (b, \nabla f)\end{aligned}$$

where the matrix $[a_{i,j}(x)]$ is called the diffusion tensor and the vector $[b_i(x)]$ is the drift. In the mathematical literature the operator $\mathscr{L}$ is usually presented in Kolmogorov's form:

$$\mathscr{L}f(x) = \sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \hat{b}_i(x) \frac{\partial f}{\partial x_i}$$

Let's point out that $\hat{b}_i(x)$ is not a drift. The standard assumption in the Itô theory is $a_{i,j}(x), b_i(x) \in \text{Lip}\left(\mathbb{R}^d\right)$:

$$\begin{aligned}\sum_{i,j=1}^{d} |a_{i,j}(x) - a_{i,j}(y)| &\leq \mathscr{L}(x-y) \\ \sum_{i=1}^{d} |b_i(x) - b_i(y)| &\leq \mathscr{L}(x-y)\end{aligned}$$

In addition, if $det[a_{i,j}(\cdot)] > 0$, then the parabolic equation

$$\begin{cases} \dfrac{\partial p(t,x,t)}{\partial t} = \mathscr{L} p(t,x,y) \\ \\ p(0,x,y) = \delta_y(x) \end{cases} \tag{9}$$

has unique strictly positive solution: the transition density of the corresponding Markov diffusion process. See details in [1].

2.4 Construction of Markov processes in Terms of Transition Density Function

Let $(X, \mathscr{F}, \mu)$ be the measure space and $p(t,x,y)$ be the transition density of some Markov process $x(t)$, $x \geq 0$, which means

$$P_x\{x(t) \in \Gamma\} = \int_\Gamma p(t,x,y)\mu(dy)$$

The transition density must satisfy Chapman-Kolmogorov relation:

$$p(t+s,x,y) = \int p(t,x,z)p(s,z,y)\mu(dz)$$

$$\int p(t,x,y)\mu(dy) = 1$$

for $0 < t_1 < t_2 < ... < t_n$ and $\Gamma_1, ..., \Gamma_n \subset \mathscr{F}$. One can define the finite dimensional distributions

$$\begin{aligned} m_{t_1,t_2,...,t_n}(\Gamma_1,...,\Gamma_n) = &\int_{\Gamma_1} p(t_1,x_1,z_1)\mu(dz_1) \\ &\cdot \int_{\Gamma_2} p(t_2-t_1,z_1,z_2)\mu(dz_2) \\ &\cdot ... \cdot \int_{\Gamma_n} p(t_n-t_{n-1},z_{n-1},z_n)\mu(dz_n) \end{aligned}$$

They satisfies the conditions of the Kolmogorov extension theorem. The symmetry is obvious, the projectivity follows from Chapman-Kolmogorov equation. The extension theorem proves the existence of the process $x(t)$ with given finite dimensional distributions in the case of the countable time t. The existence of the process with continuous trajectories or trajectories continuous from the right (for the jumping processes) one can prove under stronger restrictions on $(X, \mathscr{F}, \mu)$ and $p(t,x,y)$.

Let $x = \mathbb{R}^d$, $\mathscr{F} = \mathscr{B}(\mathbb{R}^d)$, $\mu(dx) = dx$ (Lebesgue measure on $\mathbb{R}^d$). Consider the uniformly elliptic operator $\mathscr{L}$ either in the form

$$(\mathscr{L} f)(x) = \sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{i,j}(x) \frac{\partial f}{\partial x_j} \right)$$

or non-symmetric uniformly elliptic operator

$$(\mathscr{L} f)(\mathbf{x}) = \sum_{i,j} a_{i,j}(\mathbf{x}) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_i b_i(\mathbf{x}) \frac{\partial f}{\partial x_i}$$

where $[a_{i,j}(\cdot)], \quad [b_i(\cdot)] \in \mathrm{Lip}\left(\mathbb{R}^d\right)$.

In both cases we have existence-uniqueness theorem for the fundamental solution $p(t,x,y)$ of the parabolic problem defined in Eq. (9). Since equation $\frac{\partial u}{\partial y} = \mathscr{L} u$ with initial condition $u(0,x) = 1$ has solution $u(t,x) = 1$ means that $\int\limits_{\mathbb{R}^d} p(t,x,y)dy = 1$. Solution of the problem $\frac{\partial u}{\partial y} = \mathscr{L} u$, $u(0,x) = \psi(x) \in C\left(\mathbb{R}^d\right)$ at the moment $(t+s)$ equals $\int\limits_{\mathbb{R}^d} p(t+s,x,y)\psi(y)dy$ or $\int\limits_{\mathbb{R}^d} p(\xi,x,z)dz \int\limits_{\mathbb{R}^d} p(\xi,z,y)\psi(y)dy$. It leads to the Chapman-Kolmogorov relation.

Let's recall another result by Kolmogorov: if the random process $x(t)$, $t \in [0,\tau]$ with values in $\mathbb{R}^d$ satisfies the relation:

$$E\left[X(t+s) - X(t)\right]^{\alpha} \leq cs^{1+\delta}, \quad \delta > 0, \quad \alpha > 0$$

then it has the P-a.s. continuous modification.

For the Markov process with transition density $p(t, x, y)$ the condition above holds if

$$\int\limits_{\mathbb{R}^d} |y - x|^\alpha p(t, x, y) dy \leq cs^{1+\delta}, \quad \forall \left(x, y \in \mathbb{R}^d\right), \quad \delta > 0, \quad \alpha > 0, \quad s \in [0, \delta_1]$$

Lemma 4 (Nash-Aronson estimate). In the self-ajoint case

$$\int\limits_{\mathbb{R}^d} |y - x|^4 p(t, x, y) dy \leq cs^2$$

where constant c depends only on d and the constant of the uniform ellipticity.

2.5 Hörmander's Condition

Vector fields in $\mathbb{R}^d$ can be identified with the first order differential operator:

$$(Xf)(x) = \sum_{i=1}^{d} a_i(x) \frac{\partial f}{\partial x_i}$$

$$\overrightarrow{X(x)} = \{a_i(x), \quad i = 1, 2, ..., d\}$$

$$\{a_i(x), \quad i = 1, 2, ..., d\} \in C^\infty$$

The class of such operators (vector fields) forms the Lee algebra with operations of addition, multiplication by the constant and the Poisson bracket (or commutator). if

$X = \sum\limits_{i=1}^{d} a_i(x)\frac{\partial}{\partial x_i}$ and $Y = \sum\limits_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$, then $[X,Y] = XY - YX$, which means:

$$\begin{aligned}
\big([X,Y]f\big)(x) &= \sum_{i=1}^{d} a_i(x)\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d} b_j(x)\frac{\partial f}{\partial x_j}\right) - \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d} a_j(x)\frac{\partial f}{\partial x_j}\right) \\
&= \sum_{i=1}^{d}\sum_{j=1}^{d} a_i(x)b_j(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d}\sum_{j=1}^{d} a_i(x)\frac{\partial b_j(x)}{\partial x_i}\frac{\partial f}{\partial x_j} \\
&- \sum_{i=1}^{d}\sum_{j=1}^{d} a_i(x)b_j(x)\frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{d}\sum_{j=1}^{d} b_i(x)\frac{\partial a_j(x)}{\partial x_i}\frac{\partial f}{\partial x_j} \\
&= \sum_{j=1}^{d}\left[\sum_{i=1}^{d} a_i(x)\frac{\partial b_j(x)}{\partial x_i} - \sum_{i=1}^{d} b_i(x)\frac{\partial a_j(x)}{\partial x_i}\right]\frac{\partial f}{\partial x_j}
\end{aligned}$$

Lee algebra can contain all C^∞ vector fields on $\mathbb{R}^d$ on subclass of such fields closed with respect to linear operations and multiplication.

Consider on $\mathbb{R}^d$ the degenerated elliptic operator

$$\mathscr{L} = \sum_{i=1}^{m} X_i^2 + Y \tag{10}$$

where $X_i = \sum\limits_{j=1}^{d} a_{i,j}(x)\frac{\partial}{\partial x_j}$, for $i = 1, 2, ..., m$, $m < d$ and $Y = \sum\limits_{j=1}^{d} b_i(x)\frac{\partial}{\partial x_j}$.

It is the diffusion operator with degenerated diffusion matrix of the order m.

The corresponding diffusion process $X(t)$ can be constructed as the solution of Ito's SDE with smooth coefficients, which means that we have to assume that the derivatives of a_i, b_j are bounded, i.e. $a_i, b_j \in \mathrm{Lip}\,(\mathbb{R}^2)$.

Process $X(t)$ has the transition density function, i.e. the measure:

$$P(t,x,\Gamma) = P_X\{X(t) \in \Gamma\}, \quad \Gamma \in \mathscr{B}(\mathbb{R}^d)$$

which satisfies the Chapman-Kolmogorov's equation:

$$P(t+s,s,\Gamma)=\int\limits_{\mathbb{R}^d} p(t,x,\mu(dz))p(s,z,\Gamma)$$

Theorem 5 (Hörmander's condition)*.* Consider all commutators generated by the vector fields X_1, X_2, ..., X_m except Y. which means:

$$\begin{aligned}
X_1, X_2, ..., X_m, Y\\
[X_i, Y], \quad i=1,2,...,m\\
[X_i, X_j], \quad i,j=1,2,...,m; i\neq j\\
[[X_i, Y], X_j]\,,\\
[[X_i, Y], Y]\,,\\
[[X_i, X_j], X_k]\,, ...
\end{aligned}$$

Assume that for an arbitrary point $x \in \mathbb{R}^d$, one can find a set of vector fields (from this condition) which forms the basis in the linear space with the origin x, i.e. such d-fields are linearly independent. Then, process $X(t)$ has C^∞ transition density $p(t,x,y)$.

Proof. The detailed proof can be found in [14]. □

2.6 The Parametrix Method

How to construct $p(t,\mathbf{x},\mathbf{y})$, which is the fundamental solution of

$$\begin{cases} \dfrac{\partial p(t,\mathbf{x},\mathbf{y})}{\partial t} = \mathscr{L}_{\mathbf{x}} p(t,\mathbf{x},\mathbf{y}) = \sum\limits_{ij} a_{ij}(\mathbf{x}) \dfrac{\partial^2 p(t,\mathbf{x},\mathbf{y})}{\partial x_i \partial x_j} + \sum\limits_i b_i(\mathbf{x}) \dfrac{\partial p(t,\mathbf{x},\mathbf{y})}{\partial x_i} \\ \\ p(0,\mathbf{x},\mathbf{y}) = \delta_{\mathbf{y}}(\mathbf{x}) \end{cases}$$

in the case of sufficiently smooth coefficient, say $a_{ij}, b_i \in C^1\left(\mathbb{R}^d\right)$

Let's illustrate it in the simplified situation when $\mathbf{b} = 0$. We have to solve the equation:

$$\begin{cases} \dfrac{\partial p(t,\mathbf{x},\mathbf{y})}{\partial t} = \sum\limits_{ij} a_{ij}(\mathbf{x}) \dfrac{\partial^2 p(t,\mathbf{x},\mathbf{y})}{\partial x_i \partial x_j} \\ \\ p(0,\mathbf{x},\mathbf{y}) = \delta_{\mathbf{y}}(\mathbf{x}) \end{cases} \tag{11}$$

Let's "freeze" the coefficients a_{ij} at the point $\xi \in \mathbb{R}^d$ (the singularity of fundamental solution), i.e. consider the parabolic equation (over t and $\mathbf{x} \in \mathbb{R}^d$)

$$\begin{cases} \dfrac{\partial q(t,\mathbf{x},\xi)}{\partial t} = \sum\limits_{ij} a_{ij}(\xi) \dfrac{\partial^2 q(t,\mathbf{x},\xi)}{\partial x_i \partial x_j} \\ \\ q(0,\mathbf{x},\xi) = \delta_{\xi}(\mathbf{x}) \end{cases}$$

This is an equation with constant coefficients and it can be solved by the Fourier transform. Set

$$\hat{q}(t,\mathbf{k},\xi) = \int\limits_{\mathbb{R}^d} q(t,\mathbf{x},\xi) e^{i(\mathbf{k},\mathbf{x})} dx$$

then

$$\begin{cases} \dfrac{\partial \hat{q}}{\partial t}(t,\mathbf{k},\xi) = -\sum\limits_{ij}^{d} a_{ij}(x) k_i k_j \hat{q}(t,\mathbf{k},\xi) \\ \\ \hat{q}(0,\mathbf{k},\xi) = e^{i(k,x)} \end{cases}$$

then

$$\hat{q}(t,\mathbf{x},\xi) = e^{-t \sum\limits_{ij}^{d} a_{ij}(x) k_i k_j} e^{i(k,\xi)}$$

Let $a^{ij}(\mathbf{x})$ be the inverse matrix to $a_{ij}(\mathbf{x})$. Observe that $q(t,\mathbf{x},\xi)$ is Gaussian with covariance matrix $[a_{ij}(\xi)]$ and expectation ξ. Due to the well known formulas for the d-dimensional Gaussian distribution, the inverse Fourier transform finally gives us:

$$q(t,\mathbf{x},\xi) = \frac{1}{(4\pi)^{\frac{d}{2}} \sqrt{\det(a_{ij}(\xi))}} \mathrm{Exp} \left(-\frac{1}{4t} \sum_{i,j=1}^{d} a^{ij}(\xi)(x_i - \xi_i)(x_j - \xi_j) \right) t^{-\frac{d}{2}}$$

For the equation

$$\begin{cases} \dfrac{\partial u(t,\mathbf{x})}{\partial t} = \sum_{ij}^{d} a_{ij}(\mathbf{x}) \dfrac{\partial^2 u}{\partial x_i \partial x_j} + f(t,\mathbf{x}) \\ \\ u(0,\mathbf{x}) = 0 \end{cases}$$

one can use the Duhamel's formula

$$u(t,\mathbf{x}) = \int_0^t ds \int_{\mathbb{R}^d} q(t-s,\mathbf{x},\xi) f(s,\xi) d\xi$$

then we present Eq. (11) in the following form (for a fixed ξ)

$$\begin{cases} \dfrac{\partial p(t,\mathbf{x},\xi)}{\partial t} = \sum_{i,j=1}^{d} a_{ij}(\xi) \dfrac{\partial^2 p}{\partial x_i \partial x_j} + \sum_{i,j=1}^{d} \left[a_{ij}(\mathbf{x}) - a_{ij}(\xi)\right] \dfrac{\partial^2 p}{\partial x_i \partial x_j} \\ \\ p(0,\mathbf{x},\xi) = \delta_\xi(\mathbf{x}) \end{cases}$$

note that here $f(t,\mathbf{x}) = (\mathscr{L}_{\mathbf{x}} - \mathscr{L}_\xi) p(t,\mathbf{x},\xi)$ which means

$$p(t,\mathbf{x},\xi) = q(t,\mathbf{x},\xi) + \int_0^t ds \int_{\mathbb{R}^d} q(t-s,\mathbf{x},\xi)(\mathscr{L}_{\mathbf{x}} - \mathscr{L}_\xi) p(t,\mathbf{x},\xi) d\xi \tag{12}$$

This is the integral equation of the Volterra type. Unfortunately for small $t-s$ the Kernal $q(t-s,\mathbf{x},\mathbf{y})$ is very singular, but $|a_{ij}(x) - a_{ij}(y)| \leq \mathscr{L}|x-y|$ and this fact (even $|a_{ij}(x) - a_{ij}(y)| \leq c_1|x-y|^\alpha, \quad 0 < \alpha < 1$) compensate this singularity. Calculations and estimations in the successive approximations of Eq. (12) are complicated. See details in Friedman [9].

The central result is the existence of the fundamental solution $p(t,\mathbf{x},\mathbf{y})$ and it's upper estimate

$$p(t,\mathbf{x},\mathbf{y}) \leq c_1(T,\Lambda,d)^+ \mathrm{Exp}\left(-\frac{c_2(T,\Lambda,d)^+ |\mathbf{x}-\mathbf{y}|^2}{t}\right) t^{-\frac{d}{2}}, \quad t \in [0,T] \tag{13}$$

in contrast to the self-adjoint case, the constants $C^{+}_{1,2}$ depend on T and coefficient $a_{ij}(x)$ are at least Gölder's. It is sufficient to prove the existence of the diffusion process with the generator $\mathscr{L}$.

If we add to $\mathscr{L}$ the first order term $\left(\overrightarrow{b}(x), \nabla\right)$ and assume that $|\overrightarrow{b}(x)| \leq c_0$ and $\overrightarrow{b}(x) \in Lip\left(\mathbb{R}^d\right)$, then the estimate Eq. (13) is still valid. The proof of these results is in [9].

CHAPTER 3: ASIAN AND BASKET OPTIONS

All stochastic processes discussed here are real-valued. They are defined on a common probability space $(\Omega, \mathscr{F}, P)$. Notation $X(t) \stackrel{d}{=} Y(t)$ means the equality in law of X and Y.

The distribution of a the integral of geometric Brownian motion over a finite time interval with applications to risk theory and pension funding was studied by Dufresne [5]. In this paper he showed that the integral of geometric Brownian motion has the same distribution as a random variable with inverse gamma distribution. In mathematical finance, this integral is being used in Asian option pricing. In this area Yor [28] made a significal contribution by deriving an explicit formula for the distribution and moments of the integral of geometric Brownian motion. He used a Bessel process and the Laplace transform method in the derivation of his results. Using his results we can find an alternative way to derive Bougerol's identity in law [2].

