A DYNAMIC APPROACH TO OPTIMIZING INTERVENTIONS AND MITIGATING CONTAGION IMPACTS IN FINANCIAL NETWORKS

by

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ABSTRACT

CHRISTOPHER FARTHING. A Dynamic Approach to Optimizing Interventions and Mitigating Contagion Impacts in Financial Networks. (Under the direction of Dr. Isaac Sonin)

While modern financial networks are so large, complex, and dynamic that the interactions between their members are impossible to model in their entirety, mathematical models with some simplifying assumptions allow for effective study of network dynamics that can inform academics, economists, and policymakers.

We review prior research and mathematical models examining the clearing of liabilities within financial networks, the network dynamics that affect members' abilities to clear, and the role of financial contagion in propagating defaults across a network. Implementing the Banks as Tanks model introduced by Sonin and Sonin (2017, 2020) as a coding solution to derive a network's clearing payment vector as defined by Eisenberg and Noe (2001), we develop an R program that can be used to explore clearing outcomes for the network's members based on initial information about each's cash and debt positions and support further analyses. Extending dynamics observed in the Banks as Tanks model and others, we also extend these models' analysis of outcomes to examine the factors impacting the effectiveness of attempts to rescue defaulting members through provision of outside funding and investment. Our primary contribution is development of a framework to identify optimal interventions a regulator may impose to prevent defaults caused by a bank's own illiquidity or by financial contagion from other defaulting banks. Secondary contributions include our evaluation of the impact of network structure on intervention cost through simulations and our evaluation of methods for simplification of ergodic network or sub-network structures. Our analysis also provides a framework for further analysis of interventions within more complex networks.

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GLOSSARY OF KEY TERMS

Financial network	A set of financial institutions connected by liabilities to one another. We study a network together with information about each member's cash reserves, total liabilities, and how those liabilities are payable to other network members.
Clearing payment vector	A vector specifying the amount each member of a financial network must pay to either satisfy the entirety of its debts or pay out the entirety of its resources attempting to do so.
Default	An inability to satisfy liabilities at the time they are due. A default can trigger the early sale of assets or solicitation of loans to raise capital and may result in bankruptcy in situations where sufficient capital cannot be raised.
Financial contagion	Second-order or higher-order defaults caused by disruptions in the payments scheduled to be received from defaulting firms.
Organic clearing	Clearing dependent solely on the debt and cash positions of the financial network's members and the payments each is able to make (i.e., clearing with no outside intervention).
Intervention	Any action taken to prevent the default of one or multiple financial network members. We focus on interventions defined as loans to struggling institutions from a party external to the network, intended to provide emergency liquidity to allow those institutions to satisfy their current liabilities and continue operations.
Lender of last resort	A party external to the financial network with complete information about the network and the ability to lend funds to struggling firms that would otherwise default.

CHAPTER 1: INTRODUCTION

In this introduction, we provide key background, terminology, and notation that will be used in the remainder of this thesis, much of which follows from the existing body of literature surrounding financial clearing and contagion. Special focus is given to the framework and model introduced by Eisenberg and Noe (2001), which forms the basis for much of the later research and literature, including our own. However, we note questions unanswered by Eisenberg and Noe's static model and others in the literature regarding sufficient conditions for complete clearing and optimal interventions to achieve desired clearing outcomes, and discuss our use of the dynamic Banks as Tanks model proposed by Sonin and Sonin (2017, 2020) to answer those questions.

1.1 Background

The balance sheets of modern financial institutions are densely interconnected, with debts, ownership of stock and securities, and other financial arrangements generating liabilities between institutions. At the same time, financial institutions are under pressure to invest their capital to maximize returns rather than hold it in liquid reserve to cover such liabilities. As a result, institutions rely on the scheduled receipt of payments from other institutions to varying degrees, and unexpected shocks that disrupt those scheduled payments may in turn affect an institution's ability to pay its own debts, a phenomenon described as *financial contagion.* Contagion effects have been identified and explored in the literature, including those caused by direct credit exposures and those caused indirectly (e.g., a firm in crisis negatively impacting the value of assets help by other firms) (see Allen and Gale, 2000; Freixas, Parigi, and Rochet, 2000; Gai and Kapadia, 2010; Glasserman and Young, 2016).

The size, complexity, and dynamic nature of financial networks makes it difficult to understand all the risks network members face, although proactive study of such networks and retroactive root cause analysis after crises can inform academics, economists, and policymakers about aspects of a network that can mitigate or propagate shocks.

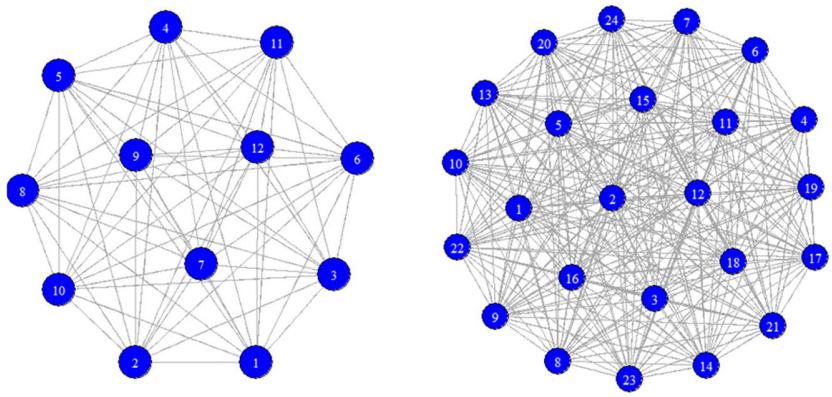

Figure 1: Network Complexity – Complete Networks ($n = 12$ and $n = 24$)

Similar to the networks that connect financial institutions to one another, financial institutions and other businesses are connected to consumers and industries in myriad ways. Economic impacts affect consumers' ability to spend, affecting small businesses and industries who profit from that spending. Harm to industries leads to layoffs and bankruptcies, affecting employment and wages.

Mathematical clearing models can describe how payments will be made and obligations satisfied within a network given details of the obligations and resources available to the network's members. While no model will perfectly forecast real-world outcomes due to real-world networks' size, complexity, and dynamic interactions, clearing models apply rules and conditions that simplify the modeling while aligning conceptually with real-world laws and conventions, allowing for effective study of the clearing process and outcomes (we will focus extensively on models introduced by Eisenberg and Noe, 2001 and Sonin and Sonin, 2017, 2020; however, see also

Elsinger, 2009; Rogers and Veraart, 2013; Fischer, 2014; Capponi and Chen, 2015; Kusnetsov and Veraart, 2018).

For example, in a network involving thousands of banks connected by obligations to one another and having sufficient liquidity to satisfy those obligations, a clearing model can summarize the necessary transactions as a vector describing the net amount each bank should pay or receive. More interesting examples include networks in which some members cannot pay their debts due to some unforeseen shock or circumstance, which may in turn affect the ability of would-be recipients to pay their own debts. In a deficient network like this, a clearing model might instead describe a fair way for solvent banks to pay their debts while defaulting members pay all they can.

Shocks that adversely impact a financial network or economy may arise from various sources. The U.S. financial crisis of 2007-2008 underscored the role of financial interdependencies in the risks facing members of a financial network. After risky subprime lending by U.S. institutions was followed by a sharp decrease in home values, massive mortgage defaults drove large losses for mortgage lenders and financial institutions holding mortgage-backed securities, instruments whose values are tied to the values of underlying mortgages.

Similarly, the current COVID-19 pandemic illustrates the unpredictable nature of economic shocks and the potential for varying impact across industries. From early 2020 to present, the COVID-19 pandemic has changed the way the U.S. economy operates, with widespread stay-at-home orders shuttering businesses deemed non-essential and causing schools and businesses to operate remotely in an effort to reduce social interactions that spread the virus. As a result, unemployment rates have skyrocketed as small businesses have ceased operations, leading to economic hardships at an individual level, while certain industries have suffered as a whole while consumers stay home (e.g., travel, hospitality, entertainment).

In both instances, the U.S. government took action to mitigate the adverse impacts to the U.S. economy, citizens, and businesses. During the 2007-2008 U.S. financial crisis, the widespread financial losses and potential bankruptcy of several prominent financial institutions prompted the U.S. Treasury to intervene to rescue, or "bail out," those institutions, preventing their bankruptcies and any subsequent default cascades (Glasserman and Young, 2016 note that AIG's bailout was likely intended to prevent such a cascade). The Federal Housing Financing Agency took Freddie Mac and Fannie Mae[1] into conservatorship, and the U.S. Treasury acquired preferred equity stakes in many other institutions in exchange for large injections of cash—essentially buying debt and making loans to those institutions—as part of its Troubled Asset Relief Program (TARP). This program was designed to bring stability to the U.S. financial sector and prevent additional bankruptcies from institutions' exposures to troubled mortgages and mortgage-backed securities or losses suffered as a result of defaults by their peers.

While successful in preventing more widespread bankruptcy in the U.S. financial sector, TARP was politically unpopular due to its use of government funds to rescue institutions in the private sector that were seen as culpable for the crisis. TARP was also unpopular due to a lack of transparency into how those funds would then be used by financial institutions and skepticism about the program's ultimate benefit to the U.S. housing market. TARP was also wide-reaching, involving the expenditure of $475 billion to purchase preferred shares of stock in qualifying U.S. financial institutions, to fund loans from a pool made available to the Federal Reserve's member banks, and eventually to purchase shares of equity in troubled U.S. auto manufacturers (see U.S. Department of the Treasury, 2016).

[1] The Federal Home Loan Mortgage Corporation (FHLMC, or Freddie Mac) and the Federal National Mortgage Association (FNMA, or Fannie Mae) are government-established enterprises chartered to expand the secondary market for mortgages by buying mortgages from depository institutions and packaging and selling them as mortgage-backed securities.

In response to the COVID-19 pandemic, in 2020 Congress passed the Coronavirus Aid, Relief, and Economic Security (CARES) Act, allotting more than $2 trillion for enhanced unemployment benefits, loans to small businesses, payroll tax credits for companies in affected industries, and even a payment of up to $1,200 directly to each U.S. individual (see Cochrane & Stolberg, 2020; Internal Revenue Service, 2020; and U.S. Senate Committee on Small Business and Entrepreneurship, 2020). This intervention was intended to support individuals and industries suffering hardship as a result of the pandemic and lessen the long-term impacts to the economy by preserving businesses and industries so that their employees could quickly return to work once the pandemic subsides. While the impact of this intervention is still being measured and will be for years to come, critics and watchdogs have noted concerns with the intervention's design, including the degree of government oversight into the deployment and use of funds, the Federal Reserve's ability to make loan facilities available to intended beneficiaries (e.g., small businesses), and a lack of provisions to ensure the health and safety of those workers whose jobs were preserved (see Judge, 2020; Kamensky, 2020; and Bivens & Shierholz, 2020).

1.2 Motivation and Goals

Learning from these historical cases of crisis and intervention by regulators, we note the potential value of a methodology to determine specific recipients, amounts, and times in which a more targeted inflow of capital could also successfully reduce default and financial contagion impacts in a network at an optimal (i.e., minimal) cost.

We note that the goal of such an intervention may vary depending on the situation and the regulator's priorities. For instance, Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) show that in networks where certain institutions are more well-connected, the failure of these well-connected

institutions may have larger consequences to the overall liquidity of the network (through contagion effects) than would the failure of smaller, less well-connected institutions. Therefore, in such a network that is facing stress, a regulator's goal may be to prevent bankruptcy of only the most central institutions or those most important by some definition. In more well-distributed networks, a regulator may prefer to prevent bankruptcy of the largest number of institutions, or simply to maximize clearing by minimizing the total amount of debts that remain unpaid before organizing restructuring of remaining debts.

We also note that potential interventions could take many forms (e.g., Chapter 11 bankruptcy involving mediation between defaulting banks and their creditors to restructure the debts). However, we wish to examine methods involving the optimal application of a loan or investment from a *lender of last resort* external to the network, who may intervene to rescue banks that would otherwise default. Thus, we define an *intervention* as a loan or other injection of capital from such an entity intended to provide emergency liquidity to allow struggling members of a financial network to pay their debts and continue operations.

For notation, we define a vector $\boldsymbol{w}$ of length n that denotes the amount and distribution of funds provided in an intervention, with w_i denoting the amount provided to bank i. We wish to derive the vector $\boldsymbol{w}$ that accomplishes a particular intervention goal with lowest cost (i.e., with $w = \sum_n w_i$ minimized).

Given information that could be obtained by a regulatory body during—or preferably prior to—such a crisis, we extend the existing line of research following from a classical clearing model introduced by Eisenberg and Noe (2001) to problems that evaluate the cost and effectiveness of interventions and methods for optimizing them. Our analysis works within the framework introduced by Eisenberg and Noe (2001) and the clearing dynamics presented in Sonin and Sonin's

(2017, 2020) Banks as Tanks model to set a desired end state and identify minimal conditions that must be met to obtain it. We then apply these methods to answer four key questions:

1) How can all defaults in a network be prevented at lowest cost?
2) How can the default of a particular network member be prevented at lowest cost?
3) How can the defaults of a group of network members be prevented at lowest cost?
4) How can our framework and simulations provide insight into risks in a way that can inform policy on network structure and reserving requirements?

The first three questions will be answered by **Theorem 1**, **Theorem 2**, and **Theorem 3**. After we explain the dynamics of interventions and introduce modifications to the BaT model framework, we introduce our algorithmic approach for measuring the benefit of investment from an external lender to one, all, or a group A of banks through **Multiplier Effects in the Benefit of Intervention to a Single Bank.** How can the benefit to a particular zero-status bank be measured as a function of the amount invested in any other bank?

Theorem 4, **Theorem 5**, and **Theorem 6**, which we then use in **Section 4.4** to select optimal recipients given any investment amount and demonstrate the algorithmic approaches that answer our questions. The final question will be addressed through observations from simulation results in **Section** 5.2.

1.3 Framework and Eisenberg & Noe Model

Eisenberg and Noe's clearing model provides a framework used by many further studies of financial networks. Eisenberg and Noe (2001) define a *financial network* as a set of n nodes representing entities with financial liabilities to one another. Eisenberg and Noe specify the network with a vector $\boldsymbol{c}$ containing elements $c_i \geq 0$ denoting each member's operating cash prior

to clearing and a matrix B whose elements $b_{i,j}$ denote the liabilities member i has to member j. Equivalently, the network can be specified by vector $\boldsymbol{c}$, a vector $\boldsymbol{b}$ consisting of elements b_i denoting each of the n network members' total liabilities, and a matrix Q whose elements $q_{i,j}$ describe the proportion of member i's total liabilities owed to member j. Matrix B uniquely defines vector $\boldsymbol{b}$ and matrix Q, and vice versa, with $q_{i,j} = \frac{b_{i,j}}{\sum_j b_{i,j}}$, $b_i = \sum_j b_{i,j}$, and $b_{i,j} = b_i q_{i,j}$ for all i, j. Using these parameters, Eisenberg and Noe seek to solve for a *clearing payment vector* $\boldsymbol{p}$, whose elements p_i describe the total amount each member i will pay.

Eisenberg and Noe note two logical conditions that must be met by this clearing payment vector:

(A) Limited liability: A member's total payments cannot exceed its available cash (i.e., members cannot make payments without having cash). Additional cash may be received from other network members during the clearing exercise. Thus,

$$p_i \leq c_i + \sum_{j=1}^{n} q_{j,i} p_j \,, \forall i \tag{1}$$

(B) Absolute priority and proportionality: Each network member either pays the entirety of its obligations or pays the entirety of its value (i.e., cash and realized receivables) towards its liabilities, paying $p_i q_{i,j}$ to each member j. Thus, for a member i the same proportion (100% or lower) of each of that member's liabilities will be paid, implying an equal priority among the debts owed by each member.[2]

Thus, the *clearing payment vector* $\boldsymbol{p}$ consists of elements p_i that satisfy the following for all i:

[2] Realistic financial networks often include debts of differing seniorities, meaning that some debts are paid with higher priorities than others (i.e., some debts are paid first). However, we note that Eisenberg and Noe's proportionality assumption represents a fair way to distribute the assets of defaulting members, and the assumption simplifies subsequent analysis.

$$p_i = min(b_i, c_i + \sum_{j=1}^{n} q_{j,i} p_j) \tag{2}$$

When $p_i < b_i$, we say that member i has *defaulted* on its liabilities.

Noting that the amount each member pays will be the minimum between the amount it owes and its starting cash plus the amount it receives, Eisenberg and Noe define a contraction that maps $[\mathbf{0}, \boldsymbol{b}] \rightarrow [\mathbf{0}, \boldsymbol{b}]$. Using a fixed point argument, Eisenberg and Noe then prove the existence of the clearing payment vector specifying the amount each member must pay to either satisfy the entirety of its liabilities or pay out the entirety of its value in an attempt to do so. Eisenberg and Noe also introduce an algorithmic method for deriving the clearing payment vector in a finite number of discrete steps, which illustrates the contraction referenced by their proof.

Eisenberg and Noe's *fictitious default algorithm* begins by examining each network member's ability to satisfy its liabilities assuming that all other members satisfy theirs. Any member that cannot satisfy its liabilities from its cash reserves and receivables (all of which are assumed to be realized) is said to default. Following an iterative process, in a next round of calculations each member is assumed to fully realize its receivables from non-defaulting members, but the cash and realized receivables of defaulting members are instead allocated to their creditors in proportions equal to the proportions of the creditors' claims (but satisfying none of them fully). Thus, the same percentage of each of a defaulting member's debts is satisfied. Each non-defaulting member's ability to satisfy its liabilities is then re-examined with the reduced payments from defaulting members, any subsequent defaults are recorded, and payments from defaulting members are again adjusted accordingly. This process continues in additional iterations until no additional defaults occur.

The fictitious default algorithm highlights the contraction supporting Eisenberg and Noe's fixed point argument, as values within the payment vector, which is bounded within the interval $[\mathbf{0}, \boldsymbol{b}]$, monotonically decrease throughout the algorithm's steps. The existence of the clearing payment vector (the algorithm's final output) also follows easily from this algorithmic approach.

1.4 Additional Questions and the Banks as Tanks Model

Eisenberg and Noe's model provides the framework used in many subsequent studies of clearing. We note that their model is static, deriving the outcome of clearing for debts due at a single point in time and providing a sequence of defaults that yields some insight into the drivers of outcomes (e.g., default or complete satisfaction of liabilities) for each member during the clearing process. However, their model does not explain how the banks will complete these payments or facilitate much insight into the conditions required for clearing or how any would-be defaults may be prevented.

To address these questions, we note the value of a dynamic model that tracks the progress of clearing and provides insight into the causes of any defaults. The Banks as Tanks model introduced by Sonin and Sonin (2017, 2020) allows for derivation of the clearing payment vector using a dynamic approach, in which payments occur over time within intervals defined by changes in a network member's status (i.e., when a member runs out of cash or clears the entirety of its debts). Similar to Eisenberg and Noe's model, Sonin and Sonin's Banks as Tanks model allows for modeling of what we term *organic clearing*—that is, the clearing payment vector and outcomes given the point-in-time cash and liabilities of network members with no outside intervention. The model demonstrates the value of continuous-time modeling, arriving at the same clearing payment vector as Eisenberg and Noe under an intuitive and transparent framework that allows examination

of payment dynamics at any stage of the clearing process. Sonin and Sonin propose the model, discuss its application through several examples, and note that the model would be easily programmable, but did not operationalize or automate the model in their paper.

Our research implements the Banks as Tanks model through an R program to output the results of organic clearing, then evaluates similar payment dynamics before, during, or after organic clearing in order to examine the impact of a loan from an external source to network members that would otherwise default. We extend dynamics observed in the Banks as Tanks model to propose a framework for cost/benefit analysis of such interventions and identification of the optimal intervention locations and amounts—those that minimize cost of preventing one, several, or all defaults or maximize the benefit of a fixed investment amount towards that goal—to protect the stability of a financial system under stress or prevent the default of particular members.

We then modify the Banks as Tanks model to specify a desired outcome and calculate the minimum pre-clearing cash positions required by each network member to achieve it, allowing us to identify any shortfalls in the actual pre-clearing cash positions that would prevent the clearing of all debts or clearing of debts by particular network members. Lastly, we explore how properties of Markov chains may be applied to simplify complex ergodic networks and derive further insights, and we examine through simulation the characteristics of a financial network's structure that impact the cost of interventions.

Simplifying Assumptions. Although realistic financial networks involve liabilities payable at different maturity dates or with differing priorities (i.e., liabilities that must be paid before others), we provide this initial evaluation under the simple conditions set forth by Eisenberg and Noe (2001)—equal priority of liabilities and analysis of clearing within a single time period—and note

the potential for future extensions of this research or the Banks as Tanks model into networks with more complex structures or different rules. We note that real-world financial networks are made even more complex by stochasticity driving uncertainty in liabilities, cash positions, and outcomes, by gaming between network members and regulators (see Bernard, Capponi, and Stiglitz, 2017; Capponi, Corell, and Stiglitz, 2020), and by changes in domestic or foreign laws, regulations, and conventions over time. While we explore elements of stochasticity through simulation, we do not explore gaming between network members and we assume the legal and regulatory environment remains static.

Additionally, we note that complete information about the cash, debts, and claimants of those debts for each member of a financial network is generally unavailable to researchers. Further, our banking experience has shown that even individual members of the network are limited in their knowledge of such details for other members. Thus, researchers have historically relied on hypothetical examples while noting that complete information may only be obtainable by regulatory agencies or other such bodies having authority to request such information.[3] However, we note that a body of literature exists describing methods for estimating unknown network connections given known values of total cash, total liabilities, and potentially some known connections (see Gandy and Veraart, 2017 and Anand et al., 2018). We note use of such estimation methods as another possible extension of our research that may allow its application by parties with incomplete information (e.g., individual network members, clearing houses).

[3] For example, Hida (2013) notes that the Dodd-Frank Act charges the U.S. Treasury's Office of Financial Research (OFR) with collecting "financial transaction data and position data from financial companies," either independently or from other regulatory bodies. Acharya et al. (2013) further discuss the OFR's responsibilities along with other provisions of the Dodd-Frank Act intended to measure and mitigate systemic risk in the U.S. financial system.

The remainder of this thesis is structured as follows: In **Chapter** 2, we frame our study through review of the existing literature, including early works from Eisenberg and Noe (2001) and Suzuki (2002) and the literature that has expanded on this research in the intervening years to present other mathematical clearing models. We also review literature discussing the 2007-2009 U.S. financial crisis from a non-mathematical economic perspective and subsequent literature that began to apply mathematical clearing models to retrospectively examine the financial crisis.