Beside finance, geometric Brownian motion is being used in accurate estimation of species divergence times from the analysis of genetic sequences relies on probabilistic models of evolution of the rate of molecular evolution [25].

Let's fix $t > 0$ and let $W = (W_s)_{s\in[0,t]}$ be a Brownian motion. Let $B = (B_s)_{s\in[0,t]} = (W_s|W_t = 0)$ be a Brownian bridge from $(0, 0)$ to $(t, 0)$. For a Brownian bridge one can find the following four equivalent definitions:

$$dB_s = dW_s - \frac{B_s}{t-s}ds \tag{14}$$

$$B_s = (t-s)\int_0^s \frac{dW_u}{t-u} \tag{15}$$

$$B_s = W_s - \frac{s}{t}W_t \tag{16}$$

$$B_s = \frac{t-s}{\sqrt{t}}W\left(\frac{s}{t-s}\right) \tag{17}$$

The Eq. (14) and Eq. (15) define the the same process. The equality between Eq. (14) and Eq. (16) is only an equality in law. The representation in Eq. (16) comes from an orthogonal decomposition of Gaussian variables. Indeed, the Brownian bridge is Gaussian with

$$E[B_s] = 0$$

$$Cov(B_u, B_v) = u \wedge v - \frac{uv}{t}$$

The Brownian bridge can be defined only up to distribution. The distribution of W and B are equivalent and orthogonal. From Eq. (14) by Girsanov's theorem [11] we get the Radon-Nykodym derivative:

$$\frac{dB_s}{dW_s} = \exp\left(-\int_0^s \frac{B_u}{t-u}dW_u - \frac{1}{2}\int_0^s \left(\frac{B_u}{t-u}\right)^2 du\right)$$

Using the self-similarity property of the Brownian motion, we can conclude the following:

$$\{B_s\}_{s\in[0,t]} \overset{d}{=} \{B_{ut}\}_{u\in[0,1]}$$
$$\frac{t-s}{\sqrt{t}}W\left(\frac{s}{t-s}\right) \overset{d}{=} \frac{t-tu}{\sqrt{t}}W\left(\frac{tu}{t-tu}\right)$$
$$\{B_s\}_{s\in[0,t]} \overset{d}{=} \sqrt{t}(1-u)W\left(\frac{u}{1-u}\right)$$

Hence, the self-similarity property of the Brownian bridge is:

$$\{B_s\}_{s\in[0,t]} \overset{d}{=} \sqrt{t}\{B_s\}_{s\in[0,1]} \tag{18}$$

Let's consider several stochastic processes related to the exponential functional $\left\{ A_t^{(\mu)} \right\}$ defined in [3]. In particular, for a continuous process $\phi : (0,\infty) \to \mathbb{R}$, we define

$$A_t^{\mu}(\phi) = \int_0^t e^{2\phi(s)+\mu s} ds \tag{19}$$

$$A_t(\phi) = \int_0^t e^{2\phi(s)} ds \quad and \quad Z_t(\phi) = e^{2\phi(t)} A_t(\phi) \tag{20}$$

Also, for two continuous processes ϕ_1 and ϕ_2 define

$$A_t(\phi_1,\phi_2) = \int_0^t e^{2\phi_1(s)-2\phi_2(s)} ds \quad and \quad Z_t(\phi_1,\phi_2) = e^{2\phi_2(t)} A_t(\phi_1,\phi_2) \tag{21}$$

3.1 Asian Options

An Asian option is a special type of option contract, where the payoff is determined by the average underlying price over some pre-set period of time. One advantage of Asian options is that these reduce the risk of market manipulation of the underlying instrument at maturity [16]. Another advantage of Asian options involves the relative cost of Asian options compared to European or American options. Because of the averaging feature, Asian options reduce the volatility inherent in the option; therefore, Asian options are typically cheaper than European or American options.

We consider a security market with a risk asset with a constant risk-free return rate $r > 0$. Let's assume that the price process dynamics is

$$dS(s) = rS(s)ds + \sigma S(s)dW(s)$$

Where $W(s)$ for $s \geq 0$ is standard Brownian motions and the volatility σ is a positive constant. The price of the asset is

$$S(s) = S(0)e^{\left(r-\frac{\sigma^2}{2}\right)s+\sigma W(s)}$$

The equation above was used in the derivation of the Black–Scholes model [6]. Merton was the first to publish a paper [24] expanding the mathematical understanding of the options pricing model, and coined the term "Black–Scholes options pricing model". Merton and Scholes received the 1997 Nobel Memorial Prize in Economic Sciences for their work.

Then the average price of underlying asset is defined in the following way:

$$\frac{1}{t}\int_0^t S(s)ds = \frac{S(0)}{t}\int_0^t e^{\nu\sigma s+\sigma W(s)}ds = \frac{4S_j(0)}{t\sigma^2}\int_0^{\frac{\sigma^2 t}{4}} e^{2\left(\frac{2\nu s}{\sigma}+\sigma W(s)\right)}ds$$

where $\nu = \frac{r}{\sigma} - \frac{1}{2}\sigma$, which is the same that

$$\frac{1}{t}\int_0^t S_j(s)ds = \frac{4S_j(0)}{t\sigma^2}A^{\frac{2\nu}{\sigma}}_{\frac{\sigma^2 t}{4}}(W) \tag{22}$$

In addition, by Girsanov's theorem [11] we can reduce $A^{\frac{2\nu}{\sigma}}_{\frac{\sigma^2 t}{4}}(W)$ to $A_{\frac{\sigma^2 t}{4}}(W)$.

3.2 Modified Asian and European-Asian Geometric Basket Options

A basket option is a type of financial derivative where the underlying asset is a group, or basket, of commodities, securities, or currencies. As with other options, a basket option gives the holder the right, but not the obligation, to buy or sell the basket at a specific price, on or before a certain date. We consider a security market with two independent risk assets with a constant risk-free return rate $r > 0$. Let's assume that the price process dynamics are

$$dS_1(s) = rS_1(s)ds + \sigma S_1(s)dW_1(s)$$

$$dS_2(s) = rS_2(s)ds + \sigma S_2(s)dW_2(s)$$

Where $W_1(s)$ and $W_2(s)$ for $s \geq 0$ are standard Brownian motions and the volatility σ is a positive constant. Further, we assume that the asset prices are uncorrelated. The price of each asset is

$$S_1(s) = S_1(0)e^{\left(r-\frac{\sigma^2}{2}\right)s+\sigma W_1(s)}$$

$$S_2(s) = S_2(0)e^{\left(r-\frac{\sigma^2}{2}\right)s+\sigma W_2(s)}$$

In general, the geometric basket option for N assets is defined in the following way:

$$\left(\prod_{i=1}^{N} S_i(t)\right)^{\frac{1}{N}}$$

where $S_i(t)$ is an asset price at time t for $i = 1, .., N$.

Let's define a modified Asian geometric basket for two assets option as

$$\int_0^t \sqrt{\frac{S_1(s)}{S_2(s)}}ds = \sqrt{\frac{S_1(0)}{S_2(0)}} \int_0^t e^{\frac{\sigma}{2}W_1(s)-\frac{\sigma}{2}W_2(s)}ds$$

Due to the self-similarity of Brownian motion, we note that $W(s) \overset{d}{=\!=} \frac{4}{\sigma} W\left(\frac{\sigma^2 s}{16}\right)$.

The price of the modified Asian geometric basket with two assets will be:

$$\sqrt{\frac{S_1(0)}{S_2(0)}} \int_0^t e^{2W_1\left(\frac{\sigma^2 s}{16}\right) - 2W_2\left(\frac{\sigma^2 s}{16}\right)} ds = \frac{16}{\sigma^2} \sqrt{\frac{S_1(0)}{S_2(0)}} \int_0^{\frac{\sigma^2 t}{16}} e^{2W_1(s) - 2W_2(s)} ds$$

which is the same that

$$\frac{16}{\sigma^2} \sqrt{\frac{S_1(0)}{S_2(0)}} A_{\frac{\sigma^2 t}{16}} \left(W_1, W_2\right) \tag{23}$$

Let's define modified European-Asian geometric basket option with two assets as

$$\int_0^t \sqrt{\frac{S_1(s) S_2(t)}{S_2(s)}} ds = \sqrt{S_1(0)} \int_0^t e^{\frac{\sigma}{2} W_1(s) + \frac{\sigma}{2} W_2(t) - \frac{\sigma}{2} W_2(s)} ds$$

The price of the modified European-Asian geometric basket option with two assets will be:

$$= \sqrt{S_1(0)} \int_0^t e^{2W_1\left(\frac{\sigma^2 s}{16}\right) + 2W_2\left(\frac{\sigma^2 t}{16}\right) - 2W_2\left(\frac{\sigma^2 s}{16}\right)} ds = \frac{16}{\sigma^2} \sqrt{S_1(0)} \int_0^{\frac{\sigma^2 t}{16}} e^{2W_1(s) + 2W_2(t) - 2W_2(s)} ds$$

which is the same that

$$\frac{16}{\sigma^2} \sqrt{S_1(0)} Z_{\frac{\sigma^2 t}{16}} \left(W_1, W_2\right) \tag{24}$$

CHAPTER 4: PROPERTIES OF EXPONENTIAL FUNCTIONALS OF BROWNIAN MOTION AND ITS APPLICATION IN ASIAN AND BASKET OPTIONS

In this chapter, we will formulate and prove several properties of asymptotic behavior of the random variables A_t and $A_t(W_1, W_2)$. Note that

$$A_t(W_1) = E\left[\int_0^t e^{W_1(s)} dW_2(s) \middle| W_1\right]$$

$$A_t(W_1, W_2) = E\left[\int_0^t e^{W_1(s) - W_2(s)} dW_3(s) \middle| W_1 W_2\right]$$

Lemma 6. Let $W(s)$ be a standard Brownian motion. Let $B(s) = (W(s)|W(t) = 0)$ be a Brownian bridge on $[0, t]$ then

$$E\left[\frac{1}{\sqrt{A_t}} \middle| W_t \in dx\right] = E\left[\left(\int_0^t e^{2B(s)} ds\right)^{-\frac{1}{2}}\right] \xrightarrow[t\to\infty]{} \frac{\sqrt{2\pi^3}}{t}$$

Proof. Per Konakov-Menozzi-Molchanov [19], we know that for two independent Brownian motions $W(s)$, $\hat{W}(s)$ for $s \in [0, t]$:

$$P\left(\int_0^t e^{W(s)} d\hat{W}(s) \in dx, W(t) \in dx\right) \sim \sqrt{\frac{\pi}{2}} t^{-\frac{3}{2}}, \quad t \to \infty$$

which means

$$P\left(\int_0^t e^{W(s)} d\hat{W}(s) \in dx \middle| W(t) \in dx\right) = \frac{P\left(\int_0^t e^{W(s)} d\hat{W}(s) \in dx, W(t) \in dx\right)}{P(W(t) \in dx)}$$

$$\sim \sqrt{\frac{\pi}{2}} t^{-\frac{3}{2}} \sqrt{2\pi t} = \frac{\pi}{t}$$

Conclude the following:

$$P\left(\int_0^t e^{B(s)} d\hat{W}(s) \in dx\right) \sim \frac{\pi}{t}, \quad t \to \infty \tag{25}$$

On the other hand $\int_0^t e^{B(s)} d\hat{W}(s)$ is a conditional centered Gaussian process for fixed B, and hence:

$$\begin{aligned} P\left(\int_0^t e^{B(s)} d\hat{W}(s) \in dx\right) &= E\left[P\left(\int_0^t e^{B(s)} d\hat{W}(s) \in dx\right) \middle| B\right] \\ &= \frac{1}{\sqrt{2\pi}} E\left[\left(\int_0^t e^{2B(s)} ds\right)^{-\frac{1}{2}}\right] \end{aligned}$$

Combing the equation above with Eq. (25), we get the statement of this lemma. □

Lemma 7. Let $W_1(s)$ and $W_2(s)$ be two independent standard Brownian motions. Let $B_1(s) = (W_1(s)|W_1(t) = 0)$ and $B_2(s) = (W_2(s)|W_2(t) = 0)$ be two independent Brownian bridges on $[0, t]$ then

$$\begin{aligned} &E\left[\frac{1}{\sqrt{A_t(W_1, W_2)}} \middle| W_1(t) \in dx, W_2(t) \in dx\right] \\ &= E\left[\left(\int_0^t e^{2B_1(s)+2B_2(t)-2B_2(s)} ds\right)^{-\frac{1}{2}}\right] \xrightarrow[t\to\infty]{} \frac{\pi}{\sqrt{2}} t^{-1} \end{aligned}$$

Proof. Per Konakov-Menozzi-Molchanov [19], we know that for two independent Brownian motions $W_1(s)$, $W_2(s)$ and $W_3(s)$ for $s \in [0, t]$:

$$P\left(\int_0^t e^{W_1(s)+W_2(t)-W_2(s)} dW_3(s) \in dx, W_1(t) \in dx, W_2(t) \in dx\right) \sim \frac{1}{4\sqrt{\pi}} t^{-2}, \quad t \to \infty$$

which means

$$P\left(\int_0^t e^{W_1(s)+W_2(t)-W_2(s)}dW_3(s)\in dx\,\middle|\, W_1(t)\in dx, W_2(t)\in dx\right)$$
$$=\frac{P\left(\int_0^t e^{W_1(s)+W_2(t)-W_2(s)}dW_3(s)\in dx, W_1(t)\in dx.w_2(t)\in dx\right)}{P(W_1(t)\in dx)P(W_2(t)\in dx)}$$
$$\sim \frac{2\pi t}{4\sqrt{\pi}}t^{-2}=\frac{\sqrt{\pi}}{2}t^{-1}$$

Conclude the following:

$$P\left(\int_0^t e^{B_1(s)-B_2(s)}dW_3(s)\in dx\right)\sim\frac{\sqrt{\pi}}{2}t^{-1},\quad t\to\infty \tag{26}$$

On the other hand, for fixed B_1 and B_2, $\int_0^t e^{B_1(s)-B_2(s)}dW_3(s)$ is a conditional centered Gaussian process and hence:

$$P\left(\int_0^t e^{B_1(s)-B_2(s)}dW_3(s)\in dx\right)=E\left[P\left(\int_0^t e^{B_1(s)-B_2(s)}dW_3(s)\in dx\,\middle|\, B_1B_2\right)\right]$$
$$=\frac{1}{\sqrt{2\pi}}E\left[\left(\int_0^t e^{2B_1(s)-2B_2(s)}ds\right)^{-\frac{1}{2}}\right]$$

Combing the equation above with Eq. (26), we get the statement of this lemma. □

Lemma 8. Let $B(s)$ be a Brownian bridge on $[0,t]$ then for any $\alpha\in\mathbb{R}^+$:

$$E\left[\left(\int_0^t e^{\alpha B(s)}ds\right)^{-1}\right]=\frac{1}{t} \tag{27}$$

Proof. Let's $\tilde{B}(s)$ for $s\in[0,1]$ be a standard Brownian bridge. Per Donati-Martin [4], for any $\alpha\in\mathbb{R}^+$:

$$E\left[\left(\int_0^1 e^{\alpha\tilde{B}(s)}ds\right)^{-1}\right]=1$$

Let's $B(s)$ for $s \in [0,t]$ be a Brownian bridge. Using the self-similarity property of Brownian bridge, we can conclude the following:

$$E\left[\left(\int_0^t e^{\alpha B(s)} ds\right)^{-1}\right] = \frac{1}{t} E\left[\left(\int_0^1 e^{\alpha\sqrt{t}\tilde{B}(s)} ds\right)^{-1}\right] = \frac{1}{t}$$

□

Lemma 9. Let $B_1(s)$ and $B_2(s)$ be two independent standard Brownian bridges for $s \in [0,t]$, then there exists a Brownian bridge $B_3(s)$ for $s \in [0,t]$ such that

$$B_1(s) - B_2(s) \overset{d}{=} \sqrt{2} B_3(s)$$

Proof. The expected value of each $B_1(s)$ and $B_2(s)$ bridge is zero, with variance $\frac{s(t-s)}{t}$. Thus:

$$\begin{aligned} E\left[B_1(s) - B_2(s)\right] &= E\left[B_1(s)\right] - E\left[B_2(s)\right] = 0 \\ Var\left(B_1(t) - B_2(t)\right) &= E\left[\left(B_1(t) - B_2(t)\right)^2\right] \\ &= E\left[B_1^2(t)\right] + E\left[B_2^2(t)\right] = \frac{2s(t-s)}{t} \end{aligned} \tag{28}$$

Hence there exists another Brownian bridge $B_3(s)$ such that

$$B_1(t) - B_2(t) \overset{d}{=} \sqrt{2} B_3(t)$$

□

The value of Asian option exponentially depends on $\sqrt{t}$, in particular $E[A_t] \sim e^{O(\sqrt{t})}$. In particular, we have the following results that were proven in [22]:

$$\frac{ln\left(\frac{A_t}{t}\right)}{2\sqrt{t}} \xrightarrow[t\to\infty]{d} max_{s\in[0,1]} W(s)$$

The value of modified European-Asian geometric basket option with two assets is very large for $t \to \infty$ exponentially depends on $\sqrt{t}$. This property was formulated in [20] without a proof, so we present its proof in the lemma bellow.

Lemma 10. Let $M(t) = \max_{s \le t} (W_1(s) + W_2(t) - W_2(s))$ where $W_1(s)$ and $W_2(s)$ are independent Brownian motions for $s \in [0, t]$, then

$$\frac{ln\left(Z_t(W_1, W_2)\right)}{2\sqrt{t}} \xrightarrow[t\to\infty]{d} M(1)$$

Proof. Due to the self-similarity of the Brownian motion, we have $W(s) \stackrel{d}{=} cW\left(\frac{s}{c^2}\right)$ for any $s \ge 0$ and $c > 0$, and hence

$$Z_t(W_1, W_2) = \frac{1}{t}\int\limits_0^t e^{2\left(W_1(s)+W_2(t)-W_2(s)\right)} ds \stackrel{d}{=} \int\limits_0^1 e^{2\sqrt{t}\left(W_1(s)+W_2(1)-W_2(s)\right)} ds$$

Per Lemma 9, there exists a Brownian bridge $B(t)$ such that

$$W_1(s) - W_2(s) \stackrel{d}{=} \sqrt{2}B(s) + \big(W_1(1) - W_2(1)\big)s$$

Denote

$$M_1 = \max_{s\in[0,1)} \left[\sqrt{2}B(s) + \big(W_1(1) - W_2(1)\big)s\right]$$

$$m_1 = \min_{s\in[0,1)} \left[\sqrt{2}B(s) + \big(W_1(1) - W_2(1)\big)s\right]$$

For any $a \in (m_1, M_1)$ define $L_1(a) = \lim\limits_{\epsilon\to 0}\int\limits_0^1 1_{|\sqrt{2}B(s)+(W_1(1)-W_2(1))s-a|\le\epsilon} ds$. Since $L_1(a) >$

0 almost surely,

$$
\begin{aligned}
A(t) &\stackrel{d}{=} e^{2\sqrt{t}W_2(1)} \int_0^1 e^{2\sqrt{t}\left(B(s)+\left(W_1(1)-W_2(1)\right)s\right)} ds \\
&= e^{2\sqrt{t}W_2(1)} \int_{m_1}^{M_1} e^{2\sqrt{t}a} L_1(a)da \geq e^{2\sqrt{t}W_2(1)} \int_{M_1-\epsilon}^{M_1} e^{2\sqrt{t}a} L_1(a)da \\
&\geq e^{2\sqrt{t}W_2(1)} e^{2\sqrt{t}(M_1-\epsilon)} \int_{M_1-\epsilon}^{M_1} L_1(a)da
\end{aligned}
$$

Then define $\delta_\epsilon = \int_{M_1-\epsilon}^{M_1} L_1(a)da$ and note that $\delta_\epsilon e^{2\sqrt{t}(W_2(1)+M_1-\epsilon)} \leq A(t) \leq e^{2\sqrt{t}W_2(1)+M_1}$, taking the log of both sides, we get the following:

$$
\begin{aligned}
ln(\delta_\epsilon) + 2\sqrt{t}\,(M_1 + W_2(1) - \epsilon) \leq ln(A(t)) \leq 2\sqrt{t}\,(M_1 + W_2(1)) \\
\frac{ln(\delta_\epsilon)}{2\sqrt{t}} + M_1 + W_2(1) - \epsilon \leq \frac{ln(A(t))}{2\sqrt{t}} \leq M_1 + W_2(1)
\end{aligned}
$$

Due to the fact that $ln(\delta_\epsilon)$ is positive and bounded for some $\epsilon > 0$, we have $\frac{ln(\delta_\epsilon)}{2\sqrt{t}} \xrightarrow[t\to\infty]{p} 0$ and hence $\frac{ln(A(t))}{2\sqrt{t}} \xrightarrow{d} \max_{s\leq 1}\big(W_1(s) + W_2(1) - W_2(s)\big)$. □

The value of Asian option is bounded if the price of the underlying asset is bounded. In particular, we have the following results that were proven in [22]:

$$
E\left[\int_0^t e^{2W_1(s)}ds \,\middle|\, max_{s\in[0,t]}W_1(s) \leq 1\right] \leq \frac{e^2}{2}\left(1 - \frac{2}{\sqrt{2\pi t}\int_1^\infty e^{-\frac{x^2}{2t}}}dx\right)^{-1}
$$

The value of modified European-Asian geometric basket option with two assets is bounded if the prices of the underlying assets are bounded. This property was formulated in [20] without a proof, so we present its proof in the lemma bellow.

Lemma 11. Let $Z_t = \int_0^t e^{2(W_1(s)+W_2(t)-W_2(s))}ds$ where $W_1(s)$ and $W_2(s)$ are independent

Brownian motions for $s \in [0, t]$. Then:

$$\lim_{t\to\infty} E\Big[\, Z_t \big|\, W_1(s) \leq 1, W_2(s) \leq 1, s \leq t\Big] = 0 \tag{29}$$

Proof. Let's define

$$M_1(t) = max_{s\in[0,t]} W_1(s)$$

$$M_2(t) = max_{s\in[0,t]} W_2(s)$$

$$a(t) = E\left[Z_t | M_1(t) \leq 1, M_2(t) \leq 1\right]$$

and notice that

$$\begin{aligned} a(t) &= E\left[\int_0^t e^{2(W_1(s)+W_2(t)-W_2(s))} ds \,\middle|\, M_1(t) \leq 1, M_2(t) \leq 1 \right] \\ &= \int_0^t E\left[e^{2(W_1(s)+W_2(t)-W_2(s))} \middle|\, M_1(t) \leq 1, M_2(t) \leq 1\right] ds \end{aligned}$$

Due to the fact that W_1 and W_2 are independent, we can simplify the formula to the following:

$$a(t) = \int_0^t E\left[e^{2W_1(s)} \middle|\, M_1(t) \leq 1\right] E\left[e^{2W_2(t)-2W_2(s)} \middle|\, M_2(t) \leq 1\right] ds$$

Since Brownian motion has independent increments, we can conclude the following:

$$E\left[e^{2W_2(t)-2W_2(s)} \middle|\, M_2(t) \leq 1\right] = E\left[e^{2W_2(t)-2W_2(s)}\right] = e^{2(t-s)}$$

Now our problem reduces to finding the following integral:

$$a(t) = e^{2t} \int_0^t e^{-2s} E\left[e^{2W_1(s)} \middle|\, M_1(t) \leq 1\right] ds$$

Due to Hölder's inequality, we can conclude the following:

$$
\begin{aligned}
a(t) &\leq \sqrt{\int_0^t e^{4t-4s} ds} \sqrt{\int_0^t \left(E\left[e^{2W_1(s)} \middle| M_1(t) \leq 1\right]\right)^2 ds} \\
&= \frac{\sqrt{e^{4t}-1}}{2} \sqrt{\int_0^t \left(E\left[e^{2W_1(s)} \middle| M_1(t) \leq 1\right]\right)^2 ds}
\end{aligned}
$$

Per Jensen's inequality, we have

$$
\left(E\left[e^{2W_1(s)} \middle| M_1(t) \leq 1\right]\right)^2 \leq E\left[e^{4W_1(s)} \middle| M_1(t) \leq 1\right]
$$

There is exists a constant C_1 for which, the following inequality is true and it was proven in [22]:

$$
E\left[\int_0^t e^{2W_1(s)} ds \middle| M_1 \leq 1\right] \leq C_1
$$

Using the same techniques that was used in [22], we can show that there exists a constant C_2 such that the following is true:

$$
E\left[\int_0^t e^{4W_1(s)} ds \middle| M_1 \leq 1\right] \leq C_2
$$

Therefore there exists a constant C such that

$$
\begin{aligned}
a(t) &\leq \frac{\sqrt{e^{4t}-1}}{2} \sqrt{\int_0^t \left(E\left[e^{2W_1(s)} \middle| M_1(t) \leq 1\right]\right)^2 ds} \\
&\leq \frac{\sqrt{e^{4t}-1}}{2} \sqrt{\int_0^t E\left[e^{4W_1(s)} \middle| M_1(t) \leq 1\right] ds} \leq C
\end{aligned}
$$

$\square$

CHAPTER 5: SOLVABLE GROUPS OF UPPER TRIANGULAR 3X3 MATRICES

1 Approximation of Diffusion by Random Walks

Consider the group T_3 of upper triangular 3×3 matrices of the form

$$T_3 = \left\{ \begin{bmatrix} e^{x_1} & y_1 & z \\ 0 & e^{x_2} & y_2 \\ 0 & 0 & e^{x_3} \end{bmatrix}, x_1, x_2, x_3, y_1, y_2, z \in \mathbb{R} \right\}$$

We are interested in the approximation of the Brownian motion by a discrete random walk on solvable groups on upper triangular matrices. Let $\epsilon > 0$ be a given small parameter. The time step of our random walk $\left(x_n^\epsilon\right)_{n\geq 0}$ will be ϵ^2. In particular for a given time $t > 0$, it makes $n_\epsilon(t) = \left\lfloor \frac{t}{\epsilon^2} \right\rfloor$ steps on the interval $[0, t]$. Set

$$e = g_{\epsilon,0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and for all $r \geq 0$ let

$$g_{\epsilon,r+1} = g_{\epsilon,r}, \quad A_{\epsilon,r+1} = \begin{bmatrix} e^{\epsilon X_{1,k}} & \epsilon Y_{1,k} & \epsilon Z_{1,k} \\ 0 & e^{\epsilon X_{2,k}} & \epsilon Y_{2,k} \\ 0 & 0 & e^{\epsilon X_{3,k}} \end{bmatrix} \tag{30}$$

where the $X_{1,k}$, $X_{1,k}$, $X_{3,k}$, $Y_{1,k}$, $Y_{2,k}$ and $Z_{1,k}$ are symmetric Bernoulli random variables, i.e. $\mathbb{P}[X_{1,k} = 1] = \mathbb{P}[X_{1,k} = -1] = \mathbb{P}[X_{2,k} = 1] = \mathbb{P}[X_{2,k} = -1] = \mathbb{P}[X_{3,k} = 1] = \mathbb{P}[X_{3,k} = -1] = \mathbb{P}[Y_{1,k} = 1] = \mathbb{P}[Y_{1,k} = -1] = \mathbb{P}[Y_{2,k} = 1] = \mathbb{P}[Y_{2,k} = -1] = \mathbb{P}[Z_{1,k} = 1] = \mathbb{P}[Z_{1,k} = -1] = \frac{1}{2}$ defined on some given probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let's also assume that they are independent for a fixed k and their sets for different k are also independent.

Note that for any r our $g_{\epsilon,r}$ has the following form:

$$g_{\epsilon,r} = \begin{bmatrix} e^{x_{1,r}} & y_{1,r} & z_{1,r} \\ 0 & e^{x_{2,r}} & y_{2,r} \\ 0 & 0 & e^{x_{3,r}} \end{bmatrix}$$

where the representation of $x_{i,r}$ is as follows:

$$\begin{cases} x_{i,1} = \epsilon X_{i,1} \\ x_{i,r} = \epsilon \sum_{j=1}^{r} X_{1,j} \end{cases} \implies x_{i,r} = \epsilon \sum_{j=1}^{r} X_{i,j}$$

.

For simplicity, let's assume that $x_{i,0} = 0$ for all i.

And, the representation of $y_{1,r}$ is as follows:

$$\begin{cases} y_{1,1} = \epsilon Y_{1,1} \\ y_{1,r} = \epsilon Y_{1,r} e^{x_{1,r-1}} + y_{1,r-1} e^{\epsilon X_{2,r}} = \epsilon Y_{1,r} e^{x_{1,r-1}} + y_{1,r-1} e^{x_{2,r} - x_{2,r-1}} \end{cases}$$

which means that the explicit formula for $y_{1,r}$ is as follows:

$$y_{1,r} = \epsilon \sum_{k=2}^{r} Y_{1,k} e^{x_{1,k-1} + x_{2,r} - x_{2,k}}$$

The representation of $y_{2,r}$ is as follows:

$$\begin{cases} y_{2,1} = \epsilon Y_{2,1} \\ y_{2,r} = \epsilon Y_{2,r} e^{x_{2,r-1}} + y_{2,r-1} e^{\epsilon X_{3,r}} \end{cases}$$

which means that the explicit formula for $y_{2,r}$ is as follows:

$$y_{2,r} = \epsilon \sum_{k=2}^{r} Y_{2,k} e^{x_{2,k-1} + x_{3,r} - x_{3,k}}$$

The representation of $z_{1,r}$ is as follows:

$$\begin{cases} z_{1,1} = \epsilon Z_{1,1} \\ z_{1,r} = z_{1,r-1} e^{\epsilon X_{3,r}} + \epsilon Y_{2,r} y_{1,r-1} + \epsilon Z_{1,r} e^{x_{1,r-1}} \end{cases}$$

which means that the explicit formula for $z_{1,r}$ is as follows:

$$z_{1,r} = \epsilon \sum_{k=2}^{r} Y_{2,k} y_{1,k-1} + \epsilon \sum_{k=1}^{r} Z_{1,k} e^{x_{1,k-1} + x_{3,r} - x_{3,k}}$$

Assume $r = [tn]$ for some $t \leq 1$, let $n \to \infty$ and then according to the functional central limit theorem by Donsker-Prohorov:

$$x_{i,r} \xrightarrow{d} e^{W_i(t)}, i \in [1,3]$$

$$y_{j,r} \xrightarrow{d} \int_0^t e^{W_j(s) + W_{j+1}(t) - W_{j+1}(s)} dW_{j+3}(s), j \in [1,2]$$

$$z_{1,r} \xrightarrow{d} \int_0^t \int_0^s e^{W_1(u) + W_2(s) - W_2(u)} dW_4(u) dW_5(s) + \int_0^t e^{W_1(s) + W_3(t) - W_3(s)} dW_6(s)$$

Hence, a Brownian motion on the T_3 group has the following form:

$$g_{T_3}(t) = \begin{bmatrix} e^{W_1(t)} & \xi_1(t) & \int\limits_0^t \xi_1(s)dW_5(s) + \xi_3(t) \\ 0 & e^{W_2(t)} & \xi_2(t) \\ 0 & 0 & e^{W_3(t)} \end{bmatrix} \tag{31}$$

where

$$\xi_1(t) = \int\limits_0^t e^{W_1(s)+W_2(t)-W_2(s)} dW_4(s)$$

$$\xi_2(t) = \int\limits_0^t e^{W_2(s)+W_3(t)-W_3(s)} dW_5(s)$$

$$\xi_3(t) = \int\limits_0^t e^{W_1(s)+W_3(t)-W_3(s)} dW_6(s)$$

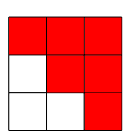
and $W_1(s)$, $W_2(s)$, $W_3(s)$, $W_4(s)$, $W_5(s)$, and $W_6(s)$ are independent standard Brownian motions for $s \in [0, t]$. The term $\xi_3(t)$ will only produce noise and for the our purpose we will zero it out in the most of our theorems. Observe the following qualities:

$$E[\xi_1(t)] = Z_t(W_1, W_2)$$

$$E[\xi_2(t)] = Z_t(W_2, W_3)$$

$$E[\xi_3(t)] = Z_t(W_1, W_3)$$

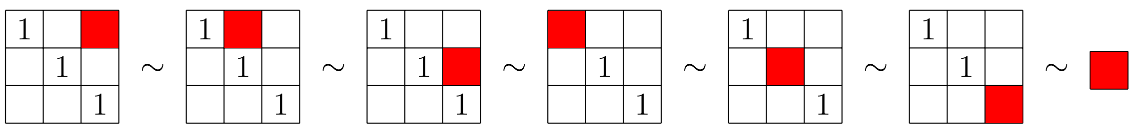
For more visual representation of all our groups of upper-triangular matrices, lets introduce the following representation of Brownian motion of T_3 group defined in Eq. (31):



We will be creating sub-groups by zeroing out some of the elements of this matrix, and visually will be replacing red squares with white ones. There are $2^6 = 64$ different matrices, but only 56 of them form a group. Let's start with the most simple set of subgroup - all groups of rank 1.

5.1 Solvable Groups of Rank 1 of Upper Triangular 3x3 Matrices

By leaving only one of the elements six elements of T_3 group in place, we will get six sub-groups of rank 1. Here is their visual representation:



Here and through out the dissertation we use $x \in dx$ to denote the notation: $x \in (x, x + dx)$. The transition probability for each of the group is the same and it has the following form:

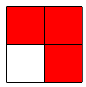
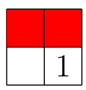
$$p(t, e, e) = P(W(t) \in dx) \sim \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}}, \quad t \to \infty$$

Note the the first three groups are nilpotent and the last three ones are solvable.