In **Chapter** 3, we discuss in greater detail the Banks as Tanks model introduced by Sonin and Sonin (2017, 2020). In particular, we discuss the key definitions and notation that form the framework for this model and most other studies of financial clearing and contagion represented in the literature, along with particulars of the algorithm, its outputs, and the insights it provides.

In **Chapter** 4, we evaluate the projected status of a network after clearing, at which time some members may be at risk of default on their liabilities as a result of illiquidity (e.g., from some external shock driving losses or from defaults on payments from other members). We define a lender of last resort with an interest in preventing these defaults by providing emergency liquidity, and we adapt the framework underlying the Banks as Tanks model to evaluate the dynamics that impact the effectiveness of such an intervention. We present several methods for deriving an optimal deployment of funds by such a lender—optimal in the sense that the benefit to the network or key members is maximized or the cost of meeting a particular goal is minimized. Lastly, we demonstrate a method by which properties of a Markov chain can be applied to simplify a complex set of liabilities or transactions, including a special case in which no liquid cash is available but clearing can still occur through multilateral cancelation of debts.

Chapter 5 presents our results from simulation evaluating the relationship between the structure of a network's connections and the cost of preventing defaults given randomized negative

shocks. We relate network structure to the minimal sufficient cash positions required for clearing, and explore the simulation results to provide insight into the underlying factors that increase or mitigate intervention costs.

Lastly, **Chapter** 6 presents several possible extensions of our research into more complex clearing problems and specific applications, and **Chapter 7** concludes.

CHAPTER 2: LITERATURE REVIEW

We review the modern literature discussing clearing problems, following from contributions by Eisenberg and Noe (2001) and Suzuki (2002) that define a financial network and a framework for studying it and introduce mathematical models to illustrate the clearing process and its outcomes. In the intervening years, additional papers have sought to extend these intuitive frameworks into more general networks and clearing problems, either to address specific limitations of Eisenberg and Noe's or Suzuki's models (e.g., their use of a single maturity date for all liabilities) or to incorporate evolving characteristics of modern networks (e.g., liabilities generated by complex derivative instruments).

The 2007-2009 U.S. financial crisis was a catalyst for renewed interest and new perspectives on the topic as the U.S. financial sector experienced widespread liquidity issues. Years of excessive risk-taking and increased availability of securitized mortgage products had led to balance sheets heavy with mortgage-related assets, which lost value after a sharp and unexpected drop in home prices. Financial interdependencies between institutions drove contagion effects that allowed the adverse impacts to spread far and wide, resulting in a global economic crisis. In subsequent papers, economists and academics have worked to dissect the crisis, evaluating the factors that caused it or affected its magnitude and offering myriad perspectives on how it should have been foreseen, prevented, or mitigated.

The literature is expansive, so we limit our review to several key papers discussing clearing problems and a set of additional papers expanding upon these initial frameworks. We organize the relevant literature into three categories: papers discussing the general theory of clearing problems and introducing relevant mathematical models; papers discussing the 2007-2009 U.S. financial

crisis, its causes, and its outcomes from a non-mathematical economic perspective; and papers evaluating clearing problems and applying mathematical models with a retrospective focus on the crisis.

2.1 Eisenberg and Noe (2001) and Subsequent Works

Eisenberg and Noe (2001) provide an early literature and framework for studying clearing problems from which much of the subsequent research stems. Using details about an interconnected set of institutions, including their point-in-time liquid cash and imminently-payable liabilities to one another, Eisenberg and Noe use a fixed point argument to show that there exists a unique clearing payment vector specifying the amount each institution must pay to either satisfy the entirety of its liabilities or pay out the entirety of its value (i.e., its cash reserves plus realized receivables) in an attempt to do so. In the event of a default, the defaulting institution pays out its full value to its creditors in amounts proportionate to their claims. Thus, the clearing payment vector $\boldsymbol{p}$ consists of elements $p_i = \min\left(b_i, c_i + \sum_{j=1}^{n} q_{j,i} p_j\right)$ denoting the amount an institution i will pay, where b_i denotes the institution's total debts due at this time, c_i its liquid cash reserves before clearing, p_j the total amount each institution j will pay, and $q_{j,i}$ the proportion of institution j's imminently-payable liabilities owed to institution i.

Specifically, Eisenberg and Noe define a mapping $\delta: [\mathbf{0}, \boldsymbol{b}] \to [\mathbf{0}, \boldsymbol{b}]$, with $\delta(\boldsymbol{p}, Q, \boldsymbol{b}, \boldsymbol{c})$ a vector whose elements are $\min\left(b_i, c_i + \sum_{j=1}^{n} q_{j,i} p_j\right)$ for each $i = 1, 2, \dots, n$, the number of institutions in the network. Any defaults for a bank i results in a decrease in p_i to take a value lower than b_i, and possibly also a decrease in p_j for some other banks j when additional defaults are caused through contagion; thus δ is a contraction. As a contraction, δ implies existence of a fixed point, the clearing vector, which also solves the system of equations $p_i = \min\left(b_i, c_i + \sum_{j=1}^{n} q_{j,i} p_j\right), i = 1, 2, \dots, n$.

As described in **Section** 1.3, Eisenberg and Noe introduce their fictitious default algorithm to derive the clearing payment vector. Eisenberg and Noe's analysis considered only liabilities of equivalent priority due at a single maturity date, although their framework has formed the basis of many subsequent papers seeking to generalize their conclusions in realistic networks involving differing debt priorities or multiple maturity dates.

With similar timing, Suzuki (2002) provides a study of clearing in financial networks in which pairs of institutions are connected both through debts and through ownership of one another's stock. Such cross-holdings of equity, termed *mochiai* in Japanese, are a common practice in Japan, and provide benefits in terms of protection from default risk or takeover and incentive for cooperation.[4] In the event of default, debts are paid before shareholders receive any value, and thus the two classes of liabilities have differing seniorities, although Suzuki did not examine more granular levels of seniority within each class (e.g., among debts). Suzuki proves the existence of a *payoff vector* in this case that either fully satisfies each institution's liabilities or fully allocates its value, first towards debts and then towards equities.

Expanding on these initial examinations of clearing, Elsinger (2009) generalizes clearing in networks in which debts have differing levels of seniority and modifies Eisenberg and Noe's fictitious default algorithm to derive a clearing vector within this framework. Fischer (2014) expands upon Suzuki's results by examining asset valuation in situations involving cross-ownership of debts, equities, and more general liabilities such as derivative instruments. In Fischer's study, liabilities within each of these classes may also have differing seniorities,

[4] Such cross-holdings have also been shown to produce double-counting of the equity in calculations of the participating institutions' market values. See McDonald (1989).

generalizing Suzuki's results. Kabanov, Mokbel, and El Bitar (2018) provide a survey of such recent papers and examine the existence and uniqueness of clearing vectors in each case.

2.2 Economic and Mathematical Papers Regarding Financial Crises

After the U.S. financial crisis provided a recent, real-world example of a financial network in peril, new literature began to offer retrospective analysis of the crisis. Economists like the Nobel laureate Paul Krugman offered conjecture on the root causes of the crisis and most economists' inability to predict it,[5] advice to the incoming Obama administration on measures to counteract it,[6] and discussion of the effectiveness of the interventions that were ultimately implemented.[7] The form of the government's response, involving government expenditure of greater than $400 billion to provide capital to banks through purchase of their preferred stock, largely aligned with Krugman's recommendations, albeit in a much lower amount than he thought necessary. Taking the Keynesian view that government expenditure may have a multiplier effect leading to broader expenditures throughout the economy, Krugman's recommendations also included ongoing government expenditure and investment in areas with long-term value such as healthcare reform, consistent with the Obama administration's later implementation of the Affordable Care Act, also dubbed Obamacare.

After the crisis, the mathematical papers also evolved, proposing application of existing or new mathematical models to study particular aspects of the crisis or the economy that produced it. In particular, authors began discussing the relationship between network structure and contagion risk and analyzing the cost of defaults or the cost of their prevention.

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[5] Paul Krugman, *How Did Economists Get It So Wrong*, 2009 and Paul Krugman, *The Profession and the Crisis*, 2011.
[6] Paul Krugman, *What to Do*, 2008 and Paul Krugman, *What Obama Must Do*, 2009.
[7] Paul Krugman and Tim Dickinson, *Obama's Bailout*, 2009 and Paul Krugman, *Earth to Ben Bernanke*, 2012.

With a retrospective view of the U.S. financial crisis, Rogers and Veraart (2013) introduce additional parameters describing the cost of a default, in which some portion of a defaulting bank's assets and receivables is used to cover liquidation expenses or is sacrificed in pursuit of early liquidation. This work highlights the general loss of value in a financial network or economy caused by defaults. Further, Rogers and Veraart describe situations in which an in-network bank or consortium of banks has an incentive to rescue banks that would otherwise default (e.g., through mergers) when the cost of absorbing the rescued banks' outstanding liabilities is less than the losses that would be incurred as a result of those banks' failures.

Noting realistic situations in which debts are due to be paid at different points in time, Capponi and Chen (2015) examine clearing problems incorporating multiple maturity dates, differentiating between illiquidity (the lack of liquid capital to cover currently-due debts) and insolvency (when the present value of cash, scheduled operating income, and scheduled receivables is less than the bank's liabilities) and including in their model the ability of a lender of last resort with complete information about the network (e.g., a central bank) to lend funds to illiquid yet solvent banks. Capponi and Chen examine the effects of network structure on systemic risk and note that the uniformness of the network's liabilities—or conversely, the centrality of particular institutions—has implications for the effectiveness of particular interventions by the lender of last resort. When certain institutions are more centrally-connected within the network, interventions to benefit those systemically important institutions are more effective in reducing systemic risk than are more general interventions to increase the network's total liquidity. Accordingly, we note that in the U.S. financial system, banks with larger total assets are subject to increased scrutiny and controls to protect the broader system's stability, with the largest banks designated Systemically Important Financial Institutions, or SIFIs (see Smith, 2016).

Kusnetsov and Veraart (2018) note that Capponi and Chen's model (as well as most previously presented single-maturity models) ignore liabilities due in the future when apportioning the value of defaulting institutions to their creditors. Modern bankruptcy law allows institutions that are owed future payments to make claims on the value of the bankrupt institution's assets, even when the bankruptcy occurs before those debts are due.[8] Allowing the value of defaulting institutions to be apportioned among claimants of current *and* future payments, Kusnetsov and Veraart present a fixed point argument for the existence of a payment vector and follow an approach similar to Eisenberg and Noe's fictitious default algorithm to determine the outcome of clearing.

Citing discrepancies in the literature's conclusions as to whether a densely-connected network mitigates or exacerbates the risk of contagion in a network, Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) seek to explain the characteristics of a network that may lead to either conclusion. Acemoglu et al. generalize the clearing problem in terms of an *interaction function*, which describes how each member's state (e.g., solvent, in default) is defined as a function of the others' states, and an *aggregation function*, which describes how the states of the members aggregate to describe the value of the overall network. Finally, Acemoglu et al. explore a financial network's ability to share and disperse a mild shock among its members—or to allow adverse effects (e.g., credit defaults) from larger shocks to cascade from one member to additional members of the network—based on attributes of the interaction and aggregation functions (e.g., the density of financial connections between its members, the cost of defaults).

In more recent literature, Roukny, Battiston, and Stiglitz (2016) and Eboli (2019) discuss cyclicality in networks and how cycles (in which a chain of debts starts and returns to a particular

[8] See Stemplewicz and Maizel (1996), Insolvency Service (2015)

bank) can lead to multiple points of equilibrium in which groups of banks may default or remain solvent as one.

2.3 Sonin and Sonin (2017, 2020) and Continuous-Time Clearing

In this existing literature, several models have been proposed to not only derive a clearing vector, but to also describe the dynamics of clearing within a financial network and the impact of a default on other members of the network. Given information about the cash reserves and liabilities between institutions in the network, these models provide information about the total payments each institution will make and each institution's cash and debt positions after the clearing process. Of particular interest is the final state of the network after all possible payments have been made, at which time each institution has either cleared the entirety of its liabilities or no longer has the cash reserves and realized receivables with which to do so.

Sonin and Sonin (2017, 2020) expand upon this literature by presenting their dynamic Banks as Tanks model. This model examines liabilities due at a single point in time, but expands this point into an artificial interval of continuous time to allow for examination of the clearing dynamics and flow of payments during the clearing process. In this intuitive model, banks are analogous to tanks containing a certain amount of liquid (representing liquid cash reserves). Net liabilities are represented by pipes between tanks, which allow a unidirectional flow. As banks make payments to satisfy their liabilities to one another, liquid flows between the corresponding tanks at particular rates until each tank is either empty with no inflows to refill it or the tank no longer has liabilities and outflows, and all flows between the tanks cease.

This dynamic model introduces a chronological component that allows insight into properties of the system at any given point in the clearing process (e.g., institutions' solvency statuses,

outstanding debts, and cash positions), the order and causes of status changes (i.e., default or complete satisfaction of liabilities), and the impact of such status changes on the subsequent payment dynamics of the system. The dynamic nature of the model could also lend itself well to problems involving multiple maturities or payments that actually occur over intervals of real time, although Sonin and Sonin (2017, 2020) leave these extensions for future research.

We explore the Banks as Tanks model framework, implement it in an R program, then extend it to examine the impact of interventions—loans or other inflows of cash from outside the network that increase liquidity and prevent defaults—arriving either before or at the time of a default. Further, we examine the dynamics of a financial network that lead to increased or decreased effectiveness of interventions. Finally, we use these observations to derive methods for designing interventions with optimal effectiveness—those that accomplish their goals at the lowest cost or that deploy a fixed amount of funds with maximal benefit.

CHAPTER 3: THE BANKS AS TANKS MODEL

To frame the study of clearing under a dynamic approach, we discuss the Banks as Tanks (hereafter, "BaT") model introduced by Sonin and Sonin (2017, 2020), which expands the instantaneous moment in which clearing occurs into an artificial interval of time to allow insight into payment dynamics that ultimately lead to either full payment or default on obligations. We discuss the model and an R program we developed to implement it, and we apply the model in a selected example.

3.1 Banks as Tanks Model – Fundamentals

We define a *financial network* F as a set of n financial institutions ("banks," to simplify) having liabilities to one another at a given point in time, together with information about each's cash reserves, total liabilities, and the proportion of those total liabilities owed to each of the other network members. Thus, consistent with the framework introduced by Eisenberg and Noe (2001) and discussed in **Section** 1.3 and **Section** 2.1, we define a financial network $F = (\boldsymbol{c}, \boldsymbol{b}, Q)$,

where $\boldsymbol{c}$ is a vector of length n whose elements are the initial cash reserves of each of the n banks, $\boldsymbol{b}$ is a vector of length n whose elements b_i are the total initial in-network liabilities currently due from each of the n banks, and $Q = \{q_{i,j}\}$ is an $n \times n$ matrix whose element $q_{i,j}$ represents the percentage of b_i that is owed to bank j.[9] Finally, a vector $\boldsymbol{p}$ describes the total amount each bank has paid during the clearing process, with initial $\boldsymbol{p}(\mathbf{0}) = (0, \dots, 0)$. The dynamic nature of the

[9] For simplicity, we assume that Q takes values of 0 along its diagonal, denoting that no bank has liabilities to itself. However, our results also generalize to networks in which Q takes nonzero values along its diagonal, denoting liabilities from a bank to itself (e.g., from one internal business unit or affiliate to another).

model then allows these parameters to vary over time, with $c_i(t)$, $b_i(t)$, and $p_i(t)$ describing the relevant details of bank i at time t. We set $\boldsymbol{c}(0) = \boldsymbol{c}$, $\boldsymbol{b}(0) = \boldsymbol{b}$, and $\boldsymbol{p}(0) = \boldsymbol{p}$.

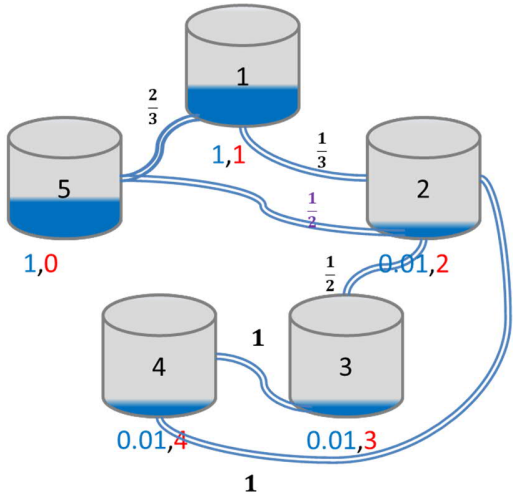


Figure 2: BaT Network Example 1A (with cash, debt, and flow rates)

Sonin and Sonin (2020) allow negative cash positions $c_i(t)$ in their model, interpreting them as cash positions reflecting lower-priority debts external to the network. This practice requires additional record-keeping to track the true cash position during clearing (c_i disregarding these external debts) and stipulate that c_i should not become *more* negative. For simplicity, we instead assume any subsequently-due or lower-priority external debts will be addressed in a subsequent clearing exercise.

When pipes open, $c_i(t)$, $b_i(t)$, and $p_i(t)$ will vary with outflow rates $u_i(t)$ and net flow rates $d_i(t)$ for each i. In particular, $b_i(t)$ will decrease by $u_i(t) \cdot t$, $p_i(t)$ will increase by $u_i(t) \cdot t$, and $c_i(t)$ will increase by $d_i(t) \cdot t$. However, banks will stop paying when their debts are satisfied and their out-pipes will be closed, or if their cash reserves are exhausted while they still have debt they will pay only as they receive cash from other banks. Thus, these dynamics $u_i(t)$ and $d_i(t)$ will change after certain intervals of time, and it is useful to define a *status* for each bank from which these payment dynamics follow.

$\mathbf{b}(t)$ and $\boldsymbol{c}(t)$ generate a partition of the set J of all banks. We describe the set each bank belongs to as that bank's *status*, describing its solvency at time t. In particular,

$$\begin{aligned} &J_+(t) \text{ contains banks with } b_i(t) > 0, c_i(t) > 0 \\ &J_0(t) \text{ contains banks with } b_i(t) > 0, c_i(t) = 0 \\ &J_*(t) \text{ contains banks with } b_i(t) = 0 \end{aligned} \tag{3}$$

As described above (respectively), these statuses are:

- *positive*, with positive-status banks having both debts and cash reserves from which to pay them
- *zero*, with zero-status banks having debts but no cash reserves, but still able to make payments toward their debts as payments are received from other banks; these banks are in danger of defaulting, but may still satisfy their debts if their receivables are sufficient
- *absorbing*, with absorbing-status banks having no debts and thus making no payments, but still able to receive payments from other banks

Following the model's tank analogy, banks are represented by tanks containing liquid, and this liquid represents the bank's available cash. Tanks are connected by pipes that represent the liabilities between banks. Liquid may flow in only one direction though a particular pipe; thus a pipe between two tanks denotes a directional liability payable by one bank to the other.

Real-world clearing problems may involve netting agreements, in which two counterparties agree in advance to allow payments they owe each other from similar deals to offset before the net liability is recorded. In situations where a pair of banks have debts to one another and a netting agreement is in place, the net liability between the two may be represented by a single pipe. Thus, in such situations, the pair of tanks has at most one pipe directly connecting them. However, we

note that the BaT model may also be applied to networks including pairs of banks without netting agreements by including two pipes between a pair of banks to allow flows in both directions.

Obviously, an absorbing-status bank with no liabilities to other banks will have no reason to make payments, and thus the corresponding tank has no pipes allowing outflows. Similarly, a zero-status bank with no cash reserves (i.e., an empty tank) can make no payments unless it receives cash from other banks. Therefore, using each bank's solvency status, we define realistic rules describing the dynamics of flows between tanks by setting a flow rate for each pipe that describes the cash flow over some arbitrary unit of time.[10] The total outflow rate $u_i(t)$ for each tank is:

$$u_i(t) = \begin{cases} 1, & i \in J_+(t) \\ n_i(t), & i \in J_0(t) \\ 0, & i \in J_*(t) \end{cases} \tag{4}$$

- a positive-status bank pays out at a baseline rate of $u_i(t) = 1$ until it runs out of cash and moves to a zero status or satisfies its debts and moves to an absorbing status
- a zero-status bank must pay out at a rate equal to its inflow rate, $u_i(t) = n_i(t)$, until it satisfies its debts and moves to an absorbing status; thus, calculating outflow rates for zero status banks may involve solving systems of equations
- an absorbing-status bank has no debts, and thus pays out at a rate of $u_i(t) = 0$

Outflow rates for each individual pipe from bank i are then given by the corresponding value of $u_i(t)q_{i,j}(t)$. For example, a positive-status bank with a flow rate of 1 that owes equal amounts to two other banks will be connected to these banks via pipes each having a flow rate of ½. We note that this practice ensures that in the event of a default, the defaulting bank's value (i.e., its

[10] A key feature of the Banks as Tanks model is the expansion of a single moment in time and the liabilities due in that moment to an artificial interval of time. Because this interval is artificial, the unit for flow rates is arbitrary (e.g., dollars per millisecond, millions of dollars per minute); ultimately, all payments may be interpreted as occurring instantaneously.

cash and receivables) has been paid to its creditors in proportions equal to the proportions of their claims on the defaulting bank's total debts. This is consistent with the proportionality assumption (B) defined by Eisenberg and Noe (2001) and represents a fair way to distribute the assets of a defaulting bank.

Under this condition, each of the debts owed by a bank is payable with the same priority. Either the bank pays the entirety of its debts in full, or the bank exhausts its resources paying an equal portion of each debt it owes. However, we note that in real-world networks, the priority with which debts are paid can vary broadly, with stipulations that some debts be paid in full prior to *any* payment toward other, more junior debts (e.g., owners of preferred stock are paid prior to owners of common stock). Payments may also be due at differing times or on flexible timelines. Banks may have a variety of reasons to prefer to pay one debt and risk default on another. We note the exploration of alternative prioritization strategies as a possible extension of this research, and for now we focus on clearing of networks under the condition of proportionality in payments.