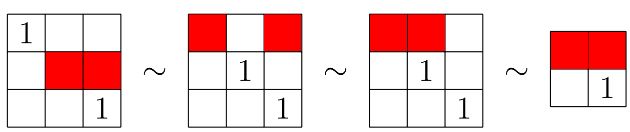
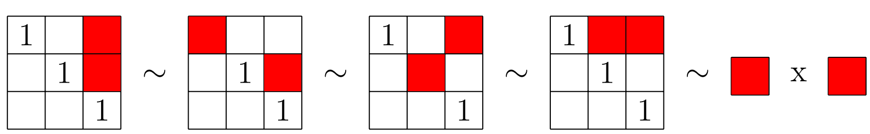
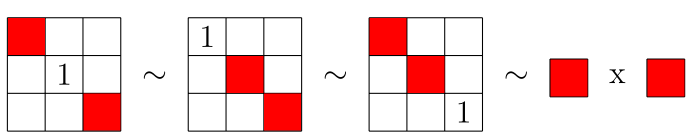
5.2 Solvable Groups of Rank 2 of Upper Triangular 3x3 Matrices

By leaving two of six elements of T_3 group in place, we will get sub-groups of rank 2. The total number of matrices will be $C_6^2 = 15$, but only 12 of them form a group.

First of all, lets visually represent results from Konakov-Menozzi-Molchanov [19]:

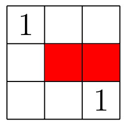
N	**Group**	**Brownian Motion**	**Decay of** $p(t,e,e)$
1		$\begin{bmatrix} e^{W_1(t)} & \int\limits_0^t e^{W_1(s)+W_2(t)-W_2(s)} dW_3(s) \\ 0 & e^{W_2(t)} \end{bmatrix}$	$\frac{1}{4\sqrt{\pi}} t^{-2}$
2		$\begin{bmatrix} e^{W_1(t)} & \int\limits_0^t e^{W_1(s)} dW_2(s) \\ 0 & 1 \end{bmatrix}$	$\sqrt{\frac{\pi}{2}} t^{-\frac{3}{2}}$

Our first seven groups shown in table bellow are simple since their transition probability can be derived directly from the table above.

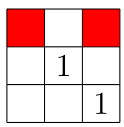
N	**Group**	**Decay of** $p(t,e,e)$
1		$\sqrt{\frac{\pi}{2}} t^{-\frac{3}{2}}$
2		$\frac{1}{2\pi} t^{-1}$
3		$\frac{1}{2\pi} t^{-1}$

Note the important fact that the first and the last group in row 2 in the table above are nilpotent group and it is known that the decay of its transition density is polynomial, but the rest of the groups in that table are solvable, and still the decay of its transition density is polynomial.

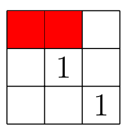
The Brownian motions on groups defined in the table above have the following form:



$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{W_2(t)} & \int_0^t e^{W_2(s)} dW_5(s) \\ 0 & 0 & 1 \end{bmatrix}$$

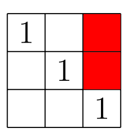


$$\sim \begin{bmatrix} e^{W_1(t)} & 0 & \int_0^t e^{W_1(s)} dW_6(s) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

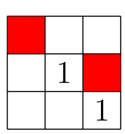


$$\sim \begin{bmatrix} e^{W_1(t)} & \int_0^t e^{W_1(s)} dW_4(s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

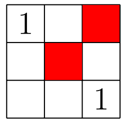
Groups on the second row:



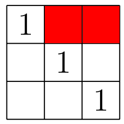
$$\sim \begin{bmatrix} 1 & 0 & W_6(t) \\ 0 & 1 & W_5(t) \\ 0 & 0 & 1 \end{bmatrix}$$



$$\sim \begin{bmatrix} 1 & 0 & W_6(t) \\ 0 & 1 & W_5(t) \\ 0 & 0 & 1 \end{bmatrix}$$

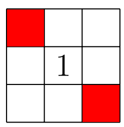


$$\sim \begin{bmatrix} 1 & 0 & W_6(t) \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

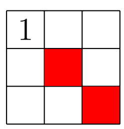


$$\sim \begin{bmatrix} 1 & W_4(t) & W_6(t) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

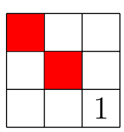
Groups on the last row:



$$\sim \begin{bmatrix} e^{W_1(t)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$$

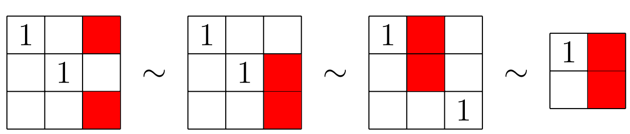


$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$$



$$\sim \begin{bmatrix} e^{W_1(t)} & 0 & 0 \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The remaining three groups have the following visual representation:



And the Brownian motions on them are defined in the following way:

$$\begin{bmatrix} 1 & 0 & \int\limits_0^t e^{W_3(t)-W_3(s)} dW_6(s) \\ 0 & 1 & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \int\limits_0^t e^{W_3(t)-W_3(s)} dW_5(s) \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & \int\limits_0^t e^{W_2(t)-W_2(s)} dW_4(s) & 0 \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that they are equivalent and have the same transition probability, the decay of which is found in the theorem bellow.

Theorem 12. Let $W_1(s)$ and $W_2(s)$ be independent Brownian motions on $[0,t]$ then the transition density for $\left(W_1(t), \int\limits_0^t e^{W_1(t)-W_1(s)} dW_2(s)\right)$ is $p(t,e,e) \sim \sqrt{\frac{2}{\pi}} t^{-\frac{3}{2}}, \quad t \to \infty$

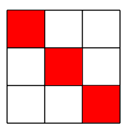
Proof. Let's define a Brownian bridge on $[0,t]$ such that: $B_1(s) = (W_1(s) | W_1(t) \in dx)$ and, using Lemma 6 we can conclude:

$$\begin{aligned} p(t,e,e) &\sim \frac{1}{\sqrt{2\pi t}} P\left(\int\limits_0^t e^{W_1(t)-W_1(s)} dW_2(s) \in dx \,\middle|\, W_1(t) \in dx \right) \\ &= \frac{1}{\sqrt{2\pi t}} P\left(\int\limits_0^t e^{-B_1(s)} dW_2(s) \in dx \right) = \frac{1}{\sqrt{2\pi t}} E\left[P\left(\int\limits_0^t e^{-B_1(s)} dW_2(s) \in dx \,\middle|\, B_1 \right) \right] \\ &= \frac{1}{2\pi\sqrt{t}} E\left[\left(\int\limits_0^t e^{-2B_1(s)} ds \right)^{-\frac{1}{2}} \right] = \sqrt{\frac{2}{\pi}} t^{-\frac{3}{2}}, \quad t \to \infty \end{aligned}$$

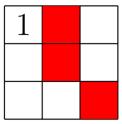
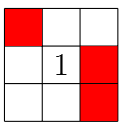
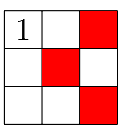
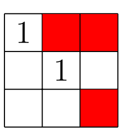
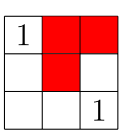
$\square$

5.3 Solvable Groups of Rank 3 of Upper Triangular 3x3 Matrices

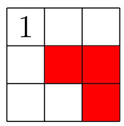
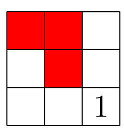
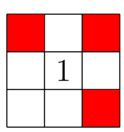
By leaving three of six elements of T_3 group in place, we will get sub-groups of rank 3. The total number of matrices will be $C_6^3 = 20$, but only 17 of them form a group. In the table bellow we show all our simple cases for which the transition probabilities are derived in a simple way.

Solvable Group	**Brownian Motion**	**Decay of** $p(t,e,e)$
	$\begin{bmatrix} e^{W_1(t)} & 0 & 0 \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	$\frac{1}{\sqrt{8\pi^3}} t^{-\frac{3}{2}}$

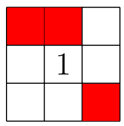
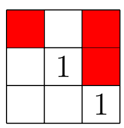
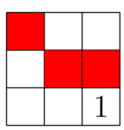
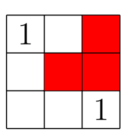
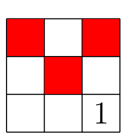
Second group of solvable subgroups of rank 3 is listed in the table bellow and their transition density's asymptotic decay is $\frac{1}{2}t^{-2}$.

Solvable Group	**Brownian Motion**	**Decay of** $p(t,e,e)$
	$\begin{bmatrix} 1 & \int_0^t e^{W_2(t)-W_2(s)}dW_4(s) & 0 \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	$\frac{1}{2}t^{-2}$
	$\begin{bmatrix} e^{W_1(t)} & 0 & 0 \\ 0 & 1 & \int_0^t e^{W_3(t)-W_3(s)}dW_5(s) \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	$\frac{1}{2}t^{-2}$
	$\begin{bmatrix} 1 & 0 & \int_0^t e^{W_3(t)-W_3(s)}dW_6(s) \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	$\frac{1}{2}t^{-2}$
	$\begin{bmatrix} 1 & W_4(t) & \int_0^t e^{W_3(t)-W_3(s)}dW_6(s) \\ 0 & 1 & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	$\frac{1}{2}t^{-2}$
	$\begin{bmatrix} 1 & \int_0^t e^{W_2(t)-W_2(s)}dW_4(s) & W_6(t) \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\frac{1}{2}t^{-2}$

Third group of solvable subgroups of rank 3 is listed in the table bellow and their transition density's asymptotic decay is $\frac{1}{4\sqrt{\pi}}t^{-2}$.

Solvable Group	**Brownian Motion**	**Decay of** $p(t,e,e)$
	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{W_2(t)} & \int\limits_0^t e^{W_2(s)+W_3(t)-W_3(s)}dW_5(s) \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	$\frac{1}{4\sqrt{\pi}}t^{-2}$
	$\begin{bmatrix} e^{W_1(t)} & \int\limits_0^t e^{W_1(s)+W_2(t)-W_2(s)}dW_4(s) & 0 \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\frac{1}{4\sqrt{\pi}}t^{-2}$
	$\begin{bmatrix} e^{W_1(t)} & 0 & \int\limits_0^t e^{W_1(s)+W_3(t)-W_3(s)}dW_6(s) \\ 0 & 1 & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	$\frac{1}{4\sqrt{\pi}}t^{-2}$

Fourth group of solvable subgroups of rank 3 is listed in the table bellow and their transition density's asymptotic decay is $\frac{1}{2}t^{-2}$.

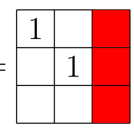
Solvable Group	**Brownian Motion**	**Decay of** $p(t,e,e)$
	$\begin{bmatrix} e^{W_1(t)} & \int_0^t e^{W_1(s)} dW_4(s) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	$\frac{1}{2}t^{-2}$
	$\begin{bmatrix} e^{W_1(t)} & 0 & \int_0^t e^{W_1(s)} dW_6(s) \\ 0 & 1 & W_5(t) \\ 0 & 0 & 1 \end{bmatrix}$	$\frac{1}{2}t^{-2}$
	$\begin{bmatrix} e^{W_1(t)} & 0 & 0 \\ 0 & e^{W_2(t)} & \int_0^t e^{W_2(s)} dW_5(s) \\ 0 & 0 & 1 \end{bmatrix}$	$\frac{1}{2}t^{-2}$
	$\begin{bmatrix} 1 & 0 & W_6(t) \\ 0 & e^{W_2(t)} & \int_0^t e^{W_2(s)} dW_5(s) \\ 0 & 0 & 1 \end{bmatrix}$	$\frac{1}{2}t^{-2}$
	$\begin{bmatrix} e^{W_1(t)} & 0 & \int_0^t e^{W_1(s)} dW_6(s) \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\frac{1}{2}t^{-2}$

Let's find transition probabilities for the remaining three groups.

5.3.1 Subgroup with Elements only in the Last Column

If we zero out X_1, X_2 and Y_1 in the T_3 group, we will get a subgroup with a Brownian motion:

$$\begin{bmatrix} 1 & 0 & \int_0^t e^{W_3(t)-W_3(s)} dW_6 \\ 0 & 1 & \int_0^t e^{W_3(t)-W_3(s)} dW_5 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix} =$$



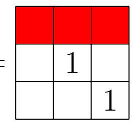
Define: $B_1(s) = (W_1(s)|W_1(t) \in dx)$, for $s \in [0,t]$, and using Lemma 8, conclude:

$$\begin{aligned}
p(t,e,e) &= P\left(W_1(t) \in dx, \int_0^t e^{W_1(t)-W_1(s)} dW_2(s) \in dx, \int_0^t e^{W_1(t)-W_1(s)} dW_3(s) \in dx\right) \\
&\sim \frac{1}{\sqrt{2\pi t}} P\left(\int_0^t e^{W_1(t)-W_1(s)} dW_2(s) \in dx, \int_0^t e^{W_1(t)-W_1(s)} dW_3(s) \in dx \,\middle|\, W_1(t) \in dx\right) \\
&= \frac{1}{\sqrt{2\pi t}} P\left(\int_0^t e^{-B_1(s)} dW_2(s) \in dx, \int_0^t e^{-B_1(s)} dW_3(s) \in dx\right) \\
&= \frac{1}{\sqrt{2\pi t}} E\left[P\left(\int_0^t e^{-B_1(s)} dW_2(s) \in dx, \int_0^t e^{-B_1(s)} dW_3(s) \in dx \,\middle|\, B_1\right)\right] \\
&= \frac{1}{\sqrt{2\pi t}} \frac{1}{2\pi} E\left[\left(\int_0^t e^{-2B_1(s)} ds\right)^{-1}\right] \\
&\sim \frac{1}{\sqrt{8\pi^3}} t^{-\frac{3}{2}}, \quad t \to \infty
\end{aligned}$$

5.3.2 Subgroup with Elements only in the First Row

Let's zero out X_2, X_3 and Y_2 in the T_3 group and we will get it's subgroup on which a Brownian motion is defined in the following way:

$$\begin{bmatrix} e^{W_1(t)} & \int\limits_0^t e^{W_1(s)} dW_4 & \int\limits_0^t e^{W_1(s)} dW_6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$



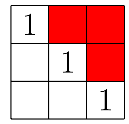
Define: $B_1(s) = (W_1(s)|W_1(t) \in dx)$, for $s \in [0, t]$, and using Lemma 8, conclude:

$$\begin{aligned}
p(t, e, e) &= P\left(W_1(t) \in dx, \int\limits_0^t e^{W_1(s)} dW_4(s) \in dx, \int\limits_0^t e^{W_1(t)} dW_6(s) \in dx \right) \\
&\sim \frac{1}{\sqrt{2\pi t}} P\left(\int\limits_0^t e^{W_1(s)} dW_4(s) \in dx, \int\limits_0^t e^{W_1(t)} dW_6(s) \in dx \,\middle|\, W_1(t) \in dx \right) \\
&= \frac{1}{\sqrt{2\pi t}} P\left(\int\limits_0^t e^{B_1(s)} dW_4(s) \in dx, \int\limits_0^t e^{B_1(t)} dW_6(s) \in dx \right) \\
&= \frac{1}{\sqrt{2\pi t}} E\left[P\left(\int\limits_0^t e^{B_1(s)} dW_4(s) \in dx, \int\limits_0^t e^{B_1(t)} dW_6(s) \in dx \,\middle|\, B_1 \right) \right] \\
&= \frac{1}{\sqrt{2\pi t}} \frac{1}{2\pi} E\left[\left(\int\limits_0^t e^{2B_1(s)} ds \right)^{-1} \right] \\
&\sim \frac{1}{\sqrt{8\pi^3}} t^{-\frac{3}{2}}, \quad t \to \infty
\end{aligned}$$

5.3.3 Heisenberg Group

A Brownian motion on the Heisenberg H group has the following form:

$$g_H(t) = \begin{bmatrix} 1 & W_1(t) & \int\limits_0^t W_1(s)dW_2(s) \\ 0 & 1 & W_2(t) \\ 0 & 0 & 1 \end{bmatrix} = \tag{32}$$



and $W_1(s)$ and $W_2(s)$ are independent standard Brownian motions for $s \in [0, t]$. Accoring to Gromov [12], Heisenberg group has a polynomial growth. Let's find what exactly it is for this group.

Consider the following process:

$$\Theta_t^{(H)} = \left(W_1(t), W_2(t), \int\limits_0^t W_1(s)dW_2(s) \right) \tag{33}$$

The fundamental solution of the parabolic equation bellow is a transition probability density of $\Theta_t^{(H)}$:

$$\begin{cases} \frac{\partial p(t,x,y)}{\partial t} = \mathscr{L} p(t,x,y) \\ p(0,x,y) = \delta_y(x) \end{cases}$$

where $\mathscr{L}$ is an infinitesimal generator of $\Theta_t^{(H)}$.

Observe that $\Theta_t^{(H)}$ is a Markov process and it satisfies the following system of stochastic differential equations:

$$\begin{cases} dx(t) = dW_1(t) \\ dy(t) = dW_2(t) \\ dz(t) = x(t)dW_2(t) \end{cases} \tag{34}$$

Let's rewrite it in a matrix form:

$$d \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \sigma(x, y, z) \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix} \tag{35}$$

where:

$$\sigma(x, y, z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x \end{bmatrix} \tag{36}$$

In order to find the infinitesimal generator $\mathscr{L}$, we need to compute the following:

$$\sigma \cdot \sigma^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & x^2 \end{bmatrix} \tag{37}$$

Hence:

$$(\mathscr{L} f)(x, y, z) = \frac{1}{2} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2x \frac{\partial^2}{\partial y \partial z} + x^2 \frac{\partial^2}{\partial z^2} \right] (x, y, z)$$

note that $det[\sigma \cdot \sigma^T] \equiv 0$, which means that the operator is a degenerator. Hörmander's form that is defined in Eq. (10) is:

$$\mathscr{L} = \frac{1}{2} \left[\frac{\partial^2}{\partial x^2} + \left(\frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \right)^2 \right] = \frac{1}{2} \left(X_1^2 + X_2^2 \right)$$

Note that in equation above $Y \equiv 0$. Now, lets find the commutators:

$$X_1 \to \frac{\partial}{\partial x} \to [1, 0, 0]$$

$$X_2 \to \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \to [0, 1, x]$$

$$[X_1, X_2] = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \right) - \left(\frac{\partial}{\partial y} + x \frac{\partial}{\partial z} \right) \frac{\partial}{\partial x} = \frac{\partial}{\partial z} \to [0, 0, 1]$$

Observe that that the commutators are linearly independent, which means that they form a basis and hence, per Theorem 5(Hörmander's condition) the operator $\mathscr{L}$ is hypoelliptic and smooth transition density exists.

Let's prove a lemma first that we will use in computing the decay of the transition density:

Lemma 13. Let $B(s)$ be a Brownian bridge on $[0,t]$, then

$$E\left[\left(\int_0^t B^2(s)ds\right)^{-\frac{1}{2}}\right] \approx \frac{3.07}{t}$$

$$E\left[\left(t\int_0^t B^2(s)ds - \left(\int_0^t B(s)ds\right)^2\right)^{-\frac{1}{2}}\right] \approx (3.01 - 1.29i)\, t^{-\frac{3}{2}}$$

Proof. Using the self-similarity property of Brownian motion $W\{(s)\}_{s\in[0,t]}$ and representing Brownian bridge as $\{B(s)\}_{s\in[0,t]} = \frac{t-s}{\sqrt{t}}W\left(\frac{s}{t-s}\right)$, we can conclude the following:

$$\begin{aligned}
E\left[\left(\int_0^t B^2(s)ds\right)^{-\frac{1}{2}}\right] &= E\left[\left(t\int_0^1 B^2(ut)du\right)^{-\frac{1}{2}}\right] \\
&= E\left[\left(t\int_0^1 \frac{(t-tu)^2}{t}W^2\left(\frac{ut}{t-ut}\right)du\right)^{-\frac{1}{2}}\right] \\
&= E\left[\left(t^2\int_0^1 (1-u)^2W^2\left(\frac{u}{1-u}\right)du\right)^{-\frac{1}{2}}\right] \\
&= \frac{1}{t}E\left[\left(\int_0^1 B^2(s)ds\right)^{-\frac{1}{2}}\right]
\end{aligned}$$

and

$$E\left[\left(t\int_0^t B^2(s)ds-\left(\int_0^t B(s)ds\right)^2\right)^{-\frac{1}{2}}\right]=t^{-\frac{3}{2}}E\left[\left(\int_0^1 B^2(s)ds-\left(\int_0^1 B(s)ds\right)^2\right)^{-\frac{1}{2}}\right]$$

Define

$$\alpha=E\left[\left(\int_0^1 B^2(s)ds\right)^{-\frac{1}{2}}\right] \tag{38}$$

$$\beta=E\left[\left(\int_0^1 B^2(s)ds-\left(\int_0^1 B(s)ds\right)^2\right)^{-\frac{1}{2}}\right] \tag{39}$$

Using the Karhunen–Loeve theorem, the Brownian bridge may also be represented as a Fourier series with stochastic coefficients, as

$$B(s)_{s\in[0,1]}=\sum_{k=1}^{\infty}\frac{\sqrt{2}sin(k\pi s)}{k\pi}\xi_k$$

where $\xi_1,\xi_2,\ldots$ are independent identically distributed standard normal random variables. Using the fact that $\frac{2}{k^2\pi^2}\int_0^1 sin^2(k\pi s)ds=\frac{1}{k^2\pi^2}$ and $\frac{2}{k_1k_2\pi^2}\int_0^1 sin(k_1\pi s)sin(k_2\pi s)ds=0$ we can compute the following:

$$\int_0^1 B^2(s)ds=\int_0^1\left(\sum_{k=1}^{\infty}\frac{\sqrt{2}sin(k\pi s)}{k\pi}\xi_k\right)^2 ds=\sum_{k=1}^{\infty}\frac{1}{k^2\pi^2}\xi_k^2$$

and

$$\begin{aligned}\int_0^1 B(s)ds&=\int_0^1\sum_{k=1}^{\infty}\frac{\sqrt{2}sin(k\pi s)}{k\pi}\xi_k ds=\sum_{k=1}^{\infty}\frac{\xi_k\sqrt{2}}{k\pi}\int_0^1 sin(k\pi s)ds\\&=\sum_{k=0}^{\infty}\frac{\sqrt{2}-\sqrt{2}cos(\pi k)}{k^2\pi^2}\xi_k=\sum_{k=0}^{\infty}N\left(0,\frac{2(1-cos(\pi k))^2}{k^4\pi^4}\right)\\&=N\left(0,\sum_{k=0}^{\infty}\frac{2(1-cos(\pi k))^2}{k^4\pi^4}\right)=N\left(0,\frac{1}{12}\right)=\frac{\chi}{\sqrt{12}},\quad\chi\sim N(0,1)\end{aligned}$$

Plugging the result of two equations above into Eq. (38) and Eq. (39), we get the following:

$$\begin{aligned} \alpha &= E\left[\left(\sum_{k=1}^{\infty}\frac{\xi_k^2}{\pi^2k^2}\right)^{-\frac{1}{2}}\right] \\ \beta &= E\left[\left(\sum_{k=1}^{\infty}\frac{\xi_k^2}{\pi^2k^2}-\frac{\chi^2}{12}\right)^{-\frac{1}{2}}\right] \end{aligned} \tag{40}$$

Let's compute the Laplace transform function:

$$\Phi_\alpha(\lambda) = E\left[e^{-\frac{\lambda}{\pi^2}\sum\limits_{k=1}^{\infty}\frac{\xi_k^2}{k^2}}\right] = E\left[\prod_{k=1}^{\infty}e^{-\frac{\lambda\xi_k^2}{\pi^2k^2}}\right] = \prod_{k=1}^{\infty}E\left[e^{-\frac{\lambda\xi_k^2}{\pi^2k^2}}\right]$$

and

$$\Phi_\beta(\lambda) = E\left[e^{-\frac{\lambda}{\pi^2}\sum\limits_{k=1}^{\infty}\frac{\xi_k^2}{k^2}-\lambda\frac{\chi^2}{12}}\right] = E\left[e^{-\lambda\frac{\chi^2}{12}}\prod_{k=1}^{\infty}e^{-\frac{\lambda\xi_k^2}{\pi^2k^2}}\right] = E\left[e^{-\lambda\frac{\chi^2}{12}}\right]\Phi_\alpha(\lambda)$$

Since each ξ_k is standard normal random variable, we have:

$$E\left[e^{-\frac{\lambda\xi_k^2}{\pi^2k^2}}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left(\frac{\lambda}{\pi^2k^2}+\frac{1}{2}\right)x^2}dx = \frac{\pi k}{\sqrt{2\lambda+\pi^2k^2}} = \frac{1}{\sqrt{1+\frac{2\lambda}{\pi^2k^2}}}$$

$$E\left[e^{-\lambda\frac{\chi^2}{12}}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left(\frac{\lambda}{12}+\frac{1}{2}\right)x^2}dx = \frac{2\sqrt{3\pi}}{\sqrt{6+\lambda}}$$

Therefore,

$$\Phi_\alpha(\lambda) = \prod_{k=1}^{\infty}\frac{1}{\sqrt{1+\frac{2\lambda}{\pi^2k^2}}}, \quad \Phi_\beta(\lambda) = \frac{2\sqrt{3\pi}}{\sqrt{6+\lambda}}\Phi_\alpha(\lambda)$$

Using the fact that $sinh(x) = x\prod\limits_{k=1}^{\infty}\left(1+\frac{x^2}{\pi^2k^2}\right)$ we can conclude that

$$\Phi_\alpha(\lambda) = \left(\prod_{k=1}^{\infty}\left(1+\frac{2\lambda}{\pi^2k^2}\right)\right)^{-\frac{1}{2}} = \left(\frac{1}{\sqrt{2\lambda}}sinh\left(\sqrt{2\lambda}\right)\right)^{-\frac{1}{2}} = \sqrt{\frac{\sqrt{2\lambda}}{sinh\left(\sqrt{2\lambda}\right)}}$$

$$\Phi_\beta(\lambda) = \sqrt{\frac{12\pi\sqrt{2\lambda}}{(6+\lambda)sinh(\sqrt{2\lambda})}}$$

Note that the probability density function will have the following form:

$$f_\alpha(\lambda) = \Phi_\alpha^{-1}(\lambda) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \sqrt{\frac{\sqrt{2x}}{sinh(\sqrt{2x})}}e^{\lambda x}dx$$

$$f_\beta(\lambda) = \Phi_\beta^{-1}(\lambda) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \sqrt{\frac{12\pi\sqrt{2x}}{(6+\lambda)sinh(\sqrt{2x})}}e^{\lambda x}dx$$

Computing these integrals is very a difficult job and instead we estimate the expectation in Eq. (40) numerically in Matlab using Monte-Carlo method [8] and get the following results:

$$\alpha \approx 3.07, \quad \beta \approx 3.01 - 1.29i$$

□

Theorem 14. Suppose that $p_H(t,e,e)$ is the transition probability density function from state e into state e of a Brownian motion on the Heisenberg group defined in Eq. (32) then

$$p_H(t,e,e) \sim C_H t^{-2} \quad as \quad t \to \infty$$

where e is the identity in G_H and $C_H \approx 0.48 - 0.2i$

Proof. We need to find a joint transition probability density function of $\int_0^t W_1(s)dW_2(s)$, $W_1(t)$ and $W_2(t)$. The transition probability density for any Brownian motion W is the probability density for $W(t+s)$ given that $W(t) = y$. Since $W(t+s) - W(t)$ is centered Gaussian, we have $E[W(t+s)] = E[W(t)] = y$ and therefore:

$$p\big(W(t+s) = x|W(t) = y\big) = \frac{1}{\sqrt{2\pi t}}e^{\frac{(x-y)^2}{2t}}$$

Hence, $P\big(W_1(t) \in dx\big) = P\big(W_2(t) \in dx\big) \sim \frac{1}{\sqrt{2\pi t}}$. Using Bayer's theorem we can conclude the following:

$$p_H(t,e,e) = P\left(\int_0^t W_1(s)dW_2(s) \in dx, W_1(t) \in dx, W_2(t) \in dx\right)$$

$$\sim \frac{1}{\sqrt{2\pi t}} P\left(\int_0^t W_1(s)dW_2(s) \in dx, W_2(t) \in dx \middle| W_1(t) \in dx\right)$$

$$\sim \frac{1}{\sqrt{2\pi t}} P\left(\int_0^t B_1(s)dW_2(s) \in dx, W_2(t) \in dx\right)$$

where $B_1(s)$ is a independent standard Brownian bridge on $[0,t]$. Let's fix B_1 and note the distribution of $\left(\int_0^t B_1(s)dW_2(s), W_2(t)\right)$ is centered Gaussian with

$$\Sigma = \begin{bmatrix} \int_0^t B_1^2(s)ds & \int_0^t B_1(s)ds \\ \int_0^t B_1(s)ds & t \end{bmatrix}$$

Hence:

$$p_H(t,e,e) = \frac{1}{\sqrt{2\pi t}} P\left(\int_0^t B_1(s)dW_2(s) \in dx, W_2(t) \in dx\right)$$

$$= \frac{1}{2\pi\sqrt{t}} E\left[\left(t\int_0^t B_1^2(s)ds - \left(\int_0^t B_1(s)ds\right)^2\right)^{-\frac{1}{2}}\right]$$

Let's change variables in the integral above as $u = \frac{s}{t}$ and use the self-similarity property of Brownian motion, i.e. $B(ut) \overset{d}{=} \sqrt{t}B(u)$, we can conclude the following:

$$p_H(t,e,e) = \frac{1}{2\pi\sqrt{t}} E\left[\left(t^3\int_0^1 B_1^2(s)ds - t^3\left(\int_0^1 B_1(s)ds\right)^2\right)^{-\frac{1}{2}}\right]$$

$$= \frac{1}{2\pi t^2} E\left[\left(\int_0^1 B_1^2(s)ds - \left(\int_0^1 B_1(s)ds\right)^2\right)^{-\frac{1}{2}}\right]$$

Per Lemma 13:

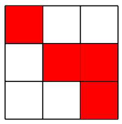
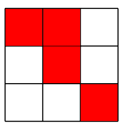
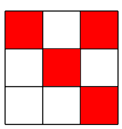
$$E\left[\left(\int_0^1 B_1^2(s)ds - \left(\int_0^1 B_1(s)ds\right)^2\right)^{-\frac{1}{2}}\right] \approx 3.01 - 1.29i$$

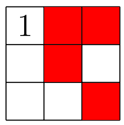
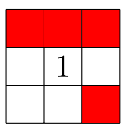
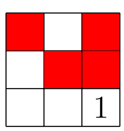
thus $p_H(t,e,e) \sim C_H t^{-2}$ where $C_H \approx \frac{3.01-1.29i}{2\pi} \approx 0.48 - 0.2i$ □

The statement of the theorem above agrees with Fischer [7], who showed that in the case of Brownian motion on the Heisenberg group the return probability decayed like t^{-2}. However, Fisher did not compute C_H which is found in Theorem 14.

5.4 Solvable Groups of Rank 4 of Upper Triangular 3x3 Matrices

By leaving four of six elements of T_3 group in place, we will get sub-groups of rank 4. The total number of matrices will be $C_6^4 = 15$, but only 12 of them form a group. In the table bellow we show all our simple cases for which the transition probabilities are derived in a simple way.

Solvable Group	**Brownian Motion**	**Decay of $p(t,e,e)$**
	$\begin{bmatrix} e^{W_1(t)} & 0 & 0 \\ 0 & e^{W_2(t)} & \int\limits_0^t e^{W_2(s)+W_3(t)-W_3(s)} dW_5(s) \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	
	$\begin{bmatrix} e^{W_1(t)} & \int\limits_0^t e^{W_1(s)+W_2(t)-W_2(s)} dW_4(s) & 0 \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	$\frac{1}{\sqrt{2}}\frac{1}{4\pi}t^{-\frac{5}{2}}$
	$\begin{bmatrix} e^{W_1(t)} & 0 & \int\limits_0^t e^{W_1(s)+W_3(t)-W_3(s)} dW_6(s) \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	

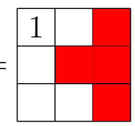
#	**Solvable Group**	**Brownian Motion**	**Decay of** $p(t,e,e)$
1		$\begin{bmatrix} 1 & \int_0^t e^{W_2(t)-W_2(s)}dW_4(s) & \int_0^t e^{W_3(t)-W_3(s)}dW_6(s) \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	$\frac{\pi}{2}t^{-3}$
2		$\begin{bmatrix} e^{W_1(t)} & \int_0^t e^{W_1(s)}dW_4(s) & \int_0^t e^{W_3(t)-W_3(s)}dW_6(s) \\ 0 & 1 & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix}$	$\frac{1}{4\sqrt{2}}t^{-\frac{7}{2}}$
3		$\begin{bmatrix} e^{W_1(t)} & 0 & \int_0^t e^{W_1(s)}dW_6(s) \\ 0 & e^{W_2(t)} & \int_0^t e^{W_2(s)}dW_5(t) \\ 0 & 0 & 1 \end{bmatrix}$	$\frac{\pi}{2}t^{-3}$

We have six more subgroups, and let's find transitional probabilities for some of them.

5.4.1 Subgroups without Middle Element in First Row and only two Elements on Main Diagonal

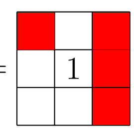
If we zero out X_1 and Y_1 in the T_3 group, we will get a subgroup with a Brownian motion having the following form:

$$\begin{bmatrix} 1 & 0 & \int_0^t e^{W_3(t)-W_3(s)} dW_6(s) \\ 0 & e^{W_2(t)} & \int_0^t e^{W_2(s)+W_3(t)-W_3(s)} dW_5(s) \\ 0 & 0 & e^{W_3(t)} \end{bmatrix} =$$



If we zero out X_2 and X_3 in the T_3 group, we will get a subgroup with a Brownian motion having the following form:

$$\begin{bmatrix} e^{W_1(t)} & 0 & \int_0^t e^{W_1(s)+W_3(t)-W_3(s)} dW_6(t) \\ 0 & 1 & \int_0^t e^{W_3(t)-W_3(s)} dW_5(t) \\ 0 & 0 & e^{W_3(t)} \end{bmatrix} =$$



The asymptotic decay of the transition density on both groups above is exactly the same. So, we will just focus on the first one.