Next, the total inflow rate $n_i(t)$ for a bank i follows from the outflow rates, with

$$n_i(t) = \sum_j u_j(t) q_{j,i}(t) \tag{5}$$

The total outflow rate $u_i(t)$ describes the rate of reduction in bank i's debt, while the net inflow rate $d_i(t) = n_i(t) - u_i(t)$ describes the change in each bank's cash position over time. Thus, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$, and $\boldsymbol{p}(t)$ will change over time, with

$$\begin{aligned} b_i(t) &= b_i(0) - \int_0^t u_i(s)ds \\ c_i(t) &= c_i(0) + \int_0^t d_i(s)ds \end{aligned} \tag{6}$$

$$p_i(t) = p_i(0) + \int_0^t u_i(s)ds$$

Recall that flow rates were calculated based on each bank's solvency status. Thus, these flow rates describe payment dynamics within a specific interval of time that ends when one or more banks change status by running out of cash or completely satisfying their debts. After a status change, flow rates must be updated to reflect the new set of statuses, and these new payment dynamics are followed until the next status change. Thus, $u_i(t)$, $n_i(t)$, and $d_i(t)$ are piecewise functions, constant within time intervals bounded by any status change affecting a bank in the network and thus the network's payment dynamics. In the absence of outside intervention, banks in a positive status may move to a zero or absorbing status, and banks in a zero status may move only to an absorbing status.[11]

The algorithm ends at a finite time T^* when $J_+(T^*) = \emptyset$ (i.e., when no bank with remaining debts to pay also has cash with which to pay them), and inflow and outflow rates become zero for all banks in the network. At this point, each bank i will have either satisfied its debts, ending in an absorbing status with $i \in J_*(T^*)$, or will have remaining debts with no means to satisfy them, ending in a zero status with $i \in J_0(T^*)$. The final clearing payment vector is given by $\boldsymbol{p}^* = \boldsymbol{p}(T^*)$, with

$$p_i(T^*) = \int_0^{T^*} u_i(s)ds \tag{7}$$

The model's key outputs include the final statuses, debt positions $\boldsymbol{b}^*$, cash positions $\boldsymbol{c}^*$, and payments made by each member of the network $\boldsymbol{p}^*$, none of which depend on the total "time"

[11] These rules apply in organic clearing; that is, in the absence of interventions. As we explore interventions that make additional cash available to institutions that would otherwise default, the timing of the intervention may allow a zero-status institution to move into a positive status by adding cash to an otherwise zero cash position.

elapsed, only the completion of the modeled cash flows; all cash flows may be interpreted as having happened instantly.

Sufficient and Deficient Models. When $\boldsymbol{p}^* = \boldsymbol{b}$, the clearing payment vector reflects that all debts are satisfied and we call the model *sufficient*. When $p_i^* < b_i$ for some i, we say the model is *deficient*. Given these definitions, it is useful to define the vector of minimum cash positions that differentiate between a sufficient model and a deficient model for a network with given $\boldsymbol{b}$ and Q. We call this vector $\boldsymbol{c^{min}}$, and explain its derivation below.

After modeling clearing with the BaT model, the final cash position of each bank is $c_i^* = c_i + \sum q_{j,i} p_j - p_i$ (initial cash plus realized receivables less payments made). Because we wish to evaluate the cash vector $\boldsymbol{c}$ that facilitates complete clearing, we set $p_i = b_i, \forall i$ and we have $c_i^* = c_i + \sum q_{j,i} b_j - b_i$. In addition, to obtain the minimal required cash positions, no bank should end with excess cash, so we set $c_i^* = 0, \forall i$ and obtain $0 = c_i^{min} + \sum q_{j,i} b_j - b_i$. Therefore, it follows that for complete clearing

$$c_i^{min} = b_i - \sum q_{j,i} b_j \tag{8}$$

or debts minus receivables (all of which are assumed realized) for each bank.

We note that $\boldsymbol{c^{min}}$ describes the difference between payments made and payments received given complete clearing, and payments made by one bank will be received by another. Thus, in all cases $\sum_i c_i^{min} = 0$. Banks with $c_i^{min} \leq 0$ can clear even with no starting cash; assuming a sufficient model, these banks' receivables alone are sufficient to allow them to pay their debts and end the clearing exercise with positive cash positions. Given a sufficient model, a bank with

$c_i^{min} \leq 0$ could take on $-c_i^{min}$ in additional debt (due at the same or a later time) and still satisfy it from its receivables during clearing, even with $c_i = 0$. With an insufficient model, banks with $c_i < c_i^{min}$ will default, while other banks with $c_i > c_i^{min}$ may still default as their receivables are not realized in full.

$\boldsymbol{c^{min}}$ can immediately be used to answer one of our key questions: How can all defaults in a network be prevented at lowest cost?

Theorem 1. *The optimal intervention to prevent all defaults can be obtained by injection of any positive amounts $c_i^{min} - c_i$ to each bank i for which $c_i < c_i^{min}$.*

As described above, from the derivation of $\boldsymbol{c^{min}}$ it follows that this intervention brings c_i^* to 0 for these banks i that would otherwise default, while setting $p_i = b_i$ for all i. Thus, this intervention satisfied all debts while providing no excess cash to the would-be defaulting banks. Use of this approach will be demonstrated later through examples in **Section** 4.4.

Interpretation as a Discrete Dynamical System. Given the iterative nature of the BaT model, Sonin and Sonin (2017, 2020) note that the model's dynamics can be fully specified by a discrete dynamical system $(\boldsymbol{X_k})$, where k denotes the time interval $\Delta_k = [T_k, T_{k+1})$ ending with a status change at time T_{k+1} in the continuous-time model (with $T_0 = 0$), and vector $\boldsymbol{X_k} = (T_k, \boldsymbol{p}(T_k), \boldsymbol{b}(T_k), \boldsymbol{c}(T_k), Q)$ contains complete information about the system at this moment in time. We note that $\boldsymbol{X_{k+1}}$ is then a function of $\boldsymbol{X_k}$, incrementing $\boldsymbol{X_k}$'s parameters $\boldsymbol{p}(T_k)$, $\boldsymbol{b}(T_k)$, and $\boldsymbol{c}(T_k)$ over $T_{k+1} - T_k$ units of time according to the dynamics $\boldsymbol{u}(T_k)$ and $\boldsymbol{d}(T_k)$ that follow from $\boldsymbol{b}(T_k), \boldsymbol{c}(T_k)$, and Q.

The next section describes an R program we developed to implement the BaT model and assist in further analysis. Our program updates the vectors $\boldsymbol{p}(T_k), \boldsymbol{b}(T_k), \boldsymbol{c}(T_k)$, a cash flow matrix $Q(T_k)$ and a status vector $\boldsymbol{j}(T_k)$ at the end of each interval and outputs $\boldsymbol{X_k} = (\boldsymbol{p}(T_k), \boldsymbol{b}(T_k), \boldsymbol{c}(T_k), Q(T_k), \boldsymbol{j}(T_k))$ for each $k = 1,2, \ldots$ until $T_k = T^*$.

3.2 Banks as Tanks Model – Programming

Using R, we developed a program to implement the BaT model and describe the status of each bank at any point in the clearing process. This chapter describes the notation and function of this program, and thus describes the BaT model's algorithmic procedure in detail as well.

Our program begins by taking inputs defining the financial network. These inputs include the number of banks (n), vectors describing each's initial cash reserves ($\boldsymbol{c}$) and imminently payable debts ($\boldsymbol{b}$), and a matrix Q describing the percentage of each bank's total debt that is payable to each of the other banks in the network. Values of these initial inputs form the basis for all subsequent calculations. Using these inputs, several initial calculations are required before the algorithm begins.

Using vectors $\boldsymbol{c}$ and $\boldsymbol{b}$, our program evaluates the debts and cash of each bank i to assign an initial status j_i for each bank, stored in a vector $\boldsymbol{j}$ for convenience. Specifically, elements j_i of the vector $\boldsymbol{j}$ are given by the following function of c_i and b_i:

$$j_i = \begin{cases} +, & b_i > 0 \; and \; c_i > 0 \\ 0, & b_i > 0 \; and \; c_i = 0 \\ *, & b_i = 0 \end{cases} \tag{9}$$

The statuses +, 0, and * correspond to the positive, zero, and absorbing statuses described in **Section** 3.1, respectively. For simplicity, we note the assumption that no bank begins in a zero

status, although a network including zero-status banks can be evaluated by the same algorithm after only minor adjustments.[12]

Next, we define an initial vector $\boldsymbol{u}$ of length n whose entries are the total payment outflow rates for each institution. Outflow rates are determined logically based on each bank i's status j_i.

- A bank i in a positive status (i.e., $j_i = +$) has both debts and cash reserves with which to pay those debts. Thus, for such a bank, u_i is assigned the baseline outflow rate of 1.
- A bank i in an absorbing status (i.e., $j_i = *$) has no debts, and therefore makes no payments. Thus, these banks are assigned outflow rates of $u_i = 0$.
- Finally, and most interesting, are banks in a zero status (i.e., $j_i = 0$). These banks have debts, but no cash reserves with which to pay them. However, these banks may still receive payments from other banks, and must pay out all that they receive. Therefore, the outflow rate for such a bank is set equal to its inflow rate. In situations where only one bank i is in a zero status, this assignment is simple, and $u_i = n_i$, where n_i is a known value. However, when several banks are in zero statuses simultaneously while having debts to one another, their outflow and inflow rates must be calculated as the solution of a system of equations. Specifically, for a zero-status bank i, u_i may be calculated as the corresponding element of

$$(I^{(0)} - Q^{(0)T})^{-1} e \tag{10}$$

where vector e is composed of elements $e_i = \sum_{j \in J_+} q_{j,i}$ for $i \in J_0$ (i.e., the sum of flow rates from positive status banks to a zero-status bank i), matrix $Q^{(0)T}$ is the transpose of matrix

[12] Sonin and Sonin (2017, 2020) describe a "Big Bang effect," in which firms in a zero status at time 0 may have net inflow rate d_i sufficient that at the next instant, they will be in a positive status. To correctly assign flow rates, such firms should be reassigned to a positive initial status. Alternatively, any firm initially in a zero status may be assumed to have a very small cash position ε, such that the algorithm assigns a positive initial status and firms experiencing the Big Bang effect remain in a positive status, while firms with lower inflow rates quickly move to a zero status.

Q with all rows and columns not corresponding to zero-status banks deleted, and $I^{(0)}$ is the identity matrix with the same dimension.

Sonin and Sonin (2017, 2020) maintain a static matrix Q describing the proportion of each bank's initial debts owed to each other bank, and use this matrix in conjunction with each bank's point-in-time status to derive inflow and outflow rates $n_i(t)$ and $u_i(t)$. We instead define $Q(t)$, a matrix where $q_{i,j}(t)$ describes the cash flow from bank i to bank j at time t. This allows for easy observation of bank-to-bank cash flow rates, and adds efficiency in calculation of inflow and outflow rates. For instance, our next step is to define a vector $\boldsymbol{n}$ of length n to describe the total payment inflow rate for each bank, where the ith entry in $\boldsymbol{n}$ is simply the sum of the ith column in $Q(0)$. Matrix $Q(t)$ will be updated at the end of each interval along with other parameters describing the network dynamics.

We note that $Q(0) = Q$ only when the network initially contains no zero-status banks. The initial matrix Q contains values between 0 and 1 describing the percentage of each bank's total debt that is payable to each other bank, which may differ from bank-to-bank flow rates in the presence of zero-status banks. For a bank i that is initially in a positive or absorbing status, the initial values $q_{i,j}$ are also the flow rates between bank i and bank j. However, for a bank that is initially in a zero status, the bank's total outflow rate must be set equal to its total inflow rate, and its total outflow rate is then portioned among the banks it owes in the same proportions as the total debt proportions in the initial matrix Q.

Therefore, in situations where the network includes zero-status banks at the start of the algorithm, we may either

1) reclassify these banks to a positive status if their net cash inflows $d_i(t)$ would be positive, meaning that in the next moment these banks would be in a positive status, and for the true

zero-status banks, derive $Q(0)$ by applying scaling factors to the corresponding rows in Q so that these rows will contain flow rates summing to the proper total outflow rate, or

2) equivalently, add a immaterial value ε to these banks' cash positions to bring them to a positive status as suggested by Sonin and Sonin (2017, 2020), after which our algorithm will adjust their cash flows in matrix $Q(t)$ as needed if their cash goes to 0 at the end of a short interval

We note that after either adjustment, all elements of the matrix $Q(0)$ can then be interpreted as bank-to-bank flow rates.

Finally, we define a vector $\boldsymbol{p}(t)$ of length n to track the total payments made by each bank throughout the clearing process. We initialize each entry in $\boldsymbol{p}(0)$ to a value of 0. Values $p_i(t)$ will be incremented after each iteration of the program.

After defining these initial inputs, our program applies the initial inflow and outflow rates to the initial cash and debt vectors to model the payment behavior of the network and derive the time until the first status change occurs (i.e., a bank's cash or debts go to 0). Using the calculated length of the interval, our program then follows an iterative process to update the statuses, corresponding inflow and outflow rates, cash and debt positions, and the total cumulative payments at this point in time, then models the payment behavior with the new payment dynamics in the interval until the next status change. The exact steps within each iteration are described in more detail below:

1) Define a vector $\boldsymbol{d}(t)$ whose entries are the *net* inflow rates for each bank, describing the change in each's cash position over time.

$$d_i(t) = n_i(t) - u_i(t) \tag{11}$$

For a bank that is paying out faster than it is receiving payments, the net inflow rate will be negative.

2) Calculate t_1, the time of the first status change. This is the first time at which either of two events occurs:

- any bank moves from its initial status into an absorbing status by paying the entirety of its debt, or
- any bank moves into a zero status by exhausting its cash reserves while still having positive debt.

Each bank's outstanding debt monotonically decreases over time in a linear fashion according to its outflow rate, while each bank's cash reserves are incremented—or decremented—over time in a linear fashion according to the bank's net inflow rate. Thus, t_1 will be the minimum finite, positive value such that

$$b_i - (t_1 \times u_i) = 0 \text{ or } c_i + (t_1 \times d_i) = 0, i \in 1, \dots, n \tag{12}$$

Solving each equation for t_1, we obtain each equation's x-intercept, and the time of the next status change is

$$t_1 = \min\left[\min_{i|u_i \neq 0, \frac{b_i}{u_i} > 0} \left(\frac{b_i}{u_i}\right), \min_{i|d_i \neq 0, -\frac{c_i}{d_i} > 0} \left(-\frac{c_i}{d_i}\right)\right] \tag{13}$$

We note that several values are excluded from these minimum calculations. Banks with outflow rates of 0 will not experience any decrease in debt and cannot be the next to satisfy their debts, and banks with no debt ($b_i = 0$) are already in an absorbing status and will not satisfy their debts in the future (because they have already) nor exhaust their cash (because they make no payments). Similarly, banks with net inflow rates of 0 or net cash inflows will not experience any decrease in cash and cannot be the next to exhaust their cash.

3) After solving for t_1, update $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{p}$ to describe each bank's remaining debts, remaining cash, and cumulative amount paid as of time t_1. In this step, each bank's debt position is

reduced by its outflow amount over the interval, while each bank's cumulative payment amount is incremented by the same amount. Each bank's cash reserves are incremented by its net cash inflow (or outflow) over the interval.

$$\begin{aligned} b_i(t_1) &= b_i - (t_1 \times u_i) \\ p_i(t_1) &= p_i + (t_1 \times u_i) \\ c_i(t_1) &= c_i + (t_1 \times d_i) \end{aligned} \tag{14}$$

4) Update the status vector $\boldsymbol{j}$ at time t_1 using the updated debt and cash vectors $\boldsymbol{b}(t_1)$ and $\boldsymbol{c}(t_1)$. We note that multiple status changes may occur simultaneously when the minimum time in (13) was obtained from multiple values of i. Similarly, a single bank i may have simultaneously run out of cash and paid off the entirety of its debt. In this case, recall from the definition of j_i that $j_i(t_1)$ will be set to *, denoting that bank i is now in an absorbing status.

5) Using the updated status vector $\boldsymbol{j}(t_1)$, update Q at time t_1 to describe the new outflow rates from each bank to each other bank.

   - A bank i in an absorbing status no longer owes other banks, thus we set each entry in row i of $Q(t_1)$ to a value of 0.
   - A bank i in a positive status still has positive debts and cash reserves. Because payments have been made proportionally to each of bank i's creditors, the same proportion of each initial debt is still owed, so we make no changes to row i of $Q(t_1)$.
   - A bank i in a zero status still has debt but has no cash reserves. Thus, such a zero-status bank must pay out what it receives, and $u_i(t_1) = n_i(t_1)$. After solving for $u_i(t_1)$, the total outflow rate for bank i—possibly part of the solution to a system

of equations as described in (10)—we update row i of $Q(t_1)$ to maintain the same proportions but sum to the new value $u_i(t_1)$.[13]

$$q_{i,k}(t_1) = u_i(t_1) \times \frac{q_{i,k}}{\sum_{k=1}^{n} q_{i,k}} \tag{15}$$

6) Update $\boldsymbol{n}$ at t_1 using the updated matrix $Q(t_1)$. As described previously, we set each entry in $\boldsymbol{n}(t_1)$ to the sum of the corresponding column in $Q(t_1)$.
7) Update $\boldsymbol{u}$ at t_1 using the updated matrix $Q(t_1)$. We set each entry in $\boldsymbol{u}(t_1)$ to the sum of the corresponding row in $Q(t_1)$.

This process then repeats itself, calculating a new vector of net outflow rates $\boldsymbol{d}(t_1)$ for the second interval and following the subsequent steps to calculate t_2, the length of the second interval, and details of the network at time $T_2 = t_1 + t_2$. We note that the times $(t_1, t_2, \dots)$ calculated represent the length of each interval, while their sum represents the total cumulative time elapsed.

Our program repeats these steps so long as at least one bank is in a positive status, having both debts and cash reserves with which to pay them. Conversely, once all banks are in an absorbing or zero status, absorbing-status banks may have cash but need make no payments, while zero status banks only make payments when receiving payments from positive status banks with cash and remaining debt, of which there are none. Accordingly, at this time $T^* = \sum_k t_k$, we see that all entries in $Q(T^*)$ ultimately take values of 0 and all payment activity ceases.

The program's outputs include the payment vector $\boldsymbol{p}(t)$, debt vector $\boldsymbol{b}(t)$, cash reserve vector $\boldsymbol{c}(t)$, status vector $\boldsymbol{j}(t)$, and flow rate matrix $Q(t)$ at the end of each interval (i.e., at T_k for

[13] When Bank m moves from a positive status to a zero status, the denominator $\sum_{h=1}^{n} q_{[m,h]} = u_m = 1$. In the next iteration for this zero-status bank, $\sum_{h=1}^{n} q_{[m,h]} = u_m$, which may take a value less than 1.

k=1,2,…), including the final values at time T^* after all payment activity has ceased. In addition, by multiplying the final payment vector $\boldsymbol{p}(T^*)$ (as a $1 \times n$ matrix) by the original $n \times n$ matrix Q (in which element $q_{i,k}$ described the percentage of bank i's original debt that was owed to bank k), we may obtain a $1 \times n$ matrix $\boldsymbol{r}(T^*)$ describing the amount each bank has received. Thus, the final vectors $\boldsymbol{p}(T^*)$ and $\boldsymbol{r}(T^*)$ summarize the outcomes of all payment activity. If the banks pay into a pool as specified in $\boldsymbol{p}(T^*)$, then take funds from the pool as specified in $\boldsymbol{r}(T^*)$, these simple transactions summarize the many bank-to-bank transactions occurring throughout the clearing exercise.

We note that the program's final output describes the results of *organic clearing* activity, which we define as clearing dependent solely on the initial state of the system, with no intervention from sources exogenous to the system. With no outside intervention, banks that remain in a zero status after organic clearing will default on their remaining outstanding debts unless they liquidate investments or assets to obtain liquid cash or secure additional funding from some in-network or out-of-network source (e.g., a loan). In **Chapter** 4, we will explore methods by which a regulatory body or central bank may use insights provided by the BaT model and our program to determine an optimal way to intervene (e.g., by providing loans) to alter the final state of the system in a desired way (i.e., preventing defaults, reducing outstanding debt).

3.3 Banks as Tanks Model – Example

To illustrate the BaT model's processing and output, we review several examples and apply the BaT model using the program described in the previous section.

Example 1A. Consider the following relatively small financial network prior to clearing.

Table 1: BaT Model Example 1A – Initial Network Parameters

Parameter	Value
n	5
$\boldsymbol{c}$	[1, .01, .01, .01, 0]
$\boldsymbol{b}$	[1, 2, 3, 4, 0]
Q	$\begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

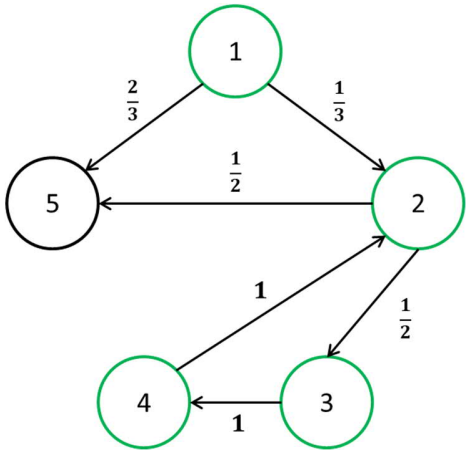


Figure 3: BaT Model Example 1A – Network Plot

Our network includes five banks. Of these, only Bank 5 is without debt, while each of the others has liabilities to other banks within the network and non-zero starting cash. In network plots like **Figure 3**, we will denote absorbing statuses by black nodes, positive statuses by green nodes, zero statuses during clearing by yellow nodes, and defaults (zero statuses when clearing ends) by red nodes. In this example, we note that Banks 2, 3, and 4 have very small cash reserves relative to their debts. We follow these banks through iterations of the algorithm described in the previous section to model how they will clear their liabilities and arrive at their final statuses.

<u>Iteration 1</u>: Comparing each bank's initial cash positions and imminently-payable debts, the four banks with positive debt and cash are each assigned a positive status (+), while Bank 5 (with no debt) is assigned an absorbing status (*). Thus, we derive an initial status vector $\boldsymbol{j} = [+,+,+,+,*]$ and corresponding outflow rate vector $\boldsymbol{u} = [1,1,1,1,0]$, and we note that with no zero-status banks, no updates are required to matrix Q to allow its interpretation as a cash flow matrix. Taking column sums from Q, we derive the initial inflow rate vector $\boldsymbol{n} = [0, \frac{4}{3}, \frac{1}{2}, 1, \frac{7}{6}]$. The net inflow vector $\boldsymbol{d}$ is then calculated as $\boldsymbol{d} = \boldsymbol{n} - \boldsymbol{u} = [-1, \frac{1}{3}, \frac{-1}{2}, 0, \frac{7}{6}]$.

With these initial dynamics, we calculate the time of the first status change as the minimum between all positive, finite values of $\frac{b_i}{u_i}$ (the time at which a bank will pay off all its debts given these dynamics) and $\frac{-c_i}{d_i}$ (the time at which a bank will run out of cash given these dynamics).[14][15]

$$\frac{b_i}{u_i} = \left[\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{0}{0}\right] = [\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \cdot]$$

$$\frac{-c_i}{d_i} = \left[\frac{-1}{-1}, \frac{-.01}{\frac{1}{3}}, \frac{-.01}{-\frac{1}{2}}, \frac{-.01}{0}, \frac{0}{\frac{7}{6}}\right] = [\mathbf{1}, -.03, , \mathbf{02}, \cdot, 0]$$

The minimum of the positive, finite values (in **bold** above) is .02, which corresponds to the time at which Bank 3 exhausts its cash reserves and thus moves to a zero status. Indeed, as we update cash and debt positions after .02 units of time, we will see that Bank 3's cash position goes to 0.

At time $T_1 = t_1 = .02$, $b_i(T_1) = b_i - t_1 \times u_i$, reflecting decreases in debt according to each bank's outflow rate, and $c_i(T_1) = c_i + t_1 \times d_i$, reflecting increases (or decreases) in cash according to each bank's net inflow rate. Further, the total payment amounts p_i are updated from their original values of 0, with $p_i(T_1) = p_i + t_1 \times u_i$. Updating the three vectors using these formulas yields

$$\boldsymbol{p}(T_1) = [.02, .02, .02, .02, 0]$$

$$\boldsymbol{b}(T_1) = [0.98, 1.98, 2.98, 3.98, 0]$$

$$\boldsymbol{c}(T_1) = [0.98, 0.02, 0, 0.01, 0.02]$$

Comparing $b_i(T_1)$ and $c_i(T_1)$, we obtain $\boldsymbol{j}(T_1) = [+,+,0,+,*]$. As a zero-status bank, Bank 3 must now pay out what it receives, so $u_3(T_1) = n_3(T_1)$. Bank 3 receives payments only from Bank 2 with a total inflow rate n_3 of $\frac{1}{2}$, so we also set $u_3(T_1)$ to $\frac{1}{2}$. Thus, the third row of cash flow matrix

[14] As described in **Section 3.2**, these values represent the solutions to the equations $b_i - (t_1 \times u_i) = 0$ and $c_i + (t_1 \times d_i) = 0$, $i \in 1, \ldots, n$, describing when debt or cash goes to 0.

[15] $\cdot$ in the calculations below denotes an infinite or nonexistent value.

Q is updated to reflect Bank 3's new total outflow rate of $\frac{1}{2}$, with the original proportions of this total maintained by the values in row 3 (i.e., 100% of this new $\frac{1}{2}$ total flow rate is allocated to Bank 4).

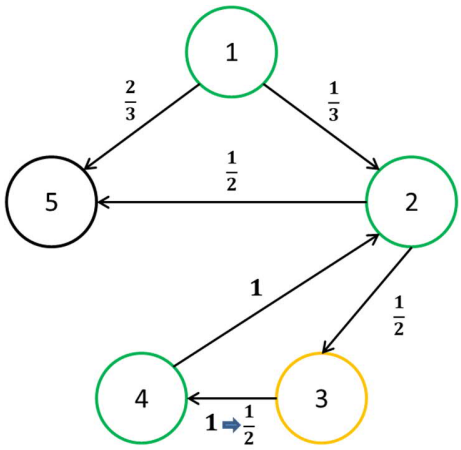


Figure 4: BaT Model Example 1A – Flow Rates $Q(T_1)$

<u>Iteration 2</u>: With a new cash vector $\boldsymbol{c}(T_1)$, debt vector $\boldsymbol{b}(T_1)$, status vector $\boldsymbol{j}(T_1)$, and cash flow matrix $Q(T_1)$, the next iteration begins with calculation of inflow and outflow rates. Specifically, from the row sums and column sums of $Q(T_1)$, $\boldsymbol{u}(T_1) = \left[1,1,\frac{1}{2},1,0\right]$ and $\boldsymbol{n}(T_1) = [0,\frac{4}{3},\frac{1}{2},\frac{1}{2},\frac{7}{6}]$, with $\boldsymbol{d}(T_1) = \boldsymbol{n}(T_1) - \boldsymbol{u}(T_1) = [-1,\frac{1}{3},0,-\frac{1}{2},\frac{7}{6}]$.

$$\frac{b_i(T_1)}{u_i(T_1)} = \left[\frac{.98}{1},\frac{1.98}{1},\frac{2.98}{\frac{1}{2}},\frac{3.98}{1},\frac{0}{0}\right] = [.\mathbf{98},\mathbf{1.98},\mathbf{5.96},\mathbf{3.98},\cdot]$$

$$\frac{-c_i(T_1)}{d_i(T_1)} = \left[\frac{-.98}{-1},\frac{-.02}{\frac{1}{3}},\frac{0}{0},\frac{-.01}{-\frac{1}{2}},\frac{-0.02}{\frac{7}{6}}\right] = [.\mathbf{98},-.05,\cdot,.\mathbf{02},-.02]$$

Calculating the minimum between all positive, finite values of $\frac{b_i(T_1)}{u_i(T_1)}$ and $\frac{-c_i(T_1)}{d_i(T_1)}$, we find that after another t_2 =.02 units of time (i.e., at time $T_2 = t_1 + t_2$), Bank 4's cash reserves are exhausted and Bank 4 moves to a zero status. As zero-status banks both Bank 3 and Bank 4 must pay out what they receive. Bank 3 receives payments only from Bank 2 with a flow rate of $\frac{1}{2}$, and Bank 3

owes only Bank 4, so $\frac{1}{2} = n_3(T_2) = u_3(T_2) = n_4(T_2) = u_4(T_2)$. Bank 4's total outflow rate is set to $\frac{1}{2}$.

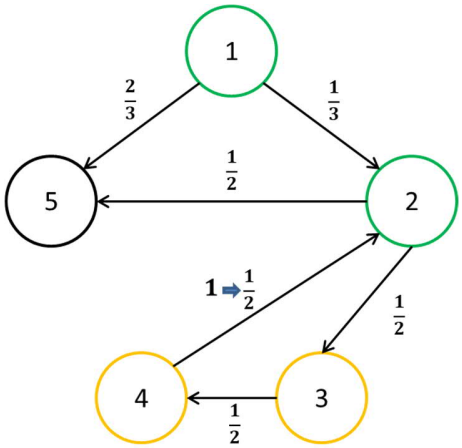


Figure 5: BaT Model Example 1A – Flow Rates $Q(T_2)$

After this iteration,

$$\boldsymbol{p}(T_2) = [.04, .04, .03, .04, 0]$$

$$\boldsymbol{b}(T_2) = [0.96, 1.96, 2.97, 3.96, 0]$$

$$\boldsymbol{c}(T_2) = [0.96, 0.02, 0, 0, 0.05]$$

$$\boldsymbol{j}(T_2) = [+, +, 0, 0, *]$$

<u>Iteration 3</u>: In the third iteration, $\boldsymbol{u}(T_2) = \left[1, 1, \frac{1}{2}, \frac{1}{2}, 0\right]$ and $\boldsymbol{n}(T_2) = [0, \frac{5}{6}, \frac{1}{2}, \frac{1}{2}, \frac{7}{6}]$, with $\boldsymbol{d}(T_2) = \boldsymbol{n}(T_2) - \boldsymbol{u}(T_2) = [-1, -\frac{1}{6}, 0, 0, \frac{7}{6}]$.

$$\frac{b_i(T_2)}{u_i(T_2)} = \left[\frac{.96}{1}, \frac{1.96}{1}, \frac{2.97}{\frac{1}{2}}, \frac{3.96}{\frac{1}{2}}, \frac{0}{0}\right] = [.\mathbf{96}, \mathbf{1.96}, \mathbf{5.94}, \mathbf{7.92}, \cdot]$$

$$\frac{-c_i(T_2)}{d_i(T_2)} = \left[\frac{-.96}{-1}, \frac{-.02}{-\frac{1}{6}}, \frac{0}{0}, \frac{0}{0}, \frac{-0.05}{\frac{7}{6}}\right] = [.\mathbf{96}, .\mathbf{14}, \cdot, \cdot, -.04]$$

After an additional $t_3 = .14$ units of time (at $T_3 = .18$), Bank 2's cash reserves are exhausted and Bank 2 moves to a zero status. As shown in $Q(T_2)$, Bank 2 receives payments from both Bank 1 (positive status, paying Bank 2 at a rate of $\frac{1}{3}$) and Bank 4 (zero status), and as a zero-status bank

Bank 2 must now pay out what it receives along with Banks 3 and 4, so $\frac{1}{3} + u_4(T_3) = n_2(T_3) = u_2(T_3)$, while $\frac{1}{2}u_2(T_3) = n_3(T_3) = u_3(T_3) = n_4(T_3) = u_4(T_3)$. Solving this system of equations, we have $n_2(T_3) = u_2(T_3) = \frac{2}{3}$ and $n_3(T_3) = u_3(T_3) = n_4(T_3) = u_4(T_3) = \frac{1}{3}$.

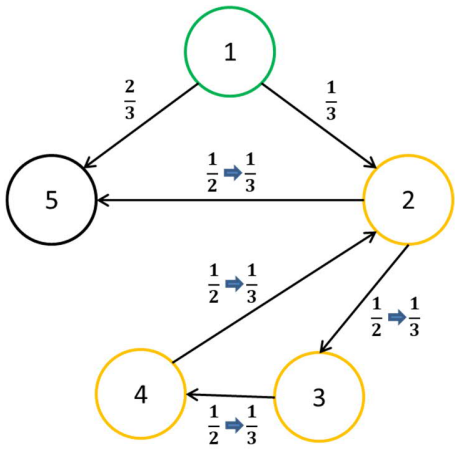


Figure 6: BaT Model Example 1A – Flow Rates $Q(T_3)$

After this iteration,

$$\boldsymbol{p}(T_3) = [.18, .18, .1, .11, 0]$$

$$\boldsymbol{b}(T_3) = [0.82, 1.82, 2.9, 3.89, 0]$$

$$\boldsymbol{c}(T_3) = [0.82, 0, 0, 0, 0.21]$$

$$\boldsymbol{j}(T_3) = [+, 0, 0, 0, *]$$

<u>Iteration 4</u>: In the fourth and final iteration, $\boldsymbol{u}(T_3) = \left[1, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0\right]$ and $\boldsymbol{n}(T_3) = [0, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 1]$, with $\boldsymbol{d}(T_3) = \boldsymbol{n}(T_3) - \boldsymbol{u}(T_3) = [-1, 0, 0, 0, 1]$.

$$\frac{b_i(T_3)}{u_i(T_3)} = \left[\frac{.82}{1}, \frac{1.82}{\frac{2}{3}}, \frac{2.9}{\frac{1}{3}}, \frac{3.89}{\frac{1}{3}}, \frac{0}{0}\right] = [.\mathbf{82}, \mathbf{2.73}, \mathbf{8.7}, \mathbf{11.67}, \cdot]$$

$$\frac{-c_i(T_3)}{d_i(T_3)} = \left[\frac{-.82}{-1}, \frac{0}{0}, \frac{0}{0}, \frac{0}{0}, \frac{-0.21}{1}\right] = [.\mathbf{82}, \cdot, \cdot, \cdot, -0.21]$$

After an additional $t_4 = .82$ units of time, Bank 1's debts are satisfied at the same time its cash reserves are exhausted, and Bank 1 moves to an absorbing status. As an absorbing-status bank,

Bank 1 will make no further payments, so we set $u_1(T_4) = 0$. Each entry in the first row of $Q(T_4)$ is set to 0, and re-evaluating the system of equations that derives inflow and outflow rates for the zero-status banks (Banks 2, 3, and 4), we find that $u_1(T_4) = 0$, $0 + u_4(T_4) = n_2(T_4) = u_2(T_4)$, $\frac{1}{2}u_2(T_4) = n_3(T_4) = u_3(T_4) = n_4(T_4) = u_4(T_4)$, and the solution is $n_2(T_4) = u_2(T_4) = n_3(T_4) = u_3(T_4) = n_4(T_4) = u_4(T_4) = 0$. Thus, we have

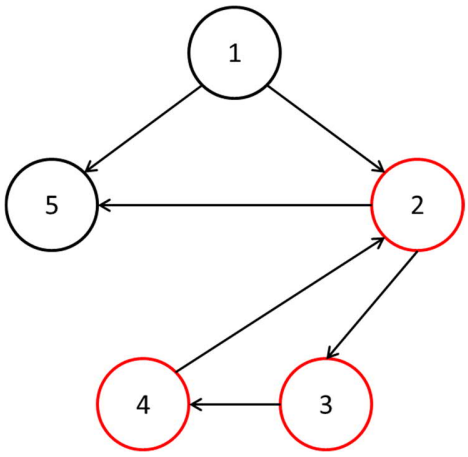


Figure 7: BaT Model Example 1A – Flow Rates $Q(T_4)$

Banks 1 and 5 are in an absorbing status and need make no payments. Banks 2, 3, and 4 are in a zero status in which they have no cash and would only make payments if payments were received from other banks that have cash and are making payments (i.e., positive-status banks). Thus, as expected since no positive-status banks remain, matrix $Q(T_4)$ reflects that all cash flows have ceased, and we have $T^* = T_4$.

After this iteration, the final output vectors are:

$$\boldsymbol{p}(T^*) = \boldsymbol{p}(T_4) = [1, .73, .37, .38, 0]$$

$$\boldsymbol{b}(T^*) = \boldsymbol{b}(T_4) = [0,1.27,2.63,3.62,0]$$

$$\boldsymbol{c}(T^*) = \boldsymbol{c}(T_4) = [0,0,0,0,1.03]$$

$$\boldsymbol{j}(T^*) = \boldsymbol{j}(T_4) = [*,0,0,0,*]$$

From these vectors, we can confirm that Banks 1 and 5 have no remaining debt (i.e., $b_1(T^*) = b_5(T^*) = 0$) and Banks 2, 3, and 4 have outstanding debt but no remaining cash (i.e., $c_2(T^*) = c_3(T^*) = c_4(T^*) = 0$), commensurate with the statuses shown in vector $\boldsymbol{j}(T^*)$.

Table 2 summarizes the parameters from each iteration. In particular, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{j}$ describe these parameters at the start of each iteration, while $\boldsymbol{u}$ and $\boldsymbol{d}$ describe the payment dynamics within the interval.

Table 2: BaT Model Example 1A – Parameters by Interval

Interval	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{j}$	$\boldsymbol{u}$	$\boldsymbol{d}$	length t_k
1	$[1, 2, 3, 4, 0]$	$[1, .01, .01, .01, 0]$	$[+, +, +, +, *]$	$[1,1,1,1,0]$	$[-1, \frac{1}{3}, \frac{-1}{2}, 0, \frac{7}{6}]$	.02
2	$[0.98, 1.98, 2.98, 3.98, 0]$	$[0.98, 0.02, 0, 0.01, 0.02]$	$[+, +, 0, +, *]$	$[1, 1, \frac{1}{2}, 1, 0]$	$[-1, \frac{1}{3}, 0, -\frac{1}{2}, \frac{7}{6}]$	.02
3	$[0.96, 1.96, 2.97, 3.96, 0]$	$[0.96, 0.02, 0, 0, 0.05]$	$[+, +, 0, 0, *]$	$[1, 1, \frac{1}{2}, \frac{1}{2}, 0]$	$[-1, -\frac{1}{6}, 0, 0, \frac{7}{6}]$	.14
4	$[0.82, 1.82, 2.9, 3.89, 0]$	$[0.82, 0, 0, 0, 0.21]$	$[+, 0, 0, 0, *]$	$[1, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0]$	$[-1, 0, 0, 0, 1]$	.82
FINAL	$[0, 1.27, 2.63, 3.62, 0]$	$[0, 0, 0, 0, 1.03]$	$[*, 0, 0, 0, *]$	-	-	-

Example 1B. Consider the same network, but with a slightly different starting cash vector $\boldsymbol{c} = [1, .01, 2, .01, 0]$. In particular, Bank 3's cash position is \$1.99 higher than in Example 1A.

Table 3: BaT Model Example 1B – Parameters by Interval

Interval	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{j}$	$\boldsymbol{u}$	$\boldsymbol{d}$	length t_k
1	$[1, 2, 3, 4, 0]$	$[1, .01, 2, .01, 0]$	$[+, +, +, +, *]$	$[1,1,1,1,0]$	$[-1, \frac{1}{3}, \frac{-1}{2}, 0, \frac{7}{6}]$	1
2	$[0, 1, 2, 3, 0]$	$[0, 0.34, 1.5, 0.01, 1.17]$	$[*, +, +, +, *]$	$[0,1,1,1,0]$	$[0, 0, -\frac{1}{2}, 0, \frac{1}{2}]$	1
3	$[0, 0, 1, 2, 0]$	$[0, 0.34, 1, 0.01, 1.67]$	$[*, *, +, +, *]$	$[0,0,1,1,0]$	$[0, 1, -1, 0, 0]$	1
4	$[0, 0, 0, 1, 0]$	$[0, 1.34, 0, 0.01, 1.67]$	$[*, *, *, +, *]$	$[0,0,0,1,0]$	$[0, 1, 0, -1, 0]$	.01
FINAL	$[0, 0, 0, 0.99, 0]$	$[0, 1.35, 0, 0, 1.67]$	$[*, *, *, 0, *]$	-	-	-

In this example, an increase of \$1.99 in Bank 3's initial cash position led to both Bank 2 and Bank 3 satisfying their debts and ending in the absorbing status, with only Bank 4 ending in the zero status, defaulting on its remaining \$0.99 debt.

Example 1C. Consider a more complex network with $n = 19$ and initial parameters:

$$\boldsymbol{c} = [3,8,0,0,9,5,0,0,0,0,12,2,1,3,8,0,0,0,0]$$

$$\boldsymbol{b} = [3,9,6,6,5,10,8,4,3,8,2,2,4,0,3,3,6,6]$$

The banks are connected by relative liabilities as follows:

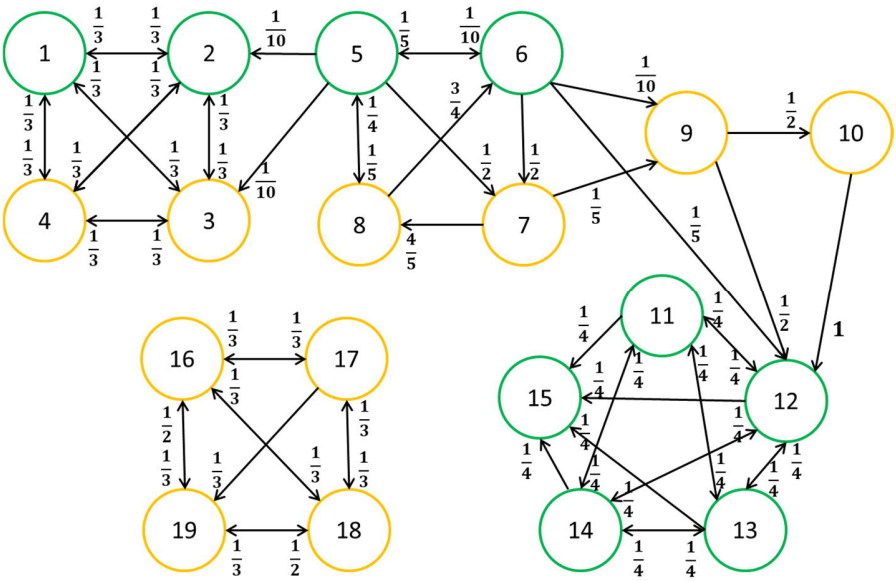


Figure 8: BaT Model Example 1C – Network Plot

Swamp: We note that Banks 16, 17, 18, and 19 are a special sub-network termed a "swamp" by Sonin and Sonin (2017, 2020). These banks have no cash, but obligations only to one another. With no cash, no cash payments can be made, but we will later describe methods by which debts can be cancelled fairly (see Example 4C in **Section** 4.5). For now, we will focus on the sub-network including Banks 1-15.

$$\boldsymbol{c} = [3,8,0,0,9,5,0,0,0,0,12,2,1,3,8]$$

$$\boldsymbol{b} = [3,9,6,6,5,10,8,4,3,8,2,2,4,0]$$

Big Bang Effect: We first note that several banks have $c_i = 0$. Noting the potential for a "Big Bang" effect as described by Sonin and Sonin (2017, 2020), some of these banks may have cash

inflows greater than 1 such that even were they assigned to a positive status (with a corresponding outflow rate of 1), these banks would have a positive cash position at the first instant after clearing begins. For these banks, a positive status should be assigned. To automate this assignment with minimal changes to our program's logic, our program adds a small value ε to each cashless bank's starting cash position, assigning each to a positive initial status and letting our algorithm evaluate whether their cash positions will increase over time or quickly fall to 0. Thus, we have initial cash vector and status vector:

$$\boldsymbol{c} = [3,8,\varepsilon,\varepsilon,9,5,\varepsilon,\varepsilon,\varepsilon,\varepsilon,12,2,1,3,8]$$

$$\boldsymbol{j} = [+,+,+,+,+,+,+,+,+,+,+,+,+,+,*]$$

<u>BaT Model</u>: After pipes open, flows begin, and we may observe the payments made in each interval ending with a status change. Parameters at the end of each interval are presented in **Appendix C**. Ultimately, after 14 intervals, the final status vector is

$$\boldsymbol{j}(\boldsymbol{T}^*) = [*,*,*,*,*,*,0,0,0,0,*,*,*,*,*]$$

and the clearing payment vector is

$$\boldsymbol{p}(\boldsymbol{T}^*) = [3,9,6,6,6,5,5.5,5.6,1.6,0.8,8,2,2,4,0]$$

We note that of the six banks that began with $c_i = 0$, two have satisfied their debts from their receivables alone (Bank 3 and Bank 4), while the remaining four ended in the zero status (Banks 7-10).

CHAPTER 4: INTERVENTION PLANNING

The BaT model derives the clearing payment vector, which describes clearing outcomes as a function of specified inputs describing the debts and cash positions of the network's members. This chapter presents our research, which instead seeks to set a desired outcome and define the sufficient (and minimally sufficient) parameters that must be present or obtained to reach that outcome. For instance, **Section** 3.1 discussed $\boldsymbol{c^{min}}$, a vector describing the minimum initial cash positions required to facilitate clearing of all obligations within the network.

In this chapter, we use the BaT model's framework and a similar dynamic approach to evaluate the impact of interventions that attempt to prevent adverse clearing outcomes (e.g., defaults) by providing emergency liquidity. Our findings allow us to address the first three key questions posed in **Section 1.2**, particularly how an intervention can be planned to prevent the default of one, all, or a group of banks within the network at lowest cost.