Let's define

$$\begin{aligned} \Theta(t) &= (x_1(t), x_2(t), x_3(t), x_4(t)) \\ &= \left(e^{W_2(t)}, e^{W_3(t)}, \int_0^t e^{W_2(s)+W_3(t)-W_3(s)} dW_5(s), \int_0^t e^{W_3(t)-W_3(s)} dW_6(s) \right) \end{aligned}$$

note that it satisfies the following PDE:

$$\begin{cases} dx_1(t) = x_1(t)dW_2(t) \\ dx_2(t) = x_2(t)dW_3(t) \\ dx_3(t) = x_3(t)dW_3(t) + x_1(t)dW_5(t) \\ dx_4(t) = x_4(t)dW_3(t) + dW_6(t) \end{cases}$$

The matrix form of it is:

$$d\mathbf{\Theta} = \sigma \cdot d\mathbf{W}, \quad \text{where} \quad \sigma = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & x_3 & x_1 & 0 \\ 0 & x_4 & 0 & 1 \end{bmatrix}$$

Define

$$A = \{a_{ij}\} = \sigma\sigma^T = \begin{bmatrix} x_1^2 & 0 & 0 & 0 \\ 0 & x_1^2 & x_2x_3 & x_2x_4 \\ 0 & x_2x_3 & x_1^2 + x_3^2 & x_3x_4 \\ 0 & x_2x_4 & x_3x_4 & 1 + x_4^2 \end{bmatrix}$$

The infinitesimal generator of the process $\Theta(t)$ has the following form:

$$\mathscr{L} = \frac{1}{2}\left[x_1^2\frac{\partial^2}{\partial {x_1}^2} + x_1^2\frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_3}^2} + 2x_2\frac{\partial^2}{\partial x_3 \partial x_4} + (x_1^2 + x_2^2)\frac{\partial^2}{\partial x_4^2}\right]$$

Note that the determinant of the diffusion tensor is $x_1^4x_2^2 > 0$ and under assumption that it is strictly positive there exists a unique and strictly positive transition density of the process $\Theta(t)$.

Define and $B_2(s) = (W_2(s)|W_2(t) = 0)$, $B_3(s) = (W_3(s)|W_3(t) = 0)$, for $s \in [0, t]$

and conclude:

$$p(t,e,e) = P\left(W_2(t) \in dx, W_3(t) \in dx, \int\limits_0^t e^{W_2(t)+W_3(t)-W_3(s)} dW_5(s) \in dx,\right.$$
$$\left.\int\limits_0^t e^{W_3(t)-W_3(s)} dW_6(s) \in dx\right)$$
$$= \frac{1}{2\pi t} P\left(\int\limits_0^t e^{B_2(s)-B_3(s)} dW_5(s) \in dx, \int\limits_0^t e^{-B_3(s)} dW_6(s) \in dx\right)$$
$$= \frac{1}{4\pi^2 t} E\left[\left(\int\limits_0^t e^{2B_2(s)-2B_3(s)} ds \int\limits_0^t e^{-2B_3(s)} ds\right)^{-\frac{1}{2}}\right]$$

Per Hölder inequality:

$$E\left[\left(\int\limits_0^t e^{2B_2(s)-2B_3(s)} ds \int\limits_0^t e^{2B_3(s)} ds\right)^{-\frac{1}{2}}\right]$$
$$\leq \sqrt{E\left[\left(\int\limits_0^t e^{2B_2(s)-2B_3(s)} ds\right)^{-1}\right] E\left[\left(\int\limits_0^t e^{2B_3(s)} ds\right)^{-1}\right]}$$

Per Lemma 8:

$$E\left[\left(\int\limits_0^t e^{2B_3(s)} ds\right)^{-1}\right] = \frac{1}{t}$$

Per Lemma 8 and Lemma 9:

$$E\left[\left(\int\limits_0^t e^{2B_2(s)-2B_3(s)} ds\right)^{-1}\right] = \frac{1}{t}$$

Therefore:

$$E\left[\left(\int\limits_0^t e^{2B_2(s)-2B_3(s)} ds \int\limits_0^t e^{2B_3(s)} ds\right)^{-\frac{1}{2}}\right] \leq \frac{1}{t}$$

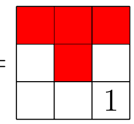
Thus, we found an upper estimate of the transition density:

$$0 < p(t,e,e) \leq \frac{1}{4\pi^2} t^{-2}, \quad t \to \infty$$

5.4.2 Subgroup without Last Two Elements in the Last Column

If we zero out X_3 and Y_2 in the T_3 group, we will get a subgroup with a Brownian motion having the following form:

$$\begin{bmatrix} e^{W_1(t)} & \int_0^t e^{W_1(s)+W_2(t)-W_2(s)} dW_4(s) & \int_0^t e^{W_1(s)} dW_6(s) \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$



Let's define

$$\begin{aligned} \Theta(t) &= (x_1(t), x_2(t), x_3(t), x_4(t)) \\ &= \left(e^{W_1(t)}, e^{W_2(t)}, \int_0^t e^{W_1(s)+W_2(t)-W_2(s)} dW_4(s), \int_0^t e^{W_1(s)} dW_6(s) \right) \end{aligned}$$

note that it satisfies the following PDE:

$$\begin{cases} dx_1(t) = x_1(t) dW_1(t) \\ dx_2(t) = x_2(t) dW_2(t) \\ dx_3(t) = x_3(t) dW_2(t) + x_1(t) dW_4(t) \\ dx_4(t) = x_1(t) dW_6(t) \end{cases}$$

The matrix form of it is:

$$d\mathbf{\Theta} = \sigma \cdot d\mathbf{W}, \quad \text{where} \quad \sigma = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & x_3 & x_1 & 0 \\ 0 & 0 & 0 & x_1 \end{bmatrix}$$

And:

$$A = \{a_{ij}\} = \sigma\sigma^T = \begin{bmatrix} x_1^2 & 0 & 0 & 0 \\ 0 & x_1^2 & x_2x_3 & 0 \\ 0 & x_2x_3 & x_1^2+x_3^2 & 0 \\ 0 & 0 & 0 & x_1^2 \end{bmatrix}$$

Hence its generator has the following form:

$$\begin{aligned} \mathscr{L} &= \frac{1}{2}\left[x_1^2\frac{\partial^2}{\partial {x_1}^2} + x_2^2\frac{\partial^2}{\partial {x_2}^2} + (x_1^2+x_3^2)\frac{\partial^2}{\partial x_3^2} + x_1^2\frac{\partial^2}{\partial {x_4}^2} + 2x_2x_3\frac{\partial^2}{\partial x_2\partial x_3}\right] \\ &= \frac{1}{2}\left[x_1^2\frac{\partial^2}{\partial {x_1}^2} + x_1^2\frac{\partial^2}{\partial x_3^2} + x_1^2\frac{\partial^2}{\partial {x_4}^2} + \left(x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right)^2\right] \end{aligned}$$

Note that the determinant of the diffusion tensor is $x_1^6x_2^2 > 0$, and under assumption that it is strictly positive exists unique and strictly positive transition density of the process $\Theta(t)$.

Define and $B_1(s) = (W_1(s)|W_1(t) = 0)$, $B_2(s) = (W_2(s)|W_2(t) = 0)$, for $s \in [0,t]$ and conclude:

$$\begin{aligned} p(t,e,e) &= P\left(W_1(t) \in dx, W_2(t) \in dx, \int_0^t e^{W_1(t)+W_2(t)-W_2(s)}dW_4(s) \in dx, \int_0^t e^{W_1(s)}dW_6(s) \in dx\right) \\ &= \frac{1}{2\pi t}P\left(\int_0^t e^{B_1(s)-B_2(s)}dW_4(s) \in dx, \int_0^t e^{B_1(s)}dW_6(s) \in dx\right) \\ &= \frac{1}{4\pi^2 t}E\left[\left(\int_0^t e^{2B_1(s)-2B_2(s)}ds \int_0^t e^{2B_1(s)}ds\right)^{-\frac{1}{2}}\right] \end{aligned}$$

Per Hölder inequality:

$$E\left[\left(\int_0^t e^{2B_1(s)-2B_2(s)}ds \int_0^t e^{2B_1(s)}ds\right)^{-\frac{1}{2}}\right]$$
$$\leq \sqrt{E\left[\left(\int_0^t e^{2B_1(s)-2B_2(s)}ds\right)^{-1}\right] E\left[\left(\int_0^t e^{2B_1(s)}ds\right)^{-1}\right]}$$

Per Lemma 8:

$$E\left[\left(\int_0^t e^{2B_1(s)}ds\right)^{-1}\right] = \frac{1}{t}$$

Per Lemma 8 and Lemma 9:

$$E\left[\left(\int_0^t e^{2B_1(s)-2B_2(s)}ds\right)^{-1}\right] = \frac{1}{t}$$

Therefore:

$$E\left[\left(\int_0^t e^{2B_1(s)-2B_2(s)}ds \int_0^t e^{2B_1(s)}ds\right)^{-\frac{1}{2}}\right] \leq \frac{1}{t}$$

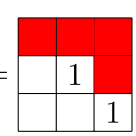
Thus, we found an upper estimate of the transition density:

$$0 < p(t, e, e) \leq \frac{1}{4\pi^2}t^{-2}, \quad t \to \infty$$

5.4.3 Subgroup with only First Element on the Main Diagonal

If we zero out X_2 and X_3 in the T_3 group, we will get a subgroup with a Brownian motion having the following form:

$$\begin{bmatrix} e^{W_1(t)} & \int_0^t e^{W_1(s)}dW_4(s) & \int_0^t\int_0^s e^{W_1(u)}dW_4(u)dW_5(s) + \int_0^t e^{W_1(s)}dW_6(s) \\ 0 & 1 & W_5(t) \\ 0 & 0 & 1 \end{bmatrix} =$$



Let's define

$$\Theta(t) = \left(e^{W_1(t)}, \int_0^t e^{W_1(s)} dW_4(s), W_5(t), \int_0^t \int_0^s e^{W_1(u)} dW_4(u) dW_5(s) + \int_0^t e^{W_1(s)} dW_6(s) \right)$$

$$\begin{aligned} X(t) &= (x_1(t), x_2(t), x_3(t), x_4(t)) \\ &= \left(e^{W_1(t)}, \int_0^t e^{W_1(s)} dW_4(s), W_5(t), \int_0^t e^{W_1(s)} dW_6(s) \right) \end{aligned}$$

Note that $\Theta(t)$ satisfies the following PDE:

$$\begin{cases} dx_1(t) = x_1(t) dW_1(t) \\ dx_2(t) = x_1(t) dW_4(t) \\ dx_3(t) = dW_5(t) \\ d\left(\int_0^t x_2(s) dW_5(s) + x_4(t) \right) = x_2(t) dW_5(t) + x_1(t) dW_6(t) \end{cases}$$

The matrix form of it is:

$$d\Theta = \sigma \cdot d\mathbf{W}, \quad \text{where} \quad \sigma = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x_2 & x_1 \end{bmatrix}$$

And:

$$A = \{a_{ij}\} = \sigma\sigma^T = \begin{bmatrix} x_1^2 & 0 & 0 & 0 \\ 0 & x_1^2 & 0 & 0 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & x_2 & x_1^2 + x_2^2 \end{bmatrix}$$

Hence the generator of the process $\Theta(t)$ has the following form:

$$\mathscr{L} = \frac{1}{2}\left[x_1^2\frac{\partial^2}{\partial {x_1}^2} + x_1^2\frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_3}^2} + 2x_2\frac{\partial^2}{\partial x_3 \partial x_4} + \left(x_1^2 + x_2^2\right)\frac{\partial^2}{\partial x_4^2}\right]$$

Hörmander's form:

$$\mathscr{L} = \frac{1}{2}\left[x_1^2\frac{\partial^2}{\partial {x_1}^2} + x_1^2\frac{\partial^2}{\partial {x_2}^2} + \left(x_2\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_3}\right)^2 + x_1^2\frac{\partial^2}{\partial x_4^2}\right]$$

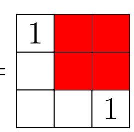
Note that the determinant of the diffusion tensor is $x_1^6 > 0$ which means that there exists a unique and strictly positive transition density of the process $\Theta(t)$. From the method of *parametrix* we know that the fundamental solution is possible to find, but is computationally complex. However, the central result is existence of it's upper estimate:

$$0 < p(t, e, e) \leq c \cdot t^{-2}, \quad t \to \infty$$

5.4.4 Subgroup with only Middle Element on the Main Diagonal

If we zero out X_1 and X_3 in the T_3 group, we will get a subgroup with a Brownian motion having the following form:

$$\begin{bmatrix} 1 & \int\limits_0^t e^{W_2(t)-W_2(s)}dW_4(s) & \int\limits_0^t\int\limits_0^s e^{W_2(s)-W_2(u)}dW_4(u)dW_5(s) + W_6(t) \\ 0 & e^{W_2(t)} & \int\limits_0^t e^{W_2(s)}dW_5(s) \\ 0 & 0 & 1 \end{bmatrix} =$$



Let's define

$$
\begin{aligned}
\Theta(t) = \Bigg(& e^{W_2(t)}, \int_0^t e^{W_2(t)-W_2(s)} dW_4(s), \int_0^t e^{W_2(s)} dW_5(s), \\
& \int_0^t \int_0^s e^{W_2(s)-W_2(u)} dW_4(u) dW_5(s) + W_6(t) \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
X(t) &= (x_1(t), x_2(t), x_3(t), x_4(t)) \\
&= \left(e^{W_2(t)}, \int_0^t e^{W_2(t)-W_2(s)} dW_4(s), \int_0^t e^{W_2(s)} dW_5(s), W_6(t) \right)
\end{aligned}
$$

Note that $\Theta(t)$ satisfies the following PDE:

$$
\begin{cases}
dx_1(t) = x_1(t) dW_2(t) \\
dx_2(t) = x_2(t) dW_2(t) + dW_4(t) \\
dx_3(t) = x_1(t) dW_5(t) \\
d\left(\int_0^t x_2(s) dW_5 + x_4(t) \right) = x_2(t) dW_5(t) + dW_6(t)
\end{cases}
$$

The matrix form of it is:

$$
d\mathbf{\Theta} = \sigma \cdot d\mathbf{W}, \quad \text{where} \quad \sigma = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2 & 1 & 0 & 0 \\ 0 & 0 & x_1 & 0 \\ 0 & 0 & x_2 & 1 \end{bmatrix}
$$

And:

$$A = \{a_{ij}\} = \sigma\sigma^T = \begin{bmatrix} x_1^2 & x_1x_2 & 0 & 0 \\ x_1x_2 & 1+x_2^2 & 0 & 0 \\ 0 & 0 & x_1^2 & x_1x_2 \\ 0 & 0 & x_1x_2 & 1+x_2^2 \end{bmatrix}$$

Hence the generator of the process Θ has the following form:

$$\mathscr{L} = \frac{1}{2}\left[x_1^2\frac{\partial^2}{\partial {x_1}^2} + (1+x_2^2)\frac{\partial^2}{\partial {x_2}^2} + x_1^2\frac{\partial^2}{\partial x_3^2} + (1+x_2^2)\frac{\partial^2}{\partial {x_4}^2} + 2x_1x_2\frac{\partial^2}{\partial x_1 \partial x_2} + 2x_1x_2\frac{\partial^2}{\partial x_3 \partial x_4}\right]$$

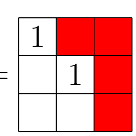
Note that the determinant of the diffusion tensor is $x_1^2 > 0$ and under assumption that it is strictly positive there exists a unique and strictly positive transition density of the process $\Theta(t)$. From the method of *parametrix* we know that the fundamental solution is possible to find, but is computationally complex. However, the central result is existence of it's upper estimate:

$$0 < p(t, e, e) \leq c \cdot t^{-2}, \quad t \to \infty$$

5.4.5 Subgroup with only Last Element on the Main Diagonal

If we zero out X_1 and X_2 in the T_3 group, we will get a subgroup where Brownian motion is defined in the following way:

$$\begin{bmatrix} 1 & W_4(t) & \int\limits_0^t W_4(s)dW_5(s) + \int\limits_0^t e^{W_3(t)-W_3(s)}dW_6(t) \\ 0 & 1 & \int\limits_0^t e^{W_3(t)-W_3(s)}dW_5(t) \\ 0 & 0 & e^{W_3(t)} \end{bmatrix} =$$



Note that in terms of calculating the transition probability from the initial state back to the initial state, the term $\int_0^t e^{W_3(t)-W_3(s)} dW_6(t)$ can be eliminated as it only creates noise.