We begin by exploring how the same interconnectedness and network dynamics that lead to financial contagion can also lead to multiplier effects in the benefit from outside investment. We then explore interventions based on details of the network after, during, or before clearing, discussing how calculated multipliers, a modified implementation of the BaT model, or comparison of initial cash positions to minimal sufficient cash positions can be used to identify the lowest-cost intervention to obtain the desired clearing outcomes. As noted in **Section** 3.1, we evaluate clearing at a particular moment in time, and thus we find that the optimal recipients and amounts required to meet a particular intervention goal do not vary with the intervention's timing. However, each method provides insight into factors that affect the effectiveness of an intervention,

and we introduce methods that could also be used in networks where clearing happens dynamically over an interval of real time.

Finally, we apply properties of Markov chains within this framework to simplify the network structure through multilateral netting; this "obligation cascade" model shows that at least one bank in the network can satisfy its debts even without having cash, and provides valuable insights about net obligations and firms most at risk.

4.1 Interventions and Key Questions

During or after a negative shock affecting many members of the financial system, a relevant regulatory body (e.g., a central bank) may wish to intervene to limit the adverse impacts of the initial shock and any resulting financial contagion. In 2008 and 2009, during and after the U.S. financial crisis, the U.S. Treasury did just that, investing hundreds of billions of dollars to provide capital to many U.S. financial institutions and successfully preventing more widespread bankruptcies.

We have discussed the BaT Model and other proposed models of liability clearing in financial networks, any of which could be used by a regulatory body in possession of the proper input data to explore the outcomes of organic clearing. During a period of financial or economic stress, given such foresight about projected clearing and the post-clearing status of each member of the financial network, a regulator can then plan an intervention to prevent defaults that would otherwise occur, and a natural question is how to accomplish the regulator's particular goal at lowest cost. As noted in **Section** 1.2, a regulator's goal may be to prevent bankruptcy of one or several key banks, or to prevent all bankruptcies.

Such interventions have garnered some discussion in the literature. Among other methods of intervention, Freixas et al. (2000) discuss the cost of bailouts from a central bank and how a network's structure may designate some more centrally-connected banks as "too big to fail." **Chapter 2** presented our review of literature, including a paper from Rogers and Veraart (2013) discussing circumstances in which an in-network consortium of banks may have an incentive to rescue a bank that would otherwise default. Bernard, Capponi, and Stiglitz (2017) expand upon this idea by modeling a regulator's decision whether to intervene as a game between the regulator and banks in the network. In addition to the options of allowing a bank's failure or directly providing it with necessary liquidity using public funds, Bernard et al. note that the regulator may also propose a partial bail-out from public funds with an additional "bail-in" from the defaulting bank's creditors (e.g., a partial forgiveness of the amounts owed). However, such creditor banks will only have incentive to participate when the threat of the regulator allowing the defaulting bank's failure (and thus the creditor recovering none of the funds it is owed) rather than bailing it out fully from public funds is credible. A bank's centrality and the perceived consequences of its failure to the broader economy increase the expectation that a regulatory body would prevent its failure, single-handedly if necessary.

In fact, Capponi, Corell, and Stiglitz (2020) note an unintended consequence of the possibility of bailouts: relatively well-connected banks may attempt to increase the likelihood of their bailouts by intentionally exposing themselves to sovereign debt (e.g., U.S. Treasury Notes or other debt owed to the government), imposing higher economic consequences if they fail while others are bailed out. Government bailouts are generally financed through issuance of additional sovereign debt, decreasing the value of existing sovereign debt assets on the would-be defaulting bank's

balance sheet and thus reducing the amount its creditors can recover through liquidation of assets if the bank is allowed to default.

Our review of the literature found that when interventions have been discussed, they are generally in the context of providing emergency liquidity to a particular bank in an amount sufficient to cover its remaining debt, with little consideration of higher-order benefits from that investment (e.g., benefit to the other banks the recipient bank will pay, additional benefit to the recipient bank when cycles in the network structure mean that payments from that bank cause additional downstream payments from other network members *to* that bank). However, the BaT model offers us a framework that is also useful to model the additional downstream effects of provided funds, and we note that understanding of these higher-order benefits can allow intervention goals to be met by lower amounts.

Applying the BaT model to a pre-clearing financial network to identify network members that will default without intervention, we use the network's projected post-clearing status to build a framework for intervention cost/benefit analysis. We then propose methods by which this framework can be used to determine the recipient banks and loan amounts that minimize the cost of preventing default for any chosen bank or set of banks, or the recipient banks and loan amounts that optimally deploy a finite amount of funds to achieve maximal reduction of uncleared debts for the chosen banks. We will then apply these methods to answer the first three questions we posed in **Section** 1.2:

1) How can all defaults in a network be prevented at lowest cost?
2) How can the default of a particular network member be prevented at lowest cost?
3) How can the defaults of a group of network members be prevented at lowest cost?

The first question was also addressed by **Theorem 1** in **Section** 3.3, and comparison of initial cash positions to a vector $\boldsymbol{c^{min}}$ is demonstrated in **Section** 4.4, along with several alternative methods of deriving the optimal intervention to rescue all banks.

The second question is addressed by **Theorem 2**.

Theorem 2. *The optimal intervention amount to prevent a bank k's default can be identified algorithmically using multipliers $m_{i,k}$.*

These multipliers and several algorithmic approaches are demonstrated in **Section** 4.4 after setup of the necessary framework in **Section** 4.3.

The third question is addressed by **Theorem 3**.

Theorem 3. *The optimal intervention amount to prevent defaults for banks $i \in A$ can be identified algorithmically using multipliers m_i^A.*

These multipliers and several algorithmic approaches are demonstrated through examples in **Section** 4.4.

Multiplier Effects in the Benefit of Intervention to a Single Bank. How can the benefit to a particular zero-status bank be measured as a function of the amount invested in any other bank? **Theorem 4**, **Theorem 5**, and **Theorem 6** in **Section** 4.3 define the main characteristics of a dynamic process for measuring the benefit of investment from an external lender to one, all, or a

group A of banks. We then present a method for deriving the lowest-cost intervention after we explain the dynamics of interventions and introduce modifications to the BaT model framework.

4.2 Preliminaries

In **Section** 4.3, we will examine the mechanism by which outside investment in any bank leads to a benefit (in the form of reduction of outstanding debt) for the recipient bank as well as other banks in the network. Before this examination, we introduce key notation and outputs from the BaT model that will be useful in our analysis. We also introduce our **Proposition 1**, which illustrates the value of a targeted intervention over more obvious (but naïve) interventions like provision of funds in the amount of each would-be defaulting bank's remaining debt.

In examining the cost of preventing defaults, we are most interested in those banks $i \in J_0(T^*)$ whose final status from the BaT model is zero. These are banks that, with no intervention, will still have outstanding debts after all clearing activity has ceased and must therefore raise funds somehow (e.g., through liquidation of assets or solicitation of loans) or default on those debts. Conversely, at the termination of organic clearing, all banks not in a zero status will be in the absorbing status $i \in J_*(T^*)$ with no remaining debts, and will therefore have already avoided default without any need for intervention. If these banks were to receive funds from an outside investor, we note from their absorbing status and corresponding outflow rate $u_i(T^*) = 0$ that such investment would lead to no bank-to-bank payments and thus no benefit in terms of debt reduction for any bank within the network.

Key inputs to our analysis include the post-clearing debt vector $\boldsymbol{b}^* = \boldsymbol{b}(T^*)$ and the *initial* relative liability matrix Q describing how those remaining debts are payable to other banks. From Eisenberg and Noe's assumption of proportionality in payments, we note that zero-status banks,

having paid out their entire values, will still owe each of their creditors the original proportion of the remaining total debt. Thus, the initial matrix Q still adequately describes the allocation of the zero-status banks' remaining debts b_i^*. We will then derive Q^* by modifying this initial matrix Q to reflect that the absorbing-status banks at time T^* have no remaining debts.

Before exploring methods for optimizing interventions in the following sections, we first note several common but simplistic methods for deriving an intervention amount to prevent a single bank's default. These methods rely only on simple observations about the bank's debt and cash positions, and do not incorporate broader information about the network and higher-order benefits of the investment. We will show that each of these methods yields a sufficient—although not minimal—intervention amount, then we will explore how the same default may be prevented at a lower cost.

1) For a bank i in a zero status after clearing, the most obvious amount that would prevent bankruptcy is b_i^*, the amount of bank i's debt still outstanding at the end of organic clearing. For instance, for a bank i still owing \$1 million when organic clearing ends, a loan of \$1 million to that bank obviously allows it to repay its remaining debt. However, this approach disregards higher-order benefits to the recipient bank (e.g., when payments from that bank will ultimately result in additional payments received by that bank from others) that may result in this intervention amount being larger than necessary.
2) A naïve but sufficient intervention amount is the difference between a bank i's initial debt and initial cash reserves ($b_i - c_i$) prior to organic clearing. A positive difference indicates an inability to cover debts solely from the bank's cash reserves. However, this "shortfall" does not account for payments that will be received from other banks during the clearing process, which could also be paid towards the bank's debts, nor does it account for higher-

order benefits to the recipient bank. Because payments may be received and then applied towards debts during the clearing process, this initial shortfall is only equal to or larger than the bank's amount of outstanding debt after organic clearing, and thus represents an even more excessive intervention amount.

Sufficient—but not minimal—interventions result in the rescued bank having excess cash after satisfying its debts, an obvious sign that the intervention was larger than necessary. The minimal intervention that we wish to obtain will leave the desired banks with debt positions b_i^* and cash positions c_i^* of 0.

The existing literature discusses interventions primarily as provision of funds in amounts consistent with the b_i^*. However, we will show that the same framework used in Eisenberg and Noe's model and the BaT model can be used to model downstream impacts of provided funds (which banks will receive funds after downstream payments that will then allow them to pay their debts), and how these payment dynamics can be used to identify a minimal intervention amount.

Proposition 1. *The minimal amount of additional capital sufficient to prevent a single zero-status bank i's default is a value less than or equal to b_i^*.*

The fact that b_i^* is a sufficient amount to satisfy a bank i's remaining debt b_i^* is trivial. However, the interesting implication of **Proposition 1** is that in some networks, the dynamics of the network—following from the relative liability matrix Q and vector $\mathbf{b}^*$—can allow a zero-status bank's remaining debts to be satisfied by a lower intervention amount. A proof is included in the following section after we define a framework for study of the cost and benefit of particular interventions. This framework will also explain the dynamics that can lead to a lower amount being sufficient.

4.3 Framework and Multiplier Effects

The study of multipliers describing the overall benefit of government investment includes works from Kahn (1931) and Keynes (1935). Kahn relates government expenditure to primary sources of employment in terms of the services purchased and the manufacture of any purchased goods, but then also to secondary sources of employment related to expenditure of profits from the primary employment. Keynes similarly relates government expenditure to income and employment, noting that the "propensity to consume," a ratio between consumption and income, impacts the secondary benefits that follow from the primary benefits of expenditure by affecting how much of any profit is subsequently spent and how much is instead saved. Similarly, in our framework the initial investment in a recipient bank provides a primary benefit to that bank, while secondary benefits are obtained when payments from that bank result in benefit to others (and potentially also additional benefit to the original recipient bank). The magnitude of these secondary benefits is driven by the values in matrix Q^*, but to our knowledge these higher-order benefits have not been incorporated extensively into studies of intervention in the literature.

In this section, we will use components of the framework used by Eisenberg and Noe and by the BaT model to explain the mechanism that can lead to multiplier effects in the benefit of outside investment, both in debt reduction for individual banks and in the network as a whole. We present a method for calculating these multipliers, then we explain their usefulness in planning a lowest-cost intervention to meet one of several goals: preventing default of a selected bank, preventing all defaults, or preventing the defaults of a group of banks.

Dynamic Intervention. To set up the framework for our algorithmic approach to measuring the benefit of a particular intervention, we first define a vector $\boldsymbol{w}$ of length n. This vector denotes the

amount and distribution of funds provided in an intervention by an external investor, with w_i denoting the amount provided to bank i. These funds could be provided prior to clearing to obtain a desired post-clearing status, during clearing as banks require funds to continue paying their debts, or after clearing (e.g., to the banks with remaining debts that would otherwise default). We wish to derive the vector $\boldsymbol{w}$ that accomplishes a particular intervention goal with lowest cost (i.e., with $w = \sum_n w_i$ minimized). We note that this intervention should leave any would-be defaulting banks that were selected for rescue and received funds with final debt and cash positions of 0.

Our algorithmic approach will use elements of a post-clearing network—namely a vector $\boldsymbol{b}^*$ describing the total remaining debts and a matrix Q^* describing the allocation of those remaining debts—to derive a matrix M of multipliers $m_{i,k}$ describing the reduction in each bank k's debt as a multiplier of an amount provided to bank i. From this matrix, multipliers describing the benefit to a group A of banks or to all defaulting banks can be derived. As with the BaT model, this intervention model is dynamic; as banks pay their debts according to $b_k^*(w) = b_k^*(0) - m_{i,k} \cdot w$, one or more will satisfy their liabilities as a result of this investment, the matrix Q^* and resultant matrix M will require updates, and thus multipliers $m_{i,k}$ are only constant within intervals defined by the recipient i and the invested amount w. Thus, the total benefit to debt reduction of a bank or set of banks will be a piecewise linear function, increasing within each interval by the relevant multiplier times w.

Also similar to the BaT model, we assume bank-to-bank payments will happen instantaneously as funds are available to banks with debts, although in this section, to evaluate the impact of different intervention amounts and different recipients, we consider provision of funds to one recipient at a time (the recipient that maximizes the benefit of the next incremental amount to the bank or banks selected for rescue), and we measure the benefit to the relevant beneficiary as a

function of the amount invested and the recipient. We will show that provision of funds to different recipients yields differing benefit to the chosen beneficiary, and these benefits (measured through multipliers) are constant within intervals of w that end with the payoff of a zero-status bank; thus, an optimal recipient can be identified to receive funds within each interval, and we define a benefit function $B(i(w), w)$ expressing the debt reduction for the bank or banks selected for rescue as a function of the cumulative investment amount w and investment recipient i selected to receive incremental funds at each w. The ultimate goal of our analysis will be to select at each cumulative intervention amount w the recipient that maximizes the marginal benefit of any incremental amount received to the chosen beneficiary bank.

The benefit of the funds provided to any recipient follows from the same payment dynamics exhibited in the BaT model. Thus, similar to the BaT model, any time component implied by the sequential selection of investment recipients is purely artificial; zero-status banks must immediately pay what they receive, so the payments that follow from an intervention may be assumed to occur instantaneously. Although we derive the intervention amount and recipient or recipients through a sequential algorithm, we may ultimately sum the sequential amounts allocated for receipt by each bank i to derive amounts w_i. These funds can be provided sequentially as they were derived (after which instantaneous payments will occur among the zero-status banks and the next amount can instantly be provided), or the w_i can be provided all at once in a post-clearing network or in a pre-clearing network (i.e., a static intervention) with no difference in outcomes; the resulting bank-to-bank payments will occur instantly in each case. As in the BaT model, our dynamic approach to deriving the intervention vector $\boldsymbol{w}$ serves to provide transparency into the dynamics of intervention that follow from the network structure, and produce an algorithm that may be useful in any real cases where intervention funds are provided gradually.

Similar to the BaT model, the benefit of funds provided by an external source can also be evaluated within a discrete dynamical system. Network dynamics are constant within intervals $d_k = [W_k, W_{k+1})$ within the cumulative intervention amount w, where $W_0 = 0$ and the W_k are the cumulative investment amounts at which a zero-status bank will satisfy its debts and network dynamics will be updated (for $k = 1,2, \ldots,)$. Because different recipients drive different benefits, we will show that within each interval d_k, choosing a different recipient can cause the next status change to occur at a different investment amount; thus the length of each interval $w_k(i) = W_{k+1} - W_k$ is a function of the selected recipient bank i in that interval. $\boldsymbol{X_k} = (W_k, \boldsymbol{b}^*(W_k), \boldsymbol{c}^*(W_k), Q^*(W_k))$ contains complete information about the network at the start of an interval, and thus $\boldsymbol{X_{k+1}}$ is a function of $\boldsymbol{X_k}$ and the recipient chosen by the regulator within interval d_k. Thus, the payment dynamics and benefit of incremental investment can be evaluated for the network described at any W_k.

Benefit Multipliers. Consider the network from Example 1A from **Section** 3.3. In this example, the network includes 5 banks, three of which were projected to end in a zero status, having remaining debts after organic clearing activities cease. At the termination of our BaT program, the final status vector $\boldsymbol{j}(T^*)$ was [*,0,0,0,*], and the final debt vector $\boldsymbol{b}(T^*)$ was $[0, 1.27, 2.63, 3.62, 0]$. Reviewing the initial matrix Q, we see the proportion of each bank's original outstanding debt that is owed to each other bank.

$$Q = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We will define the post-clearing debt vector and post-clearing relative liability matrix as $\boldsymbol{b}^* = \boldsymbol{b}(T^*)$ and Q^*, respectively. Updating the original pre-clearing matrix Q to reflect the absorbing statuses of Banks 1 and 5 after organic clearing (and thus their lack of remaining debts to other banks), we have:

$$Q^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

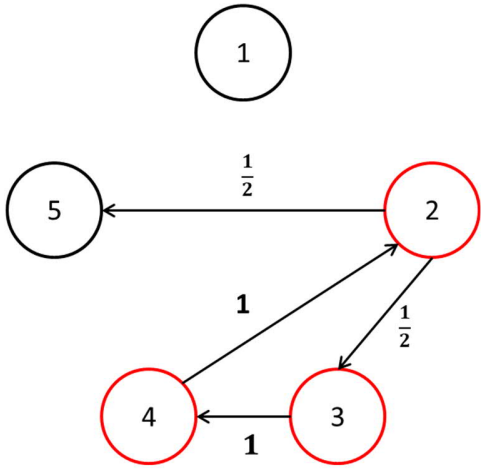


Figure 9: Post-Clearing Relative Liabilities – Network Plot Q^*

Because each bank's debts have been paid proportionally to each creditor's claim on the bank's total debt amount, once organic clearing activity has ceased, a zero-status bank i has paid the same proportion δ_i of each of its original debts, $\delta_i \in [0,1)$, and similarly still owes the complementary proportion $1 - \delta_i$ of each of its original debts. Thus, after organic clearing, each creditor k owed by a zero-status bank i is still owed the original proportion $q_{i,k}$ of bank i's *remaining* total debt, and the rows of Q^* corresponding to zero-status banks require no updates; they still accurately describe the allocation of each zero-status bank's remaining debt b_i^*.

We note that $\boldsymbol{b}^*$ and Q^* are functions of the initial $\boldsymbol{b}$, Q, and $\boldsymbol{c}$. With a different initial vector $\boldsymbol{c}$, for instance, banks may have differing levels of remaining debt after clearing and produce a

different vector $\boldsymbol{b}^*$, or some banks may have no remaining debt, resulting in a different matrix Q^* (as demonstrated by comparing Examples 1A and 1B in **Section** 3.3).

From the second row of Q^*, we see that half of Bank 2's outstanding debt of 1.27 is owed to Bank 3 and half to Bank 5. Similarly, the entirety of Bank 3's outstanding debt of 2.63 is owed to Bank 4, and the entirety of Bank 4's outstanding debt of 3.62 is owed to Bank 2.

By the BaT Model's definition of outflow rates for a zero-status bank, any cash received by Bank 2, Bank 3, or Bank 4 must be paid towards each of its debts in proportion to its share of the bank's total debts. Thus, for example, a \$0.50 loan from a central bank (which may be visualized as an external, positive-status node) to Bank 3 results in Bank 3 paying that \$0.50 to Bank 4 (because $q^*_{3,4} = 1$), Bank 4 paying that \$0.50 to Bank 2 (because $q^*_{4,2} = 1$), and Bank 2 paying \$0.25 to Bank 3 and \$0.25 to Bank 5 (because $q^*_{2,3} = q^*_{2,5} = \frac{1}{2}$). However, Bank 3 receiving \$0.25 from Bank 2 results in another cycle of payments, in which Bank 3 pays that \$0.25 towards its debt to Bank 4, Bank 4 pays \$0.25 towards its debt to Bank 2, and Bank 2 pays \$0.125 toward its debt to Bank 3 and \$0.125 toward its debt to Bank 5. Through continuation of this iterative process, Bank 2, Bank 3, and Bank 4 will each pay

$$\sum_{0}^{\infty} \$0.50 \times \left(\frac{1}{2}\right)^n = \$0.50 \times \frac{1}{1-\frac{1}{2}} = \$0.50 \times 2 = \$1 \tag{16}$$

towards satisfying their debts. To summarize, under the dynamics given by Q^*, an investment of \$$w$ in Bank 3 results in twice that amount being cleared from each of the three zero-status banks' debts.

We can similarly calculate multipliers describing the benefit to each bank resulting from an investment in Bank 2 or 4 and find that an investment in Bank 2 will also result in twice the benefit

to Bank 2, but a benefit to Banks 3 and 4 equal only to the amount of the investment (multipliers of 2, 1, and 1 for Bank 2, Bank 3, and Bank 4, respectively), as Bank 2 immediately pays half of the investment toward its debt to absorbing-status Bank 5, who makes no further payments. For similar reasons, an investment in Bank 4 results in twice the benefit to Bank 4 and Bank 2, but a benefit to Bank 3 equal only to the amount of the investment (multipliers of 2, 1, and 2 for Bank 2, Bank 3, and Bank 4, respectively).

We define $m_{i,k}$ as a multiplier describing the total reduction in Bank k's debt relative to the amount invested in Bank i under the dynamics given by matrix Q^*. As described above, multipliers $m_{i,k}$ for banks in this post-organic clearing network are:

$$M = \{m_{i,k}\} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For this simple example involving three zero-status banks, calculation of M is simple and can be performed by hand as above in (16). To generalize to post-clearing networks of any size, we first note that Q^* may be interpreted as a transition probability matrix in a Markov chain. To demonstrate this fact, we first note that each row contains proportions between 0 and 1 that sum to 1 and are analogous to probabilities. Further, as a square matrix ($n \times n$), matrix Q^* may be multiplied by itself, with powers of this matrix describing the proportion of an original investment that will be received by each bank in a later round of bank-to-bank payments. Defining $(Q^*)^n$ as $\overbrace{Q^* \times \ldots \times Q^*}^{n\ times}$, consider the following powers of Q^*:

$$(Q^*)^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (Q^*)^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(Q^*)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (Q^*)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In these matrices, a row denotes the original recipient of the investment, while each column describes the portion of that investment *received* by the corresponding bank in the nth round of subsequent bank-to-bank payments. We provide this interpretation of the first few matrices below.

$(Q^*)^0$: Before any bank-to-bank payments (i.e., after 0 bank-to-bank payments), any loan to a bank from an external source results in the recipient bank receiving the full loan amount. Accordingly, $(Q^*)^0$ has values of 1 along its diagonal.

$(Q^*)^1$: In the first round of payments after an outside investment, other banks will receive portions of the original loan amount from the original recipient as specified in the corresponding row of Q^*.

$(Q^*)^2$ and $(Q^*)^3$: In the second and third rounds of payments after an outside investment, banks continue to receive portions of the original investment amount from the recipients of the prior round of bank-to-bank payments.

To generalize, each element $\{(Q^*)^n\}_{i,k}$ is a multiplicative factor describing the amount Bank k receives in the nth round of bank-to-bank payments relative to the initial $\$w$ investment in Bank i (for $w \leq w_1(i)$, with $w_1(i)$ a minimum value at which a zero-status bank moves to an absorbing status by paying off its debts and Q^* and M will be updated; we will describe these intervals in the

remainder of this section). Summing the elements $\sum_{n=0}^{\infty}\{(Q^*)^n{}_{i,k}\} = \{[I - Q^*]^{-1}\}_{i,k}$, we then obtain multipliers describing the total amount received by Bank k after a $\$w$ investment in Bank i and a potentially infinite series of resulting bank-to-bank payments.

We note that Q^* is transient, with Bank 5 in an absorbing status and payments from any bank ultimately resulting in receipt of funds by Bank 5. $[I - Q^*]^{-1}$ yields the fundamental matrix N of the Markov chain. Classically, values $n_{i,k}$ in the fundamental matrix N describe the expected number of visits to a state k before landing in an absorbing state, given starting in a transient state i. In our application, rather than probabilities, values from matrix Q^* instead denote the proportion of a received amount that *will* be paid by bank i to bank j with probability 1 (note that zero-status banks must pay out what they receive, and by Eisenberg and Noe's proportionality assumption these payments must be made according to the proportions $q_{i,k}$), and thus the fundamental matrix describes the total amounts received by each bank after investment in a particular bank.

In this fashion, a matrix N can be calculated for any transient post-organic clearing network by defining each element $n_{i,k} = \sum_{n=0}^{\infty}\{(Q^*)^n\}_{i,k}$. For our example, N is calculated as

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 1 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

in which each element $n_{i,k}$ represents the amount *received* by Bank k as a multiplier of the amount invested in Bank i. However, only the zero-status banks will pay what they receive towards their debts, and therefore, although funds will ultimately be received by absorbing-status Bank 1 or Bank 5 given certain investments, receipt of funds by either bank will offer no benefit in terms of debt reduction. Therefore, to present the benefit in terms of debt reduction (i.e., amount paid) rather than amount received, we modify N to include values of zero in columns corresponding to

absorbing-status banks, and we obtain the same matrix M that we had previously calculated manually.

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Multiplier Effects in the Benefit of Intervention to a Single Bank. How can the benefit to a particular zero-status bank be measured as a function of the amount invested in any other bank?

Theorem 4. *(a) For any financial network with projected $J_0(T^*) \neq \emptyset$, there exist values $m_{i,k}$ describing the reduction in each bank k's remaining debts as a multiplier of the amount received by bank i from a source external to the network, with $m_{i,k} = 0$ for $k \in J_*(T^*)$ and all i and $m_{i,k} > 0$ for $k \in J_0(T^*)$ and some i.*

(b) Further, for a given investment recipient, there exist a finite number of intervals $d_k = [W_k, W_{k+1})$ within the cumulative amount invested w in which the multipliers $m_{i,k}$ are constant. Amounts W_k are the cumulative investment amounts at which a bank k with $k \in J_0(T^*)$ moves to an absorbing status and ceases to make payments, and the network dynamics and multipliers are updated.

(c) Therefore, there is a benefit function $B_k(i(w), w)$ describing the benefit to reduction of a chosen bank k's debts that is linear within each interval.

The above derivation of these values $m_{i,k}$ demonstrates part (a) of **Multiplier Effects in the Benefit of Intervention to a Single Bank.** How can the benefit to a particular zero-status bank be measured as a function of the amount invested in any other bank?

Theorem 4. Intervals and part (b) will be explained in the remainder of this section, with part (c) demonstrated through examples given in **Section** 4.4.

Lemma 1. *The maximal benefit to a zero-status bank k is obtained through investment in bank k. Therefore, $m_{i,k} \leq m_{k,k}\ \forall i$.*

However, we note that an equivalent benefit to bank k may also be obtained through investment in another upstream zero-status bank (i.e., $m_{i,k} = m_{k,k}$ for some i) if the dynamics presented in Q^* lead to bank k receiving and then paying 100% of the amount initially invested in that bank i (e.g., if $q^*_{i,k} = 1$).

Proof of Lemma 1. We note that given a matrix Q^*, the benefit of a $\$w$ investment in bank i to a zero status bank k is $m_{i,k} \cdot w = \sum_{n=0}^{\infty}\{(Q^*)^n\}_{i,k} \cdot w$.[16] As shown in $\{(Q^*)^0\}_{k,k}$, any investment in a zero-status bank k immediately benefits bank k in the amount invested, as bank k is immediately able to pay this amount towards its debts. However, when cash is invested in another zero-status bank i, bank k only receives funds when bank i pays bank k or, through other banks, payments from bank i ultimately result in payments to (and subsequently, from) bank k. In such a case, the amount first received by bank k may be less than the initial amount invested in bank i (e.g., if any of the initial investment is paid to an absorbing-status bank or if an investment in bank i benefits other zero-status banks but does not result in payments to bank k). In this case, bank k will use what it receives to make payments and clear its debts in the proportions specified in Q^*, but the nonzero values in the series $\sum_{n=0}^{\infty}\{(Q^*)^n\}_{i,k}$ will begin at a value of at most 1 (the first

[16] We will later show that a particular matrix Q^* is only valid up to a particular investment amount (a function of the selected investment recipient) at which one of the zero-status banks' debts are satisfied, that bank becomes an absorbing-status bank, and the matrix Q^* is updated as a result.

nonzero value will be the proportion of the initial investment w first received by bank k). As a result, each value in the benefit series for bank k (and the series' sum) will be proportional to but at most the corresponding value in the series $\sum_{n=0}^{\infty}\{(Q^*)^n\}_{k,k}$ where bank k received the initial investment (recall that this series begins with a value of 1). Thus, any benefit series whose nonzero values begin with a value of 1 (for example, the series resulting from an investment in bank k) yields a maximal multiplier describing reduction in bank k's debts.

While it may seem obvious that the optimal intervention to prevent bank k's default involves provision of funds to bank k, the minimal sufficient amount to prevent bank k's default may be less obvious. With **Lemma 1**, we may complete the proof of **Proposition 1** from the prior section: *The minimal amount of additional capital sufficient to prevent a zero-status bank i's default is a value less than or equal to b_i^*.*

Proof of Proposition 1. As values from Q^*or a higher power of Q^*, each term $\{(Q^*)^n\}_{i,k}$ in the series falls within the interval [0,1]. From **Lemma 1**, while a bank k has remaining debts, an initial recipient bank i can be selected such that the first term in the benefit series is 1, with potentially $i = k$. Thus, after adding later terms to the benefit series, we have $m_{i,k} = \sum_{n=0}^{\infty}\{(Q^*)^n\}_{i,k} \geq 1$ for some $i \in \{1, \dots n\}$. The benefit to bank k of an investment of $\$w$ in bank i is given by $w \cdot m_{i,k}$. Thus, under the dynamics given by Q^*, b_k^*, the amount of bank k's outstanding debt, will be satisfied by an investment amount $w = \frac{b_k^*}{m_{i,k}}$, and $m_{i,k} \geq 1$ implies that $w \leq b_k^*$. Thus, there exists

a recipient bank i—if only bank k—such that bank k's debt may be satisfied by an investment amount of at most b_k^*.[17]

Multiplier Effects in the Benefit of Intervention to the Overall Network. We now pose a similar question: How can post-clearing network dynamics be used to describe the benefit of outside investment in a particular bank to the overall network?

Theorem 5. *(a) For any financial network with projected $J_0(T^*) \neq \emptyset$, there exist values $m_i^{total} = \sum_{k=1}^{n} m_{i,k}$ (some non-zero) describing the reduction in the overall network's remaining debts as a multiplier of the amount w received by bank i from a source external to the network.*

(b) There exist a finite number of intervals $d_k = [W_k, W_{k+1})$ within the cumulative amount w invested in which the multipliers m_i^{total} are constant. Amounts W_k are the cumulative investment amounts at which a bank k with $k \in J_0(T^*)$ moves to an absorbing status, and the network dynamics are updated.

(c) Therefore, there is a benefit function $B(i(w), w)$ describing the benefit to the overall network that is linear within each interval.

Parts (a) and (b) will be demonstrated in the remainder of this section, and part (c) is demonstrated through examples given in **Section** 4.4.

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[17] Later in this section, we will show that invested funds cause banks to satisfy their debts and move to an absorbing status, causing updates to matrix Q^* and ultimately matrix M, without loss of generality in our proof; until Bank k's debts are satisfied, $\exists i$: $m_{i,k} \geq 1$, although the particular values $m_{i,k}$ may change with Q^* and M.

Reviewing matrix M from our example, in this post-organic clearing financial network, a $\$w$ investment in Bank 3 would reduce each of the three zero-status banks' debts by $\$2w$ (as shown by values of 2 for each zero-status bank in the third row of M) and thus the total outstanding debt by $\$6w$. When an investor's goal is to provide maximal benefit to the overall network rather than benefit to a single zero-status bank, such measures of the overall network benefit may be relevant in addition to measuring the benefit to individual zero-status banks.

We define m_i^{total} as the multiplier describing the overall reduction in the network's outstanding debts relative to the amount of an external investment in bank i, and note that row sums from matrix M describe this total reduction in outstanding debt. That is,

$$m_i^{total} = \sum_{k=1}^{n} m_{i,k} \tag{17}$$

From this result, (a) of **Theorem 5** is evident. For our example,

$$\boldsymbol{m^{total}} = [0,4,6,5,0]$$

When the investor's goal is reduction in the network's total amount of uncleared debt, multipliers m_i^{total} describing the reduction in the network's total outstanding debt relative to the amount invested in a particular bank provide a primary means of determining the optimal (i.e., most impactful) location for an investment.

In contrast, bank-specific multipliers $m_{i,k}$ are most useful in situations in which the investor's goal is to prevent a particular bank's default. For a zero-status bank k, the largest value of $m_{i,k}$ (usually $m_{k,k}$, but as **Lemma 1** and our example show, occasionally $m_{i,k} = m_{k,k}$ for another zero-status bank i) describes the multiplicative benefit to bank k resulting from an investment in the optimal location to maximize benefit to bank k. When the maximal benefit to bank k is obtained

by investment in either of several banks, the largest value of m_i^{total} indicates which investment is of maximal benefit to the overall network, a worthy secondary goal and a potential tie-breaker in decisions about which bank should receive the investment.

Multiplier Effects in the Benefit of Intervention to a Group of Banks. How can post-clearing network dynamics be used to describe the benefit of outside investment in a particular bank to a selected group of zero-status banks?

Theorem 6. *(a) For any financial network with projected $J_0(T^*) \neq \emptyset$, there exist values*

$$m_i^A = \sum_{k \in A} m_{i,k}$$

(some non-zero) describing the reduction in remaining debts for zero-status banks in a group A as a multiplier of the amount received by bank i from a source external to the network.

(b) There exist a finite number of intervals $d_k = [W_k, W_{k+1})$ within the cumulative amount w invested in which the multipliers m_i^{total} are constant. Amounts W_k are the cumulative investment amounts at which a zero-status bank k, $k \in J_0(T^*) \cap A$ or $k \in J_0(T^*) \backslash A$, moves to an absorbing status, and the network dynamics are updated.

(c) Therefore, there is a benefit function $B_A(i(w), w)$ describing the benefit to banks in group A that is linear within each interval.

Similar to m_i^{total} described above, when a regulator's goal is to prevent the defaults of a group A of banks, multipliers m_i^A may be calculated describing the total benefit of investment in any bank i to banks in A, and part (a) of **Theorem 6** is evident.

$$m_i^A = \sum_{k \in A} m_{i,k}$$

We note that this intervention goal generalizes the single-bank and total network cases, as A could contain a single bank, a group of banks, or all banks within the sub-network of zero-status banks. The largest multiplier m_i^A denotes the investment recipient i driving the largest benefit to zero-status banks in group A.

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In our example, if $A = \{2,3\}$, we sum the values $m_{i,2}$ and $m_{i,3}$ in each row i and obtain

$$\boldsymbol{m^A} = [0,3,4,3,0].$$

BaT-like Intervals $\boldsymbol{d_k}$ and Updated Network Dynamics. We note that the multiplicative benefits described in M, $\boldsymbol{m^{total}}$, or $\boldsymbol{m^A}$ can only be realized up to a certain investment amount. In the BaT model, after certain intervals of time and corresponding payments $u_i(t)$ and changes in cash $d_i(t)$, banks change status by satisfying their debts or running out of cash. In our analysis, similar to the BaT model's iterations, after a sufficient amount is invested one or more of the zero-status banks will satisfy its debts and move from a zero status into an absorbing status, and the matrix Q^* will require updates to reflect these new statuses and the absorbing-status bank's lack of remaining debts. Rather than defining these intervals as lengths of time, we define them as intervals $d_k = [W_k, W_{k+1})$ within the cumulative investment amount w, and for $k = 1,2, \ldots$ the W_k are cumulative investment amounts w at which status changes occur ($W_0 = 0$). With a finite number n^* of zero-status banks (and the potential for a regulator's goal to be satisfied even while some of these default), we note that network dynamics will be specified within a finite number of intervals (at most n^*), each ending with the payoff of one or more zero-status banks.

In our example, recall that $\boldsymbol{b}^* = [0, 1.27, 2.63, 3.62, 0]$ and

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider a regulator providing funds to Bank 3, either to prevent Bank 3's default or because investment in Bank 3 provides maximal benefit to the overall network (recall that $m_3^{total} = 6$). After the first \$0.64 injected into Bank 3, because $m_{3,2} = 2$ and $2 \times \$0.64 = \$1.27 = b_2^*$,[18] Bank 2's debt will be completely cleared, and Bank 2 moves to an absorbing status. Bank 2's move from a zero status to an absorbing status necessitates an update to matrix Q^*, replacing all entries in row 2 with values of 0, as Bank 2 now has no remaining debt and no reason to make payments.

$$Q^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad Q^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

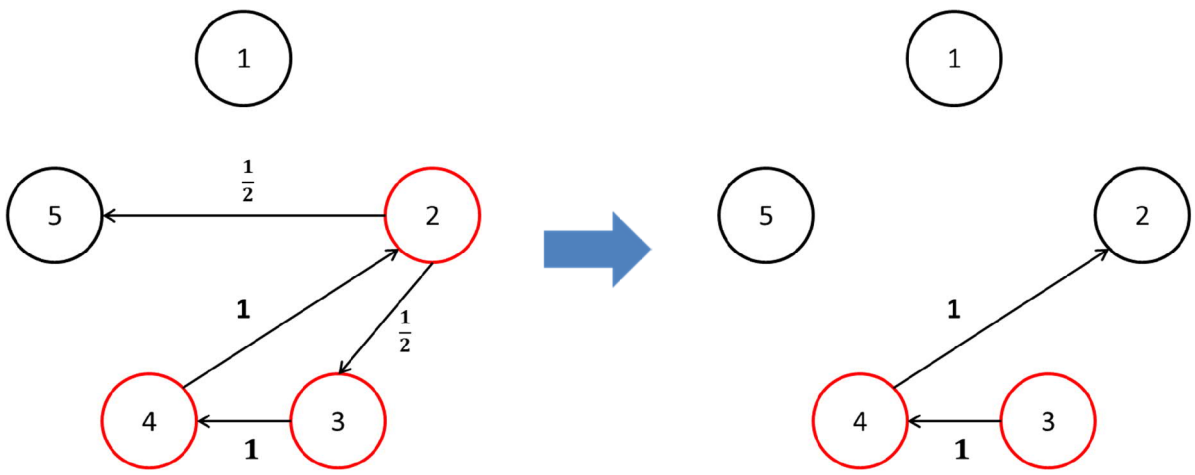


Figure 10: Post-Clearing Relative Liabilities – Network Plot Q^* Update

[18] Values are rounded to the second decimal place.

With this update to Q^*, the cyclical flow of funds between Banks 2, 3, and 4 is broken. Because Bank 2 has moved into an absorbing status and has no remaining debts, any additional amount invested directly in Bank 2 or received by Bank 2 through payments from other banks yields no additional reduction in the network's outstanding debt. However, as long as Bank 3 and Bank 4 remain in a zero status, an incremental investment of $\$w$ in Bank 3 reduces both Bank 3's and Bank 4's debts by $\$w$ and the total outstanding debt by $\$2w$ (any funds Bank 3 receives are paid to Bank 4, after which Bank 4 also pays them toward its debt to Bank 2). Similarly, a $\$w$ investment in Bank 4 would reduce Bank 4's and the network's total outstanding debt by $\$w$ (the funds would be used only to satisfy Bank 4's debt to Bank 2). Accordingly, the updated matrix M and vector $\boldsymbol{m^{total}}$ are given below.

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{m^{total}} = [0,0,2,1,0]$$

To generalize, once a recipient bank i has been selected to align with the investor's goals, the maximum amount that may be invested before the first status change occurs is

$$w_1(i) = \min_k \left(\frac{b_k^*}{m_{i,k}}\right) \tag{18}$$

among the zero-status banks k for which $m_{i,k}$ is non-zero, indicating that the investment has some benefit to bank k (note that for a zero-status bank k, the numerator b_k^* is non-zero by definition). After this amount is invested in bank i, each zero-status bank k's debts will be reduced by $w_1(i) \cdot m_{i,k}$, and the bank k for which the minimum was obtained will satisfy its debts (demonstrating part (b) of **Multiplier Effects in the Benefit of Intervention to a Single Bank.** How can the

benefit to a particular zero-status bank be measured as a function of the amount invested in any other bank?

Theorem 4, **Theorem 5**, and **Theorem 6**).

After a status change, an updated vector $\boldsymbol{b}^*$, matrix Q^*, matrix M, and vector $\boldsymbol{m^{total}}$ or $\boldsymbol{m^A}$ may be calculated, and a new best investment location may be selected if still needed to prevent bankruptcy of the desired bank or banks (i.e., if this goal was not accomplished through this first status change), or to maximize reduction of the network's overall uncleared debts through further investment.

Conclusions. In conclusion, we note that networks with denser connections can be expected to also show denser connections between any zero-status banks that remain after organic clearing. Therefore, as with potential for contagion effects, more densely connected networks may also produce larger multipliers $m_{i,k}$, m_i^{total}, and m_i^A when provided funds will be received and subsequently paid by more zero-status banks, and the potential for higher-order benefits is greater.

Having defined these multipliers and the network dynamics that drive changes in their values as banks' statuses change, we now discuss further their potential uses in designing an optimal intervention to accomplish a particular goal.

4.4 Intervention Planning in Selected Examples

How can multipliers or other methods be used to plan an optimal intervention to prevent the defaults of all, one, or a group of banks?

We begin with intervention planning methods using the multipliers $m_{i,k}$, m_i^{total}, or m_i^A corresponding to the regulator's goal (i.e., prevention of default for a bank k, prevention of all defaults, or default of a group A of banks, respectively). Selection of the highest multiplier both identifies the recipient that maximizes benefit to the relevant bank(s) within the following interval and defines a key parameter (the multiplier) used in calculation of the interval's length (i.e., the amount that can be invested before network dynamics change). We follow with several alternative methods that will yield the minimal intervention amounts and corresponding recipients.

Example 2A. For our network from Example 1A, consider a situation in which a regulatory body is interested in preventing the default of a single bank, Bank 3. In identifying an optimal investment location, we consider the matrix M, as previously calculated.

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

With preventing Bank 3's default as our goal, we are most interested in the third column of M, which describes the value of an investment in each bank in terms of reduction in Bank 3's debt. As described in **Lemma 1**, investment in Bank 3 will always yield the maximal benefit to Bank 3 by allowing the initial investment to be received (and paid out) by Bank 3 before any funds leave the sub-network of zero-status banks. Although in some networks it is possible for other upstream investment locations to yield identical benefits to a particular bank with larger benefit to the overall network, this is not the case in our example, in which investment in Bank 3 drives twice the reduction in Bank 3's debts, while investment in Bank 2 or 4 drives a reduction in Bank 3's debts of only one times the invested amount.

Thus selecting Bank 3 as the optimal investment location, we derive the maximum amount that can be invested with the benefit described in row 3 of matrix M.

$$\boldsymbol{b}^* = [0, 1.27, 2.63, 3.62, 0]$$

$$w_1(3) = \min_k(\frac{b_k^*}{m_{3,k}}) = \min\left(\frac{1.27}{2}, \frac{2.63,}{2}, \frac{3.62}{2}\right) = \$0.64$$

Once $\$0.64$ is invested in Bank 3, each bank k's debt is reduced by $\$0.64 \cdot m_{3,k}$, and we have

$$\boldsymbol{b}^* = [0, 1.27 - \$0.64 \cdot 2, 2.63 - \$0.64 \cdot 2, 3.62 - \$0.64 \cdot 2, 0]$$

$$= [0, 0, 1.35, 2.34, 0]$$

We see that Bank 2's debts are satisfied (as expected, as the minimum investment amount to drive a status change was obtained from $\frac{b_2^*}{m_{3,2}}$). With this change in Bank 2's status, we update matrix Q^* to include values of 0 in row 2 and recalculate the corresponding matrix M.

$$Q^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Again, with these new dynamics the largest benefit to Bank 3 is obtained by investment in Bank 3, now with a benefit of only one times the invested amount.

$$\boldsymbol{b}^* = [0, 0, 1.35, 2.34, 0]$$

$$w_2(3) = \min_k(\frac{b_k^*}{m_{3,k}}) = \min\left(\frac{1.35,}{1}, \frac{2.34}{1}\right) = \$1.35$$

After this investment,

$$\boldsymbol{b}^* = [0, 0, 0, 0.99, 0]$$

Bank 3's debt is satisfied, achieving the investor's goal of satisfying the original $2.63 debt at a total cost of $1.99 ($0.64 + $1.35), also rescuing Bank 2 as a by-product, and leaving Bank 4 to default on its remaining $0.99 debt. The intervention vector $\boldsymbol{w}$ is $[0,0,1.99,0,0]$. Comparing to the simplistic methods described in **Section 4.2**, an investment in the amount of Bank 3's outstanding debt after clearing would have cost $2.63, while an investment in an amount equal to the difference between Bank 3's pre-clearing debt and cash positions would have cost $3 − $0.01 = $2.99, an even larger amount.