Let's define

$$\Theta(t) = \left(e^{W_3(t)}, W_4(t), \int_0^t e^{W_3(t)-W_3(s)} dW_5(t), \int_0^t W_4(s) dW_5(s) + \int_0^t e^{W_3(t)-W_3(s)} dW_6(t) \right)$$

$$\begin{aligned} X(t) &= (x_1(t), x_2(t), x_3(t), x_4(t)) \\ &= \left(e^{W_3(t)}, W_4(t), \int_0^t e^{W_3(t)-W_3(s)} dW_5(t), \int_0^t W_4(s) dW_5(s) \right) \end{aligned}$$

Let's take a derivative of each element of the process $\Theta(t)$ with respect to t constructing a system of PDEs:

$$\begin{cases} dx_1(t) = x_1(t) dW_3(t) \\ dx_2(t) = dW_4(t) \\ dx_3(t) = x_3(t) dW_3(t) + dW_5(t) \\ d\left(\int_0^t x_2(s) dW_5 + x_4(t) \right) = x_2(t) dW_5(t) + x_4(t) dW_3(t) + dW_6(t) \end{cases}$$

The matrix form of it is:

$$d\mathbf{\Theta} = \sigma \cdot d\mathbf{W}, \quad \text{where} \quad \sigma = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x_3 & 0 & 1 & 0 \\ x_4 & 0 & x_2 & 1 \end{bmatrix}$$

And:

$$A = \{a_{ij}\} = \sigma\sigma^T = \begin{bmatrix} x_1^2 & 0 & x_1x_3 & 0 \\ 0 & 1 & 0 & 0 \\ x_1x_3 & 0 & 1+x_3^2 & x_2+x_3x_4 \\ 0 & 0 & x_2+x_3x_4 & 1+x_2^2+x_4^2 \end{bmatrix}$$

The generator of the process $\Theta(t)$ has the following form:

$$\mathscr{L} = \frac{1}{2}\left[x_1^2\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + (1+x_3^2)\frac{\partial^2}{\partial {x_3}^2} + x_2^2\frac{\partial^2}{\partial {x_4}^2}\right] + x_1x_3\frac{\partial^2}{\partial x_1\partial x_3} + x_2\frac{\partial^2}{\partial x_3\partial x_4}$$

Hörmander's form:

$$\mathscr{L} = \frac{1}{2}\left[\left(x_1\frac{\partial^2}{\partial x_1} + x_3\frac{\partial^2}{\partial x_3}\right)^2 + \left(\frac{\partial^2}{\partial x_3} + x_2\frac{\partial^2}{\partial x_4}\right)^2 + \frac{\partial^2}{\partial {x_2}^2}\right]$$

Note that the determinant of the diffusion tensor is $x_1^2 > 0$ and under assumption that it is strictly positive there exists a unique and strictly positive transition density of the process $\Theta(t)$. From the method of *parametrix* we know that the fundamental solution is possible to find, but is computationally complex. However, the central result is existence of it's upper estimate:

$$0 < p(t,e,e) \leq c \cdot t^{-2}, \quad t \to \infty$$

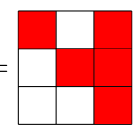
5.5 Solvable Groups of Rank 5 of Upper Triangular 3x3 Matrices

By leaving five of six elements of T_3 group in place, we will get sub-groups of rank five. The total number of matrices will be $C_6^5 = 6$, and only 5 of them form a group.

5.5.1 Subgroups without a middle Element in the First Row or in the Last Column

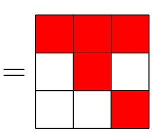
Let's assume that $Y_{1,k}$ is zero for all $k = 1, 2, ...$ in the group T_3, then it will form a subgroup of the upper triangular 3×3 matrices of rank 5 and the Brownian motion on this group will have the following form:

$$\begin{bmatrix} e^{W_1(t)} & 0 & \int_0^t e^{W_1(s)+W_3(t)-W_3(t)} dW_6(s) \\ 0 & e^{W_2(t)} & \int_0^t e^{W_2(s)+W_3(t)-W_3(s)} dW_5(s) \\ 0 & 0 & e^{W_3(t)} \end{bmatrix} =$$



Let's assume $Y_{2,k}$ is zero for all $k = 1, 2, ...$ in the group T_3, then it will form a subgroup of the upper triangular 3×3 matrices of rank 5 and the Brownian motion on this group will have the following form:

$$\begin{bmatrix} e^{W_1(t)} & \int_0^t e^{W_1(s)+W_2(t)-W_2(s)} dW_4(s) & \int_0^t e^{W_1(s)+W_3(t)-W_3(t)} dW_6(s) \\ 0 & e^{W_2(t)} & 0 \\ 0 & 0 & e^{W_3(t)} \end{bmatrix} =$$



Let's define a process that corresponds both Brownian motions defined above:

$$\Theta(t) = \left(e^{W_1(t)}, e^{W_2(t)}, e^{W_3(t)}, \int_0^t e^{W_1(s)+W_2(t)-W_2(s)} dW_4(s), \int_0^t e^{W_1(s)+W_3(t)-W_3(s)} dW_6(s) \right)$$

Note that it satisfies the following PDE:

$$\begin{cases} dx_1(t) = x_1(t)dW_1(t) \\ dx_2(t) = x_2(t)dW_2(t) \\ dx_3(t) = x_3(t)dW_3(t) \\ dx_4(t) = x_4(t)dW_2(t) + x_1(t)dW_4(t) \\ dx_5(t) = x_5(t)dW_3(t) + x_1(t)dW_6(t) \end{cases}$$

The matrix form of it is:

$$d\mathbf{\Theta} = \sigma \cdot d\mathbf{W}, \quad \text{where} \quad \sigma = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 \\ 0 & x_4 & 0 & x_1 & 0 \\ 0 & 0 & x_5 & 0 & x_1 \end{bmatrix}$$

And:

$$A = \{a_{ij}\} = \sigma\sigma^T = \begin{bmatrix} x_1^2 & 0 & 0 & 0 & 0 \\ 0 & x_2^2 & 0 & x_2x_4 & 0 \\ 0 & 0 & x_3^2 & 0 & x_3x_5 \\ 0 & x_2x_4 & 0 & x_1^2 + x_4^2 & 0 \\ 0 & 0 & x_3x_5 & 0 & x_1^2 + x_5^2 \end{bmatrix}$$

Hence the generator of $\Theta(t)$ has the following form:

$$\mathscr{L} = \frac{1}{2}\left[x_1^2\frac{\partial^2}{\partial {x_1}^2} + x_2^2\frac{\partial^2}{\partial {x_2}^2} + x_3^2\frac{\partial^2}{\partial {x_3}^2} + (x_1^2 + x_4^2)\frac{\partial^2}{\partial {x_4}^2} + (x_1^2 + x_5^2)\frac{\partial^2}{\partial {x_5}^2}\right]$$
$$+ x_2x_4\frac{\partial}{\partial x_2 \partial x_4} + x_3x_5\frac{\partial}{\partial x_3 \partial x_5}$$

Hörmander's form:

$$\mathscr{L} = \frac{1}{2}\left[x_1^2\frac{\partial^2}{\partial {x_1}^2} + x_1^2\frac{\partial^2}{\partial {x_4}^2} + x_1^2\frac{\partial^2}{\partial {x_5}^2} + \left(x_2\frac{\partial^2}{\partial x_2} + x_4\frac{\partial^2}{\partial x_4}\right)^2 + \left(x_3\frac{\partial^2}{\partial x_3} + x_3\frac{\partial^2}{\partial x_5}\right)^2\right]$$

Note that the determinant of the diffusion tensor is $x_1^6x_2^2x_3^2 > 0$, which means that there exists unique and strictly positive transition density of the process $\Theta(t)$.

Define

$$B_1(s) = (W_1(s)|W_1(t) = 0)$$

$$B_2(s) = (W_1(s)|W_1(t) = 0)$$

$$B_3(s) = (W_3(s)|W_3(t) = 0)$$

and conclude:

$$\begin{aligned} p(t,e,e) &\sim \frac{1}{\sqrt{8\pi^3}}P\left(\int\limits_0^t e^{B_2(s)-B_3(s)}dW_4(s) \in dx, \int\limits_0^t e^{B_1(s)-B_3(s)}dW_5(s) \in dx\right)t^{-\frac{3}{2}} \\ &\sim \frac{1}{\sqrt{8\pi^3}}E\left[P\left(\int\limits_0^t e^{B_2(s)-B_3(s)}dW_4(s) \in dx, \int\limits_0^t e^{B_1(s)-B_3(s)}dW_5(s) \in dx \,\middle|\, B_1B_2B_3\right)\right]t^{-\frac{3}{2}} \\ &\sim \frac{1}{\sqrt{32\pi^5}}E\left[\left(\int\limits_0^t e^{2B_2(s)-2B_3(s)}ds\int\limits_0^t e^{2B_1(s)-2B_3(s)}ds\right)^{-\frac{1}{2}}\right]t^{-\frac{3}{2}} \end{aligned}$$

Per Lemma 9 there are exist Brownian bridges $\bar{B}_1(s)_{s\in[0,t]}$ and $\bar{B}_2(s)_{s\in[0,t]}$ such that $\sqrt{2}\bar{B}_1 \overset{d}{=} 2B_1(s) - 2B_3(s)$ and $\sqrt{2}\bar{B}_2 \overset{d}{=} 2B_2(s) - 2B_3(s)$. Per Lemma 8:

$$E\left[\left(\int\limits_0^t e^{2\bar{B}_1(s)}ds\right)^{-1}\right] = E\left[\left(\int\limits_0^t e^{2\bar{B}_2(s)}ds\right)^{-1}\right] = \frac{1}{t}$$

Using Hölder's inequality, we get the following:

$$E\left[\left(\int\limits_0^t e^{2B_2(s)-2B_3(s)}ds\int\limits_0^t e^{2B_1(s)-2B_3(s)}ds\right)^{-\frac{1}{2}}\right] \le \frac{1}{t}$$

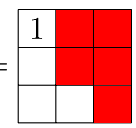
Thus, we found an upper estimate of the transition density:

$$0 < p(t,e,e) \le \frac{1}{4\sqrt{2\pi^5}} t^{-\frac{5}{2}}, \quad t \to \infty$$

5.5.2 Subgroup without a first Element in the First Row

Let's assume that $X_{1,k}$ is zero for all $k = 1, 2, ...$ in the group T_3, then it will form a subgroup of the upper triangular 3×3 matrices of rank 5 and the Brownian motion on this group will have the following form:

$$\begin{bmatrix} 1 & \int_0^t e^{W_2(t)-W_2(s)} dW_4(s) & \int_0^t \int_0^s e^{W_2(s)-W_2(u)} dW_4(u) dW_5(s) + \int_0^t e^{W_3(t)-W_3(t)} dW_6(s) \\ 0 & e^{W_2(t)} & \int_0^t e^{W_2(s)+W_3(t)-W_3(s)} dW_5(s) \\ 0 & 0 & e^{W_3(t)} \end{bmatrix} =$$



Let's define

$$\Theta(t) = \Bigg(e^{W_2(t)}, e^{W_3(t)}, \int_0^t e^{W_2(t)-W_2(s)} dW_4(s), \int_0^t e^{W_3(s)+W_3(t)-W_3(s)} dW_5(s),$$
$$\int_0^t \int_0^s e^{W_2(s)-W_2(u)} dW_4(u) dW_5(s) + \int_0^t e^{W_3(t)-W_3(t)} dW_6(s) \Bigg)$$

$$X(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t))$$
$$= \Bigg(e^{W_2(t)}, e^{W_3(t)}, \int_0^t e^{W_2(t)-W_2(s)} dW_4(s), \int_0^t e^{W_3(s)+W_3(t)-W_3(s)} dW_5(s),$$
$$\int_0^t \int_0^s e^{W_2(s)-W_2(u)} dW_4(u) dW_5(s) \Bigg)$$

Note that it satisfies the following PDE:

$$
\begin{cases}
dx_1(t) = x_1(t)dW_2(t) \\
dx_2(t) = x_2(t)dW_3(t) \\
dx_3(t) = x_3(t)dW_2(t) + dW_4(t) \\
dx_4(t) = x_4(t)dW_3(t) + x_2(t)dW_5(t) \\
d\left(\int\limits_0^t x_3(s)dW_5 + x_5(t)\right) = x_3(t)dW_5(t) + x_5(t)dW_3(t) + dW_6(t)
\end{cases}
$$

The matrix form of it is:

$$
d\mathbf{\Theta} = \sigma \cdot d\mathbf{W}, \quad \text{where} \quad \sigma = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 \\ x_3 & 0 & 1 & 0 & 0 \\ 0 & x_4 & 0 & x_2 & 0 \\ 0 & x_5 & 0 & x_3 & 1 \end{bmatrix}
$$

And:

$$
A = \{a_{ij}\} = \sigma\sigma^T = \begin{bmatrix} x_1^2 & 0 & x_1x_3 & 0 & 0 \\ 0 & x_2^2 & 0 & x_2x_4 & x_2x_5 \\ x_1x_3 & 0 & 1 + x_3^2 & 0 & 0 \\ 0 & x_2x_4 & 0 & x_2^2 + x_4^2 & x_2x_3 + x_4x_5 \\ 0 & x_2x_5 & 0 & x_2x_3 + x_4x_5 & 1 + x_3^2 + x_5^2 \end{bmatrix}
$$

Hence the generator of $\Theta(t)$ has the following form:

$$\mathscr{L} = \frac{1}{2}\left[x_1^2\frac{\partial^2}{\partial {x_1}^2} + x_2^2\frac{\partial^2}{\partial {x_2}^2} + (1+x_3^2)\frac{\partial^2}{\partial {x_3}^2} + (x_2^2+x_4^2)\frac{\partial^2}{\partial {x_4}^2} + (1+x_3^2+x_5^2)\frac{\partial^2}{\partial {x_5}^2}\right]$$
$$+ x_1x_3\frac{\partial^2}{\partial x_1\partial x_3} + x_2x_4\frac{\partial^2}{\partial x_2\partial x_4} + x_2x_5\frac{\partial^2}{\partial x_2\partial x_5}$$

Hörmander's form:

$$\mathscr{L} = \frac{1}{2}\left[\left(x_1\frac{\partial}{\partial x_3} + x_1\frac{\partial}{\partial x_3}\right)^2 + \left(x_2\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_4}\right)^2 + \left(x_2\frac{\partial}{\partial x_2} + x_5\frac{\partial}{\partial x_5}\right)^2\right]$$
$$+ \frac{1}{2}\left[\frac{\partial^2}{\partial {x_3}^2} - x_2^2\frac{\partial^2}{\partial {x_2}^2} + x_2^2\frac{\partial^2}{\partial {x_4}^2} + \frac{\partial^2}{\partial {x_5}^2} + x_3^2\frac{\partial^2}{\partial {x_5}^2}\right]$$

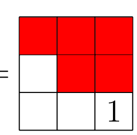
Note that the determinant of the diffusion tensor is $x_1^2x_2^2 > 0$ which means that there exists a unique and strictly positive transition density of the process $\Theta(t)$. From the method of *parametrix* we know that the fundamental solution is possible to find, but is computationally complex. However, the central result is existence of it's upper estimate:

$$0 < p(t,e,e) \le c \cdot t^{-\frac{5}{2}}, \quad t \to \infty$$

5.5.3 Subgroup without the Last Element in the Last Row

Let's assume that $X_{3,k}$ is zero for all $k = 1, 2, ...$ in the group T_3, then it will form a subgroup of the upper triangular 3×3 matrices of rank 5 and the Brownian motion on this group will have the following form:

$$\begin{bmatrix} e^{W_1(t)} & \int_0^t e^{W_1(s)+W_2(t)-W_2(s)}dW_4 & \int_0^t\int_0^s e^{W_1(u)+W_2(s)-W_2(u)}dW_4dW_5 + \int_0^t e^{W_1(s)}dW_6 \\ 0 & e^{W_2(t)} & \int_0^t e^{W_2(s)}dW_5(s) \\ 0 & 0 & 1 \end{bmatrix} =$$



Let's define

$$\Theta(t) = \Bigg(e^{W_1(t)}, e^{W_2(t)}, \int_0^t e^{W_1(s)+W_2(t)-W_2(s)} dW_4(s),$$

$$\int_0^t e^{W_2(s)} dW_5(s), \int_0^t \int_0^s e^{W_1(u)+W_2(s)-W_2(u)} dW_4(u) dW_5(s) + \int_0^t e^{W_1(s)} dW_6 \Bigg)$$

note that it satisfies the following PDE:

$$\begin{cases} dx_1(t) = x_1(t)dW_1(t) \\ dx_2(t) = x_2(t)dW_2(t) \\ dx_3(t) = x_3(t)dW_2(t) + x_1(t)dW_4(t) \\ dx_4(t) = x_2(t)dW_5(t) \\ d\left(\int_0^t x_3(s)dW_5(s) + x_5(t)\right) = x_3(t)dW_5(t) + x_1(t)dW_6(t) \end{cases}$$

The matrix form of it is:

$$d\boldsymbol{\Theta} = \sigma \cdot d\mathbf{W}, \quad \text{where} \quad \sigma = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 \\ 0 & x_3 & x_1 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 & x_1 \end{bmatrix}$$

And:

$$A = \{a_{ij}\} = \sigma\sigma^T = \begin{bmatrix} x_1^2 & 0 & 0 & 0 & 0 \\ 0 & x_2^2 & x_2x_3 & 0 & 0 \\ 0 & x_2x_3 & 1+x_3^2 & 0 & 0 \\ 0 & 0 & 0 & x_2^2 & x_2x_3 \\ 0 & 0 & 0 & x_2x_3 & x_1^2+x_5^2 \end{bmatrix}$$

Hence the generator of $\Theta(t)$ has the following form:

$$\mathscr{L} = \frac{1}{2}\left[x_1^2\frac{\partial^2}{\partial {x_1}^2} + x_2^2\frac{\partial^2}{\partial {x_2}^2} + (1+x_3^2)\frac{\partial^2}{\partial {x_3}^2} + x_2^2\frac{\partial^2}{\partial {x_4}^2} + (x_1^2+x_5^2)\frac{\partial^2}{\partial {x_5}^2}\right]$$
$$+ x_2x_3\frac{\partial^2}{\partial x_2\partial x_3} + x_2x_3\frac{\partial^2}{\partial x_5\partial x_6}$$