In this intervention, the investor's goal was preventing Bank 3's default, and so the intervention's success is measured in terms of reduction in Bank 3's debt. **Figure 11** presents the reduction in Bank 3's debt as a function of the invested amount.

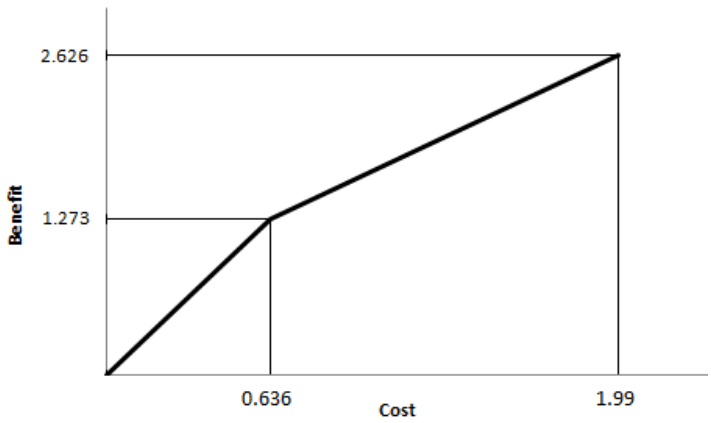


Figure 11: Intervention Example 2A – Bank 3 Benefit

From this figure, we can observe that the benefit of this intervention to Bank 3 is a piecewise function of the invested amount within intervals defined by status changes affecting the network dynamics. Within each interval, the function's slope is given by the interval-specific value $m_{i,k}$ corresponding to the selected recipient i (Bank 3 in both intervals) and relevant beneficiary k (also Bank 3 in both intervals).

Indeed, the benefit of any intervention (with "benefit" defined according to the intervention's goals) will be a piecewise function of the selected intervention locations (i.e., recipient banks), cumulative intervention amount, vector $\boldsymbol{b}^*$, and the matrix Q^* from which each interval's values of m, m^{total}, or m^A follow, as noted in part (c) of **Multiplier Effects in the Benefit of Intervention to a Single Bank.** How can the benefit to a particular zero-status bank be measured as a function of the amount invested in any other bank?

Theorem 4, **Theorem 5**, and **Theorem 6**.

Thus, a benefit function like the above can be expressed as a function of the selected recipient at each cumulative amount w and the amount:

$$B_k(i(w), w) = \int_0^w m_{i,k}(i(s), s)ds \tag{19}$$

where $m_{i,k}(i(s), s)$ denotes the interval-specific benefit multiplier given an intervention recipient i in the interval corresponding to cumulative amount s. In our example, $i(w) = 3$ for w in (0,1.99], denoting the investment recipient within each of the two intervals.

Similarly, for interventions with other goals, the integrand from (19) may be modified to align with a relevant measure of the benefit. For instance, when benefit is measured through reduction/elimination of the network's overall outstanding debt or debts of a group A of banks, the integrand will instead be $m_i^{total}(i(s), s)ds$ or $m_i^A(i(s), s)ds$.

Example 2B. Consider an intervention in which the investor wishes to prevent all defaults in the network. Because the benefit will be measured in terms of overall debt reduction in the network, optimal point-in-time recipients can be selected and the intervention's overall benefit calculated using the largest interval-specific values of m_i^{total}. **Table 4** presents the matrix M, corresponding

vector $\boldsymbol{m^{total}}$, the recipient that would be selected in each interval to provide maximal benefit to the network, and the amount that can be invested in the interval before a bank's status and the network's dynamics change.

Table 4: Intervention Example 2B – Overall Network Benefit

Interval k	M	$\boldsymbol{m^{total}}$	Recipient $i(s)$	Amount $w_k(i)$ until status change
1	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	[0,4,6,5,0]	Bank 3	\$0.64 (Bank 2 rescued)
2	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	[0,0,2,1,0]	Bank 3	\$1.35 (Bank 3 rescued)
3	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	[0,0,0,1,0]	Bank 4	\$0.99 (Bank 4 rescued)

Figure 12 summarizes the cumulative benefit as a function of the cumulative investment amount, with

$$B(i(w), w) = \int_0^w m_i^{total}(i(s), s)ds \tag{20}$$

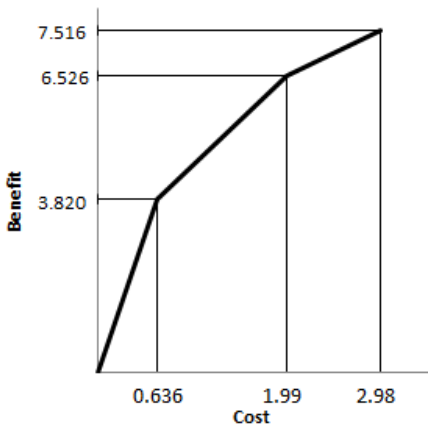


Figure 12: Intervention Example 2B – Overall Network Benefit

This intervention prevents all defaults by eliminating the entirety of the network's \$7.52 outstanding debt at a cost of \$2.98. $\boldsymbol{w} = [0,0,1.99,0.99,0]$.

Example 2C. Consider an intervention in which the investor wishes to prevent the defaults of only banks 2 and 4, so $A = \{2,4\}$. Table **5** presents the matrix M, corresponding vector $\boldsymbol{m^A}$, the recipient that would be selected in each interval to provide maximal benefit to the network, and the amount that can be invested in the interval before a bank's status and the network's dynamics change. In several intervals, the largest value of m_i^A can be obtained for multiple i, so we also consider $\boldsymbol{m^{total}}$ in these intervals to select the recipient that would drive the larger benefit to the overall network.

Table 5: Intervention Example 2C – Group $A = \{2,4\}$ Benefit

Interval k	M	$\boldsymbol{m^A}$	$\boldsymbol{m^{total}}$	Recipient $i(s)$	Amount $w_k(i)$ until status change
1	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	[0,3,4,4,0]	[0,4,6,5,0]	Bank 3	\$0.64 (Bank 2 rescued)
2	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	[0,0,1,1,0]	[0,0,2,1,0]	Bank 3	\$1.35 (Bank 3 rescued)
3	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	[0,0,0,1,0]	[0,0,0,1,0]	Bank 4	\$0.99 (Bank 4 rescued)

Figure 13 summarizes the cumulative benefit as a function of the cumulative investment amount, with

$$B_A(i(w), w) = \int_0^w m_i^A(i(s), s)ds \tag{21}$$

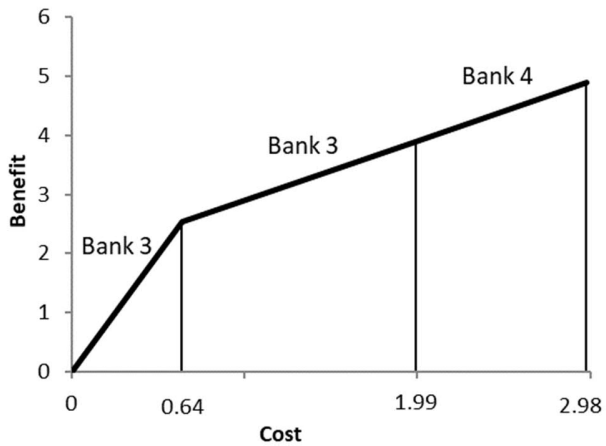


Figure 13: Intervention Example 2C – Group A Benefit

This intervention prevents the defaults of Banks 2 and 4 (and ultimately all defaults) by eliminating the entirety of group A's \$4.89 outstanding debt at a cost of \$2.98. $\boldsymbol{w} = [0,0,1.99,0.99,0]$.

Example 2D. Finally, consider an intervention in which the investor wishes to prevent the defaults of only banks 2 and 3, so $A = \{2,3\}$. Table **6** presents the matrix M, corresponding vectors $\boldsymbol{m^A}$ and $\boldsymbol{m^{total}}$**,** the recipient that would be selected in each interval to provide maximal benefit to the network, and the amount that can be invested in the interval before a bank's status and the network's dynamics change.

Table 6: Intervention Example 2D – Group $A = \{2,3\}$ Benefit

Interval k	M	$\boldsymbol{m^A}$	$\boldsymbol{m^{total}}$	Recipient $i(s)$	Amount $w_k(i)$ until status change
1	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	[0,3,4,3,0]	[0,4,6,5,0]	Bank 3	\$0.64 (Bank 2 rescued)
2	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	[0,0,1,0,0]	[0,0,2,1,0]	Bank 3	\$1.35 (Bank 3 rescued)

Figure 14 summarizes the cumulative benefit as a function of the cumulative investment amount.

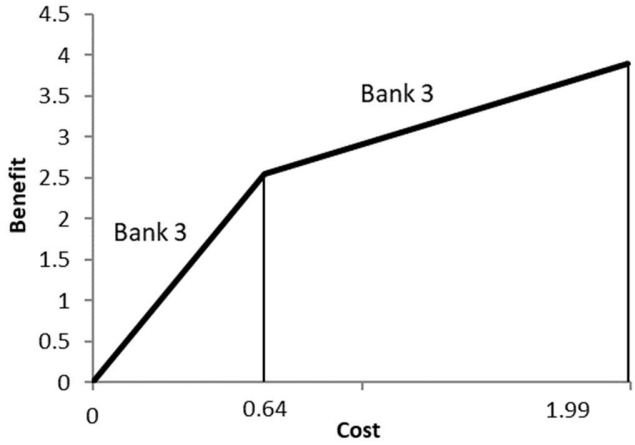


Figure 14: Intervention Example 2D – Group A Benefit

This intervention prevents the defaults of Banks 2 and 3 by eliminating the entirety of group A's \$3.90 outstanding debt at a cost of \$1.99. $\boldsymbol{w} = [0,0,1.99,0,0]$.

Proposition 2. *For interventions that aim to prevent the default of a particular bank or set of banks (including preventing all defaults by clearing all debts), the slope of the benefit function* $B(i(w), w)$ *monotonically decreases with the cumulative investment amount* w *(i.e., the benefit function is concave down). In other words, investments have diminishing marginal returns.*

Proof of Proposition 2. Benefit multipliers $m_{i,k}$ are calculated as $\{\sum_{n=0}^{\infty}(Q^*)^n\}_{i,k}$ for zero-status banks k, while the total network benefit multiplier m_i^{total} is calculated as

$$m_i^{total} = \sum_{k=1}^{n} m_{i,k} = \sum_{k}\{\sum_{n=0}^{\infty}(Q^*)^n\}_{i,k}$$

and the benefit to a group A of banks is given by

$$m_i^A = \sum_{k \in A} m_{i,k} = \sum_{k \in A} \{ \sum_{n=0}^{\infty} (Q^*)^n \}_{i,k}$$

As the cumulative investment amount allows a zero-status bank to pay its debts and move to an absorbing status, values in that bank's corresponding row of Q^* become 0 while all other values remain static. Thus, all values $q_{i,k}^*$ either retain their original values or become 0, and it follows that values $\{(Q^*)^n\}_{i,k}$ monotonically decrease after a status change (although not strictly). Monotonic decreases in $\{(Q^*)^n\}_{i,k}$ lead to monotonic decreases in $\{\sum_{n=0}^{\infty}(Q^*)^n\}_{i,k}$, $\sum_k \{\sum_{n=0}^{\infty}(Q^*)^n\}_{i,k}$, and $\sum_{k \in A}\{\sum_{n=0}^{\infty}(Q^*)^n\}_{i,k}$, equivalent to $m_{i,k}$, m_i^{total}, and m_i^A respectively.

In these examples, we have taken a "greedy" approach, at each interval selecting the investment location i with the largest corresponding benefit (i.e., the maximal value of multiplier $m_{i,3}$ or the largest value of m_i^{total} or m_i^A). As explained by **Proposition 2**, any benefit function $B(a, w)$ decreases in slope at the end of intervals. Thus, such an approach maximizes benefit while the highest multipliers are available. When the investor's available funds do not allow complete satisfaction of the investment goal, a greedy approach (i.e., selecting recipients that yield the highest multipliers) can often ensure maximal benefit has been obtained at the given investment amount. However, when available funds are sufficient to accomplish the regulator's goal, we note the possibility for selection of alternate recipients to accomplish the goal at lower cost (e.g., networks in which a recipient bank can be selected with a lower multiplier, but for which investments can be made in higher cumulative amounts before a status change occurs and multipliers decrease). However, we have yet to identify an example where the greedy approach does not produce the lowest cost, and note identification of the conditions in which the greedy approach is not optimal (or a proof that it always is optimal) as an opportunity for further research.

This framework may be used to evaluate the benefit of any selected recipients at each investment amount w, and the scenario producing the largest benefit may be selected, if not the recipients selected by the greedy approach.

Thus, with understanding of the network dynamics that lead to multiplier effects, we will describe several alternative intervention planning methods in the remainder of this section that will always produce the minimal investment amounts to meet the investment goal.

Modified BaT Approach. One convenient alternate approach for deriving the optimal intervention involves modifications to the BaT Model. This approach assumes that cash will be made available as needed to selected banks during the clearing process (as with a line of credit).

- First, because we assume that funds will be available as needed to the group A of banks the regulator intends to rescue, these banks will not cease to make payments while still having debts, so we remove the logic to define their zero status and only allow these banks to take a positive or absorbing status. With positive statuses, banks with remaining debts will pay them at the baseline rate of 1 (potentially with the regulator's money instead of their own).
- Second, as the rescued banks will not move to a zero status, we calculate the time elapsed in each interval as the time until one of the banks in A satisfies its debts and moves to the absorbing status or a bank in A^C moves to a zero or absorbing status. Because intervals are no longer defined by when a bank in A's cash is exhausted, this approach allows cash positions to become negative for banks in A.

Banks in A that would default under the unmodified BaT Model are those that end with negative cash positions in the modified model. Thus, negative ending cash positions indicate banks that should receive funds during the clearing process to end at a zero cash position, and any

negative ending cash positions are exactly the minimal incremental amounts the corresponding banks would require in order for their debts to be satisfied.

Proposition 3. *The optimal intervention* $\boldsymbol{w}$ *to prevent defaults for banks* $i \in A$ *can be identified algorithmically by negative ending cash amounts from this modified BaT model, with* $w_i = -c_i^*$ *for* $c_i^* < 0,\ i \in A$.

During the modified BaT algorithm, a bank $i \in A$ will pay towards its debts with the positive-status outflow rate of 1 until satisfying its debts and changing to the absorbing status, with an outflow rate of 0. Thus, we assume $p_i = b_i$ for all $i \in A$ because we will intervene on these banks' behalf if needed, and the ending cash for each bank $i \in A$ is given by

$$c_i^* = c_i + \sum_j q_{j,i} p_j - b_i \tag{22}$$

where p_j are the total payments made by bank j given $p_i = b_i$ for $i \in A$. Thus, a negative value of c_i^* indicates that $b_i > c_i + \sum_j q_{j,i} p_j$, while $c_i^* = 0$ indicates that $b_i = c_i + \sum_j q_{j,i} p_j$. Because we already assume $p_i = b_i$ for all $i \in A$, adding to c_i for $i \in A$ does not impact $\sum_j q_{j,i} p_j$ in (22) in any way, and incrementing c_i by an amount w_i has a direct impact on the cash that remains after clearing (the left side of (22)).

$$c_i^* + w_i = c_i + w_i + \sum_j q_{j,i} p_j - b_i$$

Therefore, when $c_i^* < 0$ for $i \in A$, the addition of $w_i = -c_i^* = -c_i + b_i - \sum_j q_{j,i} p_j$ to starting cash c_i results in the left side of (22), which denotes ending cash, being $c_i^* + w_i = 0$. Any smaller amount w_i^- will be insufficient and leave ending cash negative and debts unpaid, with $b_i > c_i +$

$\sum_j q_{j,i} p_j + w_i^-$, and any larger amount w_i^+ will result in ending cash $c_i^* + w_i^+ > 0$, denoting a sufficient, but excessive, intervention.

Example 3A. Table **7** presents the output from each iteration of this modified BaT Model for our $n = 5$ network from Example 1A, with the goal of preventing the defaults of group $A = \{2,3\}$.

Table 7: Intervention Example 3A – Modified BaT Model

Iteration	Time Elapsed	***c***	***b***	***j***
Initial	0	[1, .01, .01, .01, 0]	[1, 2, 3, 4, 0]	[+,+,+,+,*]
1	1	[0, .34, -.49, .01, 1.17]	[0, 1, 2, 3, 0]	[*,+,+,+,*]
2	2	[0, .34, -.99, .01, 1.67]	[0, 0, 1, 2, 0]	[*,*,+,+,*]
3	3	[0, 1.34, -1.99, .01, 1.67]	[0, 0, 0, 1, 0]	[*,*,*,+,*]
4	3.01	[0, 1.35, -1.99, 0, 1.67]	[0, 0, 0, .99, 0]	[*,*,*,0,*]

Of the two banks in A, only Bank 3 is left with a negative cash position, indicating that provision of $1.99 to Bank 3 will prevent the defaults of both Bank 2 and Bank 3, while Bank 4 is left to default.

How can all defaults in a network be prevented at lowest cost? How can the default of a particular network member be prevented at lowest cost?

The answers to our other two key questions can be answered as specific cases of **Proposition 3**, with $A = \{i\}$ or $A = \{1, \dots, n\}$.

Example 3B. Table 8 presents the output from each iteration of this modified BaT Model for our $n = 5$ network from Example 1A, with the goal of preventing all defaults.

Table 8: Intervention Example 3B – Modified BaT Model

Iteration	Time Elapsed	***c***	***b***	***j***
Initial	0	[1, .01, .01, .01, 0]	[1, 2, 3, 4, 0]	[+,+,+,+,*]
1	1	[0, .34, -.49, .01, 1]	[0, 1, 2, 3, 0]	[*,+,+,+,*]
2	2	[0, .34, -.99, .01, 1.5]	[0, 0, 1, 2, 0]	[*,*,+,+,*]
3	3	[0, 1.34, -1.99, .01, 1.5]	[0, 0, 0, 1, 0]	[*,*,*,+,*]
4	4	[0, 2.34, -1.99, -.99, 1.5]	[0, 0, 0, 0, 0]	[*,*,*,*,*]

From these results, we note that the negative final cash positions (-$1.99 for Bank 3 and -$0.99 for Bank 4) are exactly the required investments calculated previously using m_i^{total} and presented in **Table 4**.

Example 3C. Table 8 presents the output from each iteration of this modified BaT Model for our $n = 5$ network from Example 1A, with the goal of preventing the default of only Bank 2.

Table 9: Intervention Example 3C – Modified BaT Model

Iteration	Time Elapsed	***c***	***b***	***j***
Initial	0	[1, .01, .01, .01, 0]	[1, 2, 3, 4, 0]	[+,+,+,+,*]
1	0.02	[.98, .02, 0, .01, 0.02]	[.98, 1.98, 2.98, 3.98, 0]	[+,+,0,+,*]
2	0.04	[.96, .02, 0, 0, 0.05]	[.96, 1.96, 2.97, 3.96, 0]	[+,+,0,0,*]
3	1	[0, −.14, 0, 0, 1.17]	[0, 1, 2.49, 3.48, 0]	[*,+,0,0,*]
4	2	[0, −.64, 0, 0, 1.67]	[0, 0, 1.99, 2.98, 0]	[*,*,0,0,*]

From these results, provision of $0.64 to Bank 2 will prevent Bank 2's default, leaving both Bank 3 and Bank 4 to default. While this intervention is optimal given the goal of rescuing only Bank 2, we note from Example 2B (Table **4**) that the same amount provided to Bank 3 could also prevent Bank 2's default with additional benefit to Banks 3 and 4.

Minimum Cash Vector Approach. In Examples 2A, 2B, and 2C, we began by modeling the organic clearing that can occur given each bank's initial cash position, total debt, and allocation of that debt. Noting which banks would default without intervention, we explored the mechanism that leads to multiplier effects of any invested funds and demonstrated how, after organic clearing, a largest multiplier may be chosen at each interval to maximize the impact of the invested funds towards particular goals.

Often, choosing the maximal multiplier in each interval leads to the optimal overall intervention. In Examples 3A, 3B, and 3C, we also introduced a modified execution of the BaT model allowing cash positions to become negative during the clearing process if needed until complete clearing occurs. Negative cash positions in the final cash vector then denote cash shortfalls that would need to be remediated through provision of outside funds in order to facilitate the clearing that has already been modeled for the selected banks while bringing these negative cash positions back to 0. Because such intervention brings cash positions of such banks to 0 (and not higher) while also bringing debt positions to 0, this method identifies the optimal (i.e., minimal) intervention amounts and corresponding recipient banks to allow complete clearing.

Each of these approaches began by modeling clearing with the initial cash position available to each bank, then derived the incremental amounts necessary to rescue one or all banks in the network. Rather than deriving the optimal intervention amount using the post-clearing status of the network, we ask a natural next question: What must the initial pre-clearing cash position of each bank be to facilitate complete clearing or clearing by particular banks? With an answer to this question, pre-clearing cash positions for each bank can be compared to the corresponding value in the vector of minimum sufficient cash positions and any shortfall addressed prior to clearing, without need for a clearing model to project the post-clearing network status.

Section 3.3 discussed the derivation of $\boldsymbol{c^{min}}$, a vector describing the minimum initial cash positions needed to allow complete clearing within the network (i.e., for all banks). In particular, $c_i^{min} = b_i - \sum_j q_{j,i} b_j$ for each bank i. After deriving this key vector, actual cash positions c_i in the pre-clearing network can then be compared to the corresponding value of c_i^{min}, and when $c_i < c_i^{min}$, the positive difference $c_i^{min} - c_i$ provides the amount each bank i must receive in an intervention to allow complete clearing while bringing these banks' final cash positions to 0.

Using $\boldsymbol{b}$ and Q from our $n = 5$ example in **Section 3.3**, we calculate $\boldsymbol{c^{min}} = \left[1, -\frac{7}{3}, 2, 1, -\frac{5}{3}\right]$. With an initial $\boldsymbol{c} = [1, 0.01, 0.01, 0.01, 0]$, we may immediately see that the model is deficient; Bank 3 and Bank 4 show shortfalls of 1.99 and 0.99, respectively, consistent with the minimal intervention amounts derived for complete clearing using multipliers in Example 2B and using the modified BaT algorithm in Example 3A. The ultimate intervention cost is given by $\sum_i \max(c_i^{min} - c_i, 0) = \sum_i \max(b_i - \sum_j q_{j,i} b_j - c_i, 0)$, or 2.98 for this example.

$\boldsymbol{c^{min}}$ for complete network clearing may also be derived through the modified BaT algorithm described in **Section 4.4** by initializing the cash position of each bank at a value of 0 and allowing each bank's cash position to become negative. Any negative final cash positions denote the minimum amounts of starting cash required for complete clearing, while positive or zero final cash positions identify the banks that will clear solely from their receivables (if all are realized). In the absence of initial cash, final cash positions are simply $c_i^* = \sum q_{j,i} b_j - b_i$ (receivables minus debts), and thus $c_i^{min} = b_i - \sum q_{j,i} b_j$ can be derived by multiplying each of the final cash positions from this modified BaT algorithm by a value of negative 1.

Initializing cash positions at 0 in the modified BaT algorithm, allowing the cash positions of only certain banks $i \in A$ to go negative, and preventing the zero status for only these banks can

produce final cash positions in which negative values describe the minimal cash positions required for clearing of debts *by those rescued banks.* In this way, we may generalize $\boldsymbol{c^{min}}$ to describe the minimal sufficient cash positions for any intervention goal, not only complete clearing, and a shortfall $\max(c_i^{min} - c_i, 0)$ for $i \in A$ can be addressed through intervention prior to clearing.

In **Chapter** 5, we will discuss the impact of network structure on $\boldsymbol{c^{min}}$ and the summation of positive values of $\boldsymbol{c^{min}}$ (the amount of cash required in the network for clearing) as a relative measure of risk, and thus the ability of cash reserves to mitigate this risk under simulated shocks.

4.5 Special Cases and the Obligation Cascade

Section 3.1 and **Section** 4.4 discussed $\boldsymbol{c^{min}}$, the minimal sufficient cash vector such that when $c_i \geq c_i^{min}\ \forall i$, all banks in the network will be able to clear the entirety of their debts. We ask a natural next question: What is the minimal cash vector $\boldsymbol{c}$ such that at least one bank will be able to pay its debts? We will show that in any network, even one where no cash is present, there is a bank whose debts can be satisfied.

Theorem 7. *(a) Within any network, there is at least one bank i whose debts may be satisfied ($p_i = b_i$), potentially through restructuring of debts.*

(b) Further, any sufficient network may be simplified through restructuring to produce a simplified "obligation cascade" structure, defined as a partition of the set A of the network's banks into subsets $(A_1, A_2, \dots, A_m)$ such that banks in A_1 owe no one, banks in A_2 owe only those in A_1, banks in A_3 owe only those in A_2 and A_1, and so on, with banks in A_m ($m \leq n$) owed by none but potentially owing all others.

With a network having $J_*(0) \neq \emptyset$ (i.e., a paid-off bank prior to clearing), result (a) is trivial; we can clearly see that a bank's debts have already been satisfied. However, when no bank is initially in the absorbing status, this result is less trivial and warrants examination. Derivation of the clearing payment vector using the BaT model or other clearing models can reveal a bank that is able to pay its debts in many networks. However, in a special case—termed a "swamp" by Sonin and Sonin (2017, 2020)—in which such a sub-network B of zero-status banks exists with the condition that banks in the sub-network are connected only by obligations to or from other zero-status banks in the sub-network, the clearing payment vector is not unique; (2) can be satisfied by a clearing payment vector reflecting a range of possible payments within the swamp.

While there is no cash in a swamp and therefore no transactions involving transfers of cash can occur, we propose an algorithm for deriving a fair method for "payments" (i.e., multilateral cancellation of obligations) among the swamp's members using properties of ergodic Markov chains. The resulting network is simplified, and equivalent in the sense that debts have been restructured through multilateral cancellation via "payments" to a point where at least one bank will be without remaining debts, demonstrating (a) of **Theorem 7**. Further, we will complete the cascade structure and explain the cascade's need for sufficiency in the network (or an intervention that makes cash positions in the network sufficient) to avoid breaches of Eisenberg and Noe's assumption of proportionality in payments.

As noted in **Section 4.3**, interpreting Q or Q^* as a transition probability matrix in a Markov chain allows us to apply certain properties of Markov chains. A Markov chain is said to be ergodic when it is irreducible and aperiodic, meaning that any state can be arrived at from any other state and no state has a period (>1) for which every return to that state must occur in a number of steps that is a multiple of that period. Given a transition probability matrix P in an ergodic Markov

chain, there exists an invariant distribution, a vector $\boldsymbol{\pi}$ such that $\boldsymbol{\pi}P = \boldsymbol{\pi}$ and further transitions from the distribution of states expressed in $\boldsymbol{\pi}$ result in an identical distribution. Classically, values in this vector describe the proportions of future time that will be spent in each state, regardless of the starting state. Accordingly, each row of P^n will converge to vector $\boldsymbol{\pi}$ as $n \to \infty$.

In our framework, the matrix Q describes how a bank's debts are allocated among the other members of the financial network, and thus the proportions in which its payments will be made to those banks. Under the additional condition that zero-status banks owe and are owed only by other zero-status banks, we note that no bank in B has debts to an absorbing-status bank, and the flow of payments between these banks is an ergodic Markov chain with an invariant distribution. In particular, the invariant distribution corresponding to matrix Q will describe the percentage of the sub-network's total future payments that would be received by each bank (and therefore paid in turn by each zero-status bank) within an infinite series of bank-to-bank payments under the dynamics given in Q, were a non-zero amount of cash available within the sub-network B to facilitate such payments. However, even in the absence of cash, payment dynamics that follow from the invariant distribution can allow for fair cancellation of bank-to-bank obligations.

A zero-status bank must pay out what it receives, and therefore $u_i = n_i$ for $i \in B$. Accordingly, net inflow rates $d_i = n_i - u_i = 0$ for $i \in B$. Thus, during restructuring, there will be no change to the cash positions of each bank i, and the only changes in status will be zero-status banks whose debts are satisfied. We specify the particular inflow and outflow rates by corresponding values of the invariant distribution $\boldsymbol{\pi}(Q)$, $u_i = n_i = \pi_i$ for $i \in B$. Given these payment dynamics, the time to the first payoff will thus be calculated through (13) as a function of Q (through $\boldsymbol{\pi}$) and $\boldsymbol{b}$. After identifying the time of the first payoff, the debt vector $\boldsymbol{b}$ can be updated according to (14), and the

updated vector will reflect that one of the banks (which we denote i_1) has satisfied its debts through restructuring.

The same process can then be repeated within the sub-network of remaining banks $B^1 = B \backslash i_1$, considering only their debts to one another (a vector $\boldsymbol{b}^{(\boldsymbol{B^1})}$ of dimension $n - 1$) and disregarding their debts to i_1 (a vector $\boldsymbol{b}^{(\boldsymbol{i_1})}$ of dimension $n - 1$), although we note that B^1 is only an ergodic sub-network with its own invariant distribution when we assume that debts $\boldsymbol{b}^{(\boldsymbol{i_1})}$ have been paid and we can thus remove absorbing-status i_1 from the sub-network. Payment from banks in B^1 to i_1 without payment on their other debts violates Eisenberg and Noe's assumption of proportionality in payments unless the network is sufficient or made sufficient through intervention, allowing all other debts to be paid as well. In an insufficient model, enforcing such an order of payments means that i_1 may receive all it is owed from another bank i while other banks owed by bank i will not.

We illustrate the first round of restructuring that results in a bank with no remaining debts and the remainder of the cascade structure development through an example.

Example 4A. Consider a sub-network of five banks, each with no cash but having obligations to other banks within this sub-network, no obligations to banks outside the sub-network, and possibly obligations to itself.

$$\boldsymbol{c} = [0, 0, 0, 0, 0]$$

$$\boldsymbol{b} = [2, 1, 3, 2, 1]$$

$$Q = \begin{bmatrix} 0.4 & 0.1 & 0.3 & 0.1 & 0.1 \\ 0.1 & 0.4 & 0.2 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.3 & 0.1 \\ 0.1 & 0.3 & 0.2 & 0.1 & 0.3 \\ 0.6 & 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$$

Given this matrix Q, the corresponding invariant distribution $\boldsymbol{\pi}$ is [0.28, 0.21, 0.21, 0.14, 0.15]. Considering payments received—and thus subsequently paid out—at these rates, cash positions are unchanged over time, and the first bank to satisfy its debts may be calculated according to (13). After 4.67 units of time, Bank 2 satisfies its debts and the updated debt vector is

$$\boldsymbol{b}(\boldsymbol{T_1}) = [0.69, 0.00, 2.00, 1.33, 0.30]$$

This result demonstrates (a) from **Theorem 7**, as the simplified network shows that Bank 2's debts are satisfied in this simplified network. We denote the index of the first payoff i_1, with $i_1 = 2$ for this example. In networks where cash positions are sufficient (i.e., $c_i \geq c_i^{min}, \forall i$) or made sufficient through intervention, we may continue to simplify the network and build the cascade structure, which assumes that the other banks have first paid their debts to Bank 2, and to satisfy Eisenberg and Noe's assumption of proportionality in payments, must also assume that the banks will also pay their other debts in full.

Observing the percentages in column 2 of Q, we calculate the amount of each remaining zero-status bank's debts owed to Bank 2, $\boldsymbol{b}^{(i_1)} = [0.07, 0.40, 0.40, 0.03]$, and we consider these debts separately from the debts within the sub-network B_1 of the remaining four zero-status banks, in which the remaining debts are $\boldsymbol{b}^{(\boldsymbol{B^1})} = [0.62, 1.60, 0.93, 0.27]$. Note that $\boldsymbol{b}(\boldsymbol{T_1}) = \boldsymbol{b}^{(i_1)} + \boldsymbol{b}^{(\boldsymbol{B^1})}$. Assuming the debts to Bank 2 can be paid (e.g., after provision of funds through an intervention to obtain a sufficient model), we remove the 2nd row and column from Q, standardize the values

in each row to sum to 1 (and therefore represent the proportions of the remaining debts $b_i^{(B^1)}$ owed to each other member in B^1), and we obtain a new swamp with four members.[19]

$$Q^{(B^1)} = \begin{bmatrix} \frac{4}{9} & \frac{3}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{7} & \frac{2}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{2}{3} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix}$$

$$\boldsymbol{b}^{(\boldsymbol{B^1})} = [0.62, 1.60, 0.93, 0.27]$$

Repeating the process, we find that the invariant distribution $\boldsymbol{\pi}^{\boldsymbol{B^1}}$ is [0.38, 0.26, 0.19, 0.17], and under these payment dynamics Bank 5 (now represented by the 4th index) is the next to satisfy its debts after 1.55 units of time ($i_2 = 5$), at which time the updated debt vector is $\boldsymbol{b}(\boldsymbol{T_2})$ = [0.04, 1.20, 0.64, 0]. Of these debts, Banks 1, 3, and 4 owe $\boldsymbol{b}^{(\boldsymbol{i_2})}$ = [0.004, 0.15, 0.28] to Bank 5, and $\boldsymbol{b}^{(\boldsymbol{B^2})}$ = [0.04, 1.05, 0.37] to one another.

As the process continues, we build a diagram from the amounts owed by the remaining banks to the paid-off bank in each round. After a total of four ($n - 1$) steps, we have:

[19] If the debts to Bank 2 cannot be paid, the sub-network of remaining zero-status banks will have debts to an absorbing status bank, and that four-member sub-network will not meet the definition of a swamp. Further, the Markov chain will not be ergodic, as no transitions (i.e., payments) can occur from Bank 2. Thus, further iterations require that payments be made to Bank 2 first (breaching the assumption of proportionality in payments), or more ideally, that sufficient cash is made available (e.g., through an intervention) to facilitate clearing of all debts.

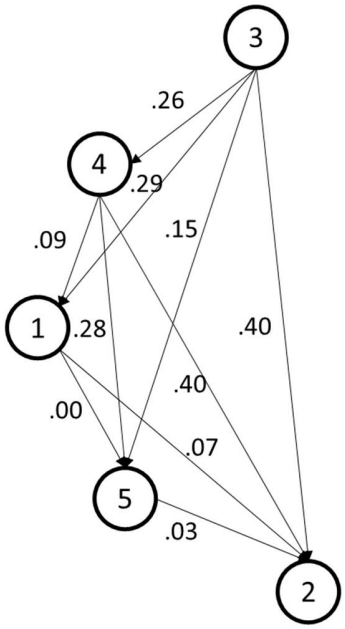


Figure 15: Obligation Cascade Diagram

We note that:

- Each of the other banks has remaining debts to Bank 2,
- Banks 1, 3, and 4 have remaining debts to Bank 5,
- Banks 3 and 4 have remaining debts to Bank 1,
- Bank 3 has remaining debts to Bank 4, and
- No bank owes Bank 3

We may also present the obligation cascade in a table, in which negative values in a column represent payments *from* the corresponding bank to other banks and positive values represent payments received from the other banks. Similarly, row indices denote the bank that is counterparty to the payment.

Table 10: Example 4A – Obligation Cascade Table

	$i_5 = 3$	$i_4 = 4$	$i_3 = 1$	$i_2 = 5$	$i_1 = 2$
$i_5 = 3$	0	0.26	0.29	0.15	0.40
$i_4 = 4$	-0.26	0	0.09	.28	0.40
$i_3 = 1$	-0.29	-0.09	0	0.00	0.07
$i_2 = 5$	-0.15	-0.28	-0.00	0	0.03
$i_1 = 2$	-0.40	-0.40	-0.07	-0.03	0
Total	-1.1	-0.5	0.3	0.4	0.9

The banks are indexed according to their rank within the cascade, with index 1 (i_1) describing the bank with no remaining debts (Bank 2 in our example) and index 5 (i_5) describing the bank with remaining debts to all others. Column totals describe the final cash position of each bank if all debts are paid, and therefore negative totals describe the incremental amounts needed by banks to facilitate complete clearing. Thus, we note that multiplying these column sums by negative 1 yields exactly the values of $\boldsymbol{c^{min}}$.

This "obligation cascade" represents the outcome of multilateral netting among the network's members, a simplification of the debt structure presented by vector $\boldsymbol{b}$ and matrix Q after cancellation of debts within each sub-network; note that in the diagram, each bank's inflows minus its outflows are identical to the amount each should receive minus the amount each should pay according to the original $\boldsymbol{b}$ and Q. However, we note that the proportions of these simplified debts are no longer those given by the original matrix Q.

With these results, we note that even in a network in which no member has cash, there is a member whose debts can be completely satisfied through "payments" that multilaterally cancel debts. In this example, Bank 2's debts will be satisfied after fair cancellation of debts within the swamp. Additional levels of cancellation may occur within the remaining sub-network without violating Eisenberg and Noe's assumption of proportionality provided that cash is sufficient (or is

made sufficient through outside intervention) to allow payment of all the simplified debts. Accordingly, we note that this algorithm does not rely on particular values from the initial cash vector $\boldsymbol{c}$. Example 4A applied the algorithm to a swamp, but this algorithm can also be applied to any ergodic network with an invariant distribution to derive key insights about the structure of debts, and the sufficiency of $\boldsymbol{c}$ can then be evaluated to determine whether the simplified cascade model also provides a fair way to schedule cash payments.

Example 4B. Consider the same sub-network, this time with starting cash positions given by:

$$\boldsymbol{c} = [1, 0, 2, 1, 0.5]$$

The construction of the obligation cascade is a function of only Q and $\boldsymbol{b}$, and thus the cascade structure is exactly that given by Figure **15** and **Table 10**. From the column totals in **Table 10**, we see that after restructuring, Bank 3 must pay \$1.1 and Bank 4 must pay \$0.5 for all debts to be satisfied. In other words, $c_3^{min} = \$1.1$ and $c_4^{min} = \$0.5$.

	$i_5 = 3$	$i_4 = 4$	$i_3 = 1$	$i_2 = 5$	$i_1 = 2$
Total	-1.1	-0.5	0.3	0.4	0.9

Reviewing $\boldsymbol{c}$, we may immediately see that the network is sufficient, with $c_i > c_i^{min}, \forall i$.

Example 4C. Consider our $n = 19$ network from Example 1C (see **Figure 8**). This network included two sub-networks:

1) A swamp in Banks 16-19, with debts to one another but no cash with which to satisfy them.
2) A sub-network of 15 banks with cash (Banks 1-15), many of which can clear their debts, but some of which will ultimately default.

Applying our obligation cascade program to the swamp, we obtain the following obligation cascade:

Table 11: Example 4C – Obligation Cascade Table (Swamp)

	$i_4 = 19$	$i_3 = 18$	$i_2 = 17$	$i_1 = 16$
$i_4 = 19$	0	$\frac{1}{3}$	0	$\frac{5}{3}$
$i_3 = 18$	$-\frac{1}{3}$	0	$\frac{1}{3}$	1
$i_2 = 17$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$
$i_1 = 16$	$-\frac{5}{3}$	-1	$-\frac{1}{3}$	0
Total	-2	-1	0	3

From these results, we note that after simplification (and assuming the network is made sufficient through intervention providing $2 to Bank 19 and $1 to Bank 18), Bank 19 has the largest outflows ($2), while Bank 16 has the largest inflows ($3). With no cash in this sub-network, Bank 18 and Bank 19 will require intervention from an outside source to facilitate clearing.

Recall that the remaining sub-network of 15 banks includes one absorbing status bank, Bank 15. With this bank in the sub-network, the sub-network is not ergodic and does not have an invariant distribution or cascade structure. Although no fair restructuring is possible for the $n = 15$ sub-network as Bank 15 cannot benefit with the other banks, we may calculate $\boldsymbol{c^{min}}$ as in (8).

For the entire $n = 19$ network, we have

$$\boldsymbol{c^{min}} = [-4, 3.4, -0.6, 0, 3, -1.6, 4.5, -1.2, 1.5, 1, 6, -7.5, -1.5, 1, -4, -3, 0, 1, 2]$$

and we can see that the last four values are the additive inverses of the corresponding column totals in **Table 11**.

In the context of interventions, an obligation cascade diagram such as the one presented above may be useful to a regulatory body wishing to intervene in the network. In our swamp from

Example 4A, after arranging multilateral cancellation of debts as above, the obvious optimal location for a cash infusion (i.e., a loan) with the largest benefit is Bank 3, as payments from Bank 3 would also provide cash to each of the other banks to use in satisfaction of their debts. Some portion of the amount loaned to Bank 3 would be paid from Bank 3 to Bank 4 to Bank 1 to Bank 5 to Bank 2. A regulator wishing to drive the largest reduction in outstanding debt may enforce violation of the proportionality assumption in payments by prescribing that payments be made only along this longest path through the obligation cascade diagram. However, as noted in **Section 4.3**, the amount that can be paid along this path is at most the minimum bank-to-bank debt within the path (in our diagram, 0.004 owed by Bank 1 to Bank 5), after which a bank in the chain will no longer have incentive to pay what it receives to the next bank in the chain.

Adjacent to intervention planning, such a diagram may be useful to simply inform a regulator of the true structure of debts in the network. For instance, higher levels of the obligation cascade clearly identify banks with the largest net liabilities, which may present higher risk and call for a higher level of supervision or stricter controls. Conversely, lower levels of the obligation cascade identify banks who are owed by many others, and may be solvent in benign environments but be threatened by financial contagion in stressed environments.

CHAPTER 5: SIMULATIONS

In this chapter, we apply Monte Carlo simulation to explore the relationship between a network's structure (i.e., the size and density of connections) and the cost of interventions to prevent defaults given simulated shocks to the cash positions of the network's banks. Network structures resulting in consistently higher intervention costs given the same shocks can be considered riskier. Relating network structure to minimally sufficient cash positions, we show that these intervention costs are a function of the network's structure Q and debts $\boldsymbol{b}$ (through $\boldsymbol{c^{min}}$) and the initial or shocked cash positions c_i of each bank, specifically how they compare to the corresponding values c_i^{min}. These analyses can help regulators to answer two questions:

1) How can network structures (i.e., the density and magnitude of debts) be regulated to reduce risk as measured through potential intervention cost?
2) What cash reserves should be required to ensure all debts can be paid or intervention costs are held below a certain amount under most shocks?

5.1 Simulations in the Literature

Prior research has shown that the structure of connections within a network has implications for the network's stability when faced with shocks. Cont, Moussa, and Santos (2013) note that under business-as-usual circumstances, a shock affecting a single network member has limited impact, while under larger, more pervasive shocks affecting many members, contagion effects are larger. Cont et al. apply Monte Carlo simulation within a data set of exposures from the Brazilian financial system to derive a Contagion Index, and use this index to describe the risk posed by Brazilian financial institutions in historical time periods and under stress scenarios.

Acemoglu, Ozdaglar, and Tahbaz-Salehi (2013) evaluate the stability and resilience of densely connected and less well-connected financial networks under shocks of different sizes. Acemoglu et al. note that given sufficient interbank liabilities, more densely-connected networks are the most stable and resilient under small shocks (those below a particular value ε^*),[20] with higher stability defined as a lower expected number of defaults and higher resilience defined as a lower maximum number of defaults when shocks of a given size and number are applied to the cash positions of the network's members. Conversely, under large shocks, the most densely connected networks are equally as unstable as the least densely connected networks.

To illustrate this effect, Acemoglu et al. perform a simulation to evaluate the expected number of defaults as a function of the size of a negative shock within a network of 20 banks, but under several different network structures. Specifically, Acemoglu et al. evaluate the following extreme cases in addition to several networks of intermediate connectedness:

- a *ring network* in which each bank has the same total liability amount owed to only one other bank and each bank in the network is owed by exactly one other bank; thus a plot of the network's liabilities resembles a ring
- a *complete network* in which each bank has the same total liability amount that is owed equally among the other banks in the network

[20] ε^* is a measure of the amount of excess liquidity in the network. When the size of the shock (or total size of multiple shocks) exceeds this value, Acemoglu found that the complete network shifts from being the most stable to the least stable network structure.

Figure 16: Ring and Complete Network Structures

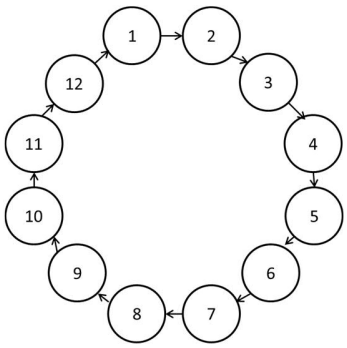
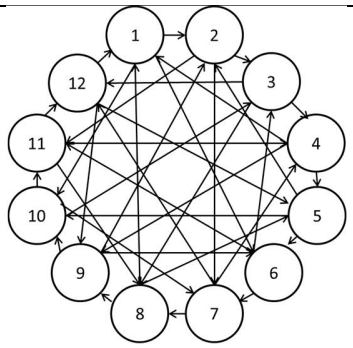