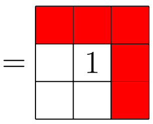
Note that the determinant of the diffusion tensor is $x_1^6x_2^4 > 0$ which means that there exists a unique and strictly positive transition density of the process $\Theta(t)$. From the method of *parametrix* we know that the fundamental solution is possible to find, but is computationally complex. However, the central result is existence of it's upper estimate:

$$0 < p(t,e,e) \leq c \cdot t^{-\frac{5}{2}}, \quad t \to \infty$$

5.5.4 Subgroup without the Middle Element in the Middle Row

Let's assume that $X_{2,k}$ is zero for all $k = 1, 2, ...$ in the group T_3, then it will form a subgroup of the upper triangular 3×3 matrices of rank 5 and the Brownian motion on this group will have the following form:

$$\begin{bmatrix} e^{W_1(t)} & \int\limits_0^t e^{W_1(s)} dW_4(s) & \int\limits_0^t \int\limits_0^s e^{W_1(u)} dW_4(u) dW_5(s) + \int\limits_0^t e^{W_1(s)+W_3(t)-W_3(t)} dW_6(s) \\ 0 & 1 & \int\limits_0^t e^{e^{W_3(t)-W_3(s)}} dW_5(s) \\ 0 & 0 & e^{W_3(t)} \end{bmatrix} =$$



Let's define $\Theta(t)$ as the following process:

$$\Theta(t) = \Bigg(e^{W_1(t)}, e^{W_3(t)}, \int_0^t e^{W_1(s)} dW_4(s),$$
$$\int_0^t e^{W_3(t)-W_3(s)} dW_5(s), \int_0^t \int_0^s e^{W_1(u)} dW_4(u) dW_5(s) + \int_0^t e^{W_1(s)+W_3(t)-W_3(t)} dW_6(s) \Bigg)$$

Note that it satisfies the following PDE:

$$\begin{cases} dx_1(t) = x_1(t) dW_1(t) \\ dx_2(t) = x_2(t) dW_3(t) \\ dx_3(t) = x_1(t) dW_4(t) \\ dx_4(t) = x_4(t) dW_3(t) + dW_5(t) \\ d\left(\int\limits_0^t x_3(s) dW_5 + x_5(t) \right) = x_3(t) dW_5(t) + x_5(t) dW_3(t) + x_1(t) dW_6(t) \end{cases}$$

Let's rewrite it in a vector form:

$$d\mathbf{X} = \sigma d\mathbf{W}, \quad \text{where} \quad \sigma = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 \\ 0 & x_4 & 0 & 1 & 0 \\ 0 & x_5 & 0 & x_3 & x_1 \end{bmatrix}$$

Then:

$$A = \{a_{i,j}(\mathbf{x})\}_{i,j\in[1,5]} = \sigma \cdot \sigma^T = \begin{bmatrix} x_1^2 & 0 & 0 & 0 & 0 \\ 0 & x_2^2 & 0 & x_2x_4 & x_2x_6 \\ 0 & 0 & x_1^2 & 0 & 0 \\ 0 & x_2x_4 & 0 & 1+x_4^2 & x_3+x_4x_5 \\ 0 & x_2x_5 & 0 & x_3+x_4x_5 & x_1^2+x_3^2+x_5^2 \end{bmatrix}$$

Note that $x_1, x_2 > 0$, $x_3, x_4, x_5 \geq 0$. The matrix A is positive-definite and its eigenvalues are $(1, x_1, x_1, x_2, x_3)$.

Hence the generator of $\Theta(t)$ has the following form:

$$\mathscr{L} = \frac{1}{2}\left[x_1^2\frac{\partial^2}{\partial {x_1}^2} + x_2^2\frac{\partial^2}{\partial {x_2}^2} + x_1^2\frac{\partial^2}{\partial {x_3}^2} + (1+x_4^2)\frac{\partial^2}{\partial {x_4}^2} + (x_1^2+x_3^2+x_5^2)\frac{\partial^2}{\partial {x_5}^2}\right]$$
$$+ x_2x_4\frac{\partial^2}{\partial x_2 \partial x_4} + x_2x_5\frac{\partial^2}{\partial x_2 \partial x_5} + (x_3+x_4x_5)\frac{\partial^2}{\partial x_4 \partial x_5}$$

Note that the determinant of the diffusion tensor is $x_1^6 x_2^2 > 0$ which means that there exists a unique and strictly positive transition density of the process $\Theta(t)$. From the method of *parametrix* we know that the fundamental solution is possible to find, but is computationally complex. However, the central result is existence of it's upper estimate:

$$0 < p(t, e, e) \leq c \cdot t^{-\frac{5}{2}}, \quad t \to \infty$$

5.6 Solvable Groups of of Upper Triangular 3x3 Matrices: General Case

We will not going to zero out any elements of the T_3 group and will find the decay of the transition density of the following process:

$$\Theta(t) = \Big(e^{W_1(t)}, e^{W_2(t)}, e^{W_3(t)},$$
$$\int_0^t e^{W_1(s)+W_2(t)-W_2(s)} dW_4(s),$$
$$\int_0^t e^{W_2(s)+W_3(t)-W_3(s)} dW_5(s),$$
$$\int_0^t \int_0^s e^{W_1(u)+W_2(s)-W_2(u)} dW_4(u) dW_5(s) + \int_0^t e^{W_1(s)+W_3(t)-W_3(s)} dW_6(s).\Big)$$

Let's define

$$\begin{cases} x_1(t) = e^{W_1(t)} \\ x_2(t) = e^{W_2(t)} \\ x_3(t) = e^{W_3(t)} \\ x_4(t) = \int_0^t e^{W_1(s)+W_2(t)-W_2(s)} dW_4(s) \\ x_5(t) = \int_0^t e^{W_2(s)+W_3(t)-W_3(s)} dW_5(s) \\ x_6(t) = \int_0^t e^{W_1(s)+W_3(t)-W_3(s)} dW_6(s) \end{cases}$$

Let's take a derivative of each element of the process $\Theta(t)$ with respect to t constructing a system of PDEs:

$$dx_1(t) = e^{W_1(t)}dW_1(t) = x_1(t)dW_1(t)$$

$$dx_2(t) = e^{W_2(t)}dW_2(t) = x_2(t)dW_2(t)$$

$$dx_3(t) = e^{W_3(t)}dW_3(t) = x_3(t)dW_3(t)$$

$$dx_4(t) = e^{W_2(t)}dW_2(t)\int_0^t e^{W_1(s)-W_2(s)}dW_4(s) + e^{W_1(t)}dW_4(t)$$

$$= x_4(t)dW_2(t) + x_1(t)dW_4(t)$$

$$dx_5(t) = e^{W_3(t)}dW_3(t)\int_0^t e^{W_2(s)-W_3(s)}dW_5(s) + e^{W_2(t)}dW_5(t)$$

$$= x_5(t)dW_3(t) + x_2(t)dW_5(t)$$

$$d\left(\int_0^t x_4(s)dW_5(s) + x_6(t)\right) = x_4(t)dW_5(t) + x_6(t)dW_3(t) + x_1(t)dW_6(t)$$

Let's rewrite it in a vector form:

$$d\mathbf{\Theta} = \sigma d\mathbf{W}, \quad \text{where} \quad \sigma = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 \\ 0 & x_4 & 0 & x_1 & 0 & 0 \\ 0 & 0 & x_5 & 0 & x_2 & 0 \\ 0 & 0 & x_6 & 0 & x_4 & x_1 \end{bmatrix}$$

Define $A = \{a_{i,j}(\mathbf{x})\}_{i,j\in[1,6]}$ then

$$A = \sigma \cdot \sigma^T = \begin{bmatrix} x_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2^2 & 0 & x_2x_4 & 0 & 0 \\ 0 & 0 & x_3^2 & 0 & x_3x_5 & x_3x_6 \\ 0 & x_2x_4 & 0 & x_1^2+x_4^2 & 0 & 0 \\ 0 & 0 & x_3x_5 & 0 & x_2^2+x_5^2 & x_2x_4+x_5x_6 \\ 0 & 0 & x_3x_6 & 0 & x_2x_4+x_5x_6 & x_1^2+x_4^2+x_6^2 \end{bmatrix}$$

Per Theorem 2, the infinitesimal generator of $\Theta(t)$ has the following form:

$$\begin{aligned}\hat{\mathscr{L}} = &\frac{1}{2}\Bigg[x_1^2\frac{\partial^2}{\partial {x_1}^2} + x_2^2\frac{\partial^2}{\partial {x_2}^2} + x_3^2\frac{\partial^2}{\partial {x_3}^2} + \left(x_1^2+x_4^2\right)\frac{\partial^2}{\partial {x_4}^2} \\ &+ \left(x_2^2+x_5^2\right)\frac{\partial^2}{\partial {x_5}^2} + \left(x_1^2+x_4^2+x_6^2\right)\frac{\partial^2}{\partial {x_6}^2}\Bigg] \\ &+ x_2x_4\frac{\partial^2}{\partial x_2\partial x_4} + x_3x_5\frac{\partial^2}{\partial x_3\partial x_5} + (x_2x_4+x_5x_6)\frac{\partial^2}{\partial x_5\partial x_6} + x_3x_6\frac{\partial^2}{\partial x_3\partial x_6}\end{aligned}$$

$$\begin{aligned}\hat{\mathscr{L}} = &\frac{1}{2}\Bigg[\left(x_2\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_4}\right)^2 + \left(x_3\frac{\partial}{\partial x_3} + x_5\frac{\partial}{\partial x_5}\right)^2 \\ &+ \left(x_2\frac{\partial}{\partial x_5} + x_4\frac{\partial}{\partial x_6}\right)^2 + x_6^2\frac{\partial^2}{\partial {x_6}^2}\Bigg] \\ &+ \frac{x_1^2}{2}\left[\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_4}^2} + \frac{\partial^2}{\partial {x_6}^2}\right] + x_5x_6\frac{\partial^2}{\partial x_5\partial x_6} + x_3x_6\frac{\partial^2}{\partial x_3\partial x_6}\end{aligned}$$

Note that $x_1, x_2, x_3 > 0$ and $x_4, x_5, x_6 \geq 0$. The matrix A is a positive-definite matrix and $det[A] = x_1^6x_2^4x_3^2 > 0$, which means per [1] that the parabolic equation

$$\begin{cases} \dfrac{\partial p(t,x,t)}{\partial t} = \mathscr{L}p(t,x,y) \\ p(0,x,y) = \delta_y(x) \end{cases} \tag{41}$$

has unique strictly positive solution, which is the transition density of the $\Theta(t)$ diffusion process.

Using the Parametrix method described in the section 2.6, we can try to construct the solution, but this exercise is very complicated and is out of scope for this dissertation. However we are able to use Eq. (13) and by letting $t \to \infty$ we get an upper estimate of the transition probability density:

$$0 < p(t, 0, 0) \leq ct^{-\frac{5}{2}}, \quad t \to \infty$$

where the constant c depend only on the dimension d.

5.7 Conclusions and Future Work

5.7.1 Conclusions

Starting from M. Yor [28], exponential functionals of the Brownian motion were studied in mathematical finance, in particular in Asian option pricing. At the same time, they play a significant role in different settings: the analysis of diffusions on the class of solvable Lie groups, in particular on the group of upper-triangular 3x3 matrices, with positive diagonal elements.

Diffusion processes on solvable groups of upper-triangular 2x2 matrices studied in a few papers by S. Molchanov, V. Konakov, S. Menozzi; the Brownian motion on thse groups is studied in [21], the approximation of diffusion on the these groups is studied in [19] and the local and quasi-local limit theorems on these groups are studied in [20]. Brownian motions on these groups are constructed by using the multiplicative stochastic integral. In this thesis, we extended this research and these results are summarized bellow:

- Brownian motions were constructed on all 52 sub-groups of the solvable groups of upper-triangular 3x3 matrices, including the Heisenberg group. There are six solvable sub-groups of rank 1, 12 solvable sub-groups of rank 2, 17 solvable sub-groups of rank 3, 12 solvable sub-groups of rank 4, five solvable sub-groups of rank 5 and one in general case.

- We proved that the asymptotic decay of the return probabilities in the continuous model is polynomial for all sub-groups of rank 1, rank 2 and rank 3.

- For 6 out of 12 sub-group of rank 4, we proved the asymptotic decay of the return probabilities is polynomial. For the remaining 6 solvable sub-groups, we proved that the existence and uniqueness of positive return probabilities. Moreover, we found a polynomial upper bound of the asymptotic decay of return probabilities.

- For general case and all solvable sub-groups of rank 5 we proved the existence and uniqueness of positive return probabilities and found a polynomial upper bound of the asymptotic decay of return probabilities.

- We have proven that for modified Asian-European geometric basket options with two assets, the value of the option is bounded if the underline asset prices are bounded. This fact implies that there is more risk in certain type of basket options.

- We have also proven that the price of modified Asian-European geometric basket options with two assets depends on $\sqrt{t}$.

5.7.2 Future Work

For future work, we plan to prove the polynomial behaviour of the asymptotic decay of return probabilities of Brownian motion defined on the remaining ten solvable Lie sub-groups of upper-triangular $3x3$ matrices. We also plan to expend the research to the general case of solvable groups of upper-triangular NxN matrices.

To extend work in [19], we plan to compute the return probabilities in discrete models of solvable group of upper-triangular 3x3 matrices that have been defined in this dissertation. Also, we plan establish additional properties for Asian and European-Asian geometric basket options in the general case of N assets.

References

[1] D. G. Aronson. The fundamental solution of a linear parabolic equation containing a small parameter. *Illinois J. Math.*, 3(4):580–619, 12 1959.

[2] P. Bougerol. Exemples de théorèmes locaux sur les groupes résolubles. *Annales de l'Institut Henri Poincaré. Nouvelle Série. Section B. Calcul des Probabilités et Statistique*, 19, 01 1983.

[3] C. Donati-Martin, H. Matsumoto, and M. Yor. On positive and negative moments of the integral of geometric brownian motions. *Statistics Probability Letters*, 49:45–52, 08 2000.

[4] C. Donati-Martin, H. Matsumoto, and M. Yor. On striking identities about the exponential functionals of the brownian bridge and brownian motion. *Periodica Mathematica Hungarica*, 41:103–119, 11 2000.

[5] D Dufresne. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scandinavian Actuarial Journal*, 1990(1):39–79, 1990.

[6] Black F. and Scholes M. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[7] D. Fischer. The gaussian random walk on the heisenberg group. *Illinois J. Math.*, 24(2):264–286, 06 1980.

[8] G. Fishman. *Monte Carlo - Concepts, Algorithms, and Application.* Springer-Verlag New York, 1996.

[9] A. Friedman. *Partial Differential Equations of Parabolic Type.* Englewood Cliffs, N.J.: Prentice-Hall, New York, 1992.

[10] G. Gel'fand, I.; Shilov. *Generalized functions*, volume 1. Academic Press, New York, 1996-1968.

[11] I. Girsanov. On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. *Theor. Prob. Appl.*, 5:285–301, 1960.

[12] M. Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, 53:183–215, 01 1981.

[13] B. Hall. *Lie Groups, Lie Algebras, and Representations. An Elementary Introduction.* Springer, 2015.

[14] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[15] M. Ibero. Intégrales stochastiques multiplicatives et construction de diffusions sur un groupe de lie. *Bull. Sci. Math*, 100:175–191, 1976.

[16] A. G. Z. Kemna and A. C. F. Vorst. A pricing method for options based on average asset values. *Journal of Banking & Finance*, 14(1):113–129, March 1990.

[17] H. Kesten. Full banach mean values on countable groups. *Math. Scand*, 7:146–159, 1959.

[18] H. Kesten. Symmetric random walks on groups. *Trans. Amer. Math. Soc.*, 92:336–354, 1959.

[19] V. Konakov, S. Menozzi, and S. Molchanov. Diffusion processes on solvable groups of upper triangular 2 x 2 matrices and their approximation. *Dokl. Math*, 84(527), 2011.

[20] V. Konakov, S. Menozzi, and S. Molchanov. Approximation of diffusion processes on solvable lie groups by random walks. local and quasi-local limit theorems. *Analytical and computational methods in probability theory and its application (ACMPT-17)*, pages 202–206, 2017.

[21] V. Konakov, S. Menozzi, and S. Molchanov. The brownian motion on aff(r) and quasi-local theorems. 09 2017.

[22] A. Kumukova. Asymptotic properties of the brownian motion exponential functionals and asian options. Master's thesis, Higher School of Economics, Moscow, Russia, 2018.

[23] H. McKean. *Stochastic integrals*. Academic Press, New York, 1969.

[24] R. Merton. Theory of rational option pricing. *Bell Journal of Economics*, 4(1):141–183, 1973.

[25] N Privault and S. Guindon. Closed form modeling of evolutionary rates by exponential brownian functionals. *Journal of mathematical biology*, 71, 02 2015.

[26] A. Tempelman. *Ergodic Theorems for Group Actions: Informational and Thermodynamical Aspects*. Kluwer Academic Publicshers, 2010.

[27] N. Varopoulos. A potential theoretic property of soluble groups. *Bull. Sci. Math.*, 108:263–273, 1984.

[28] M. Yor. On some exponential functionals of brownian motion. *Advances in Applied Probability*, 24(3):509–531, 1992.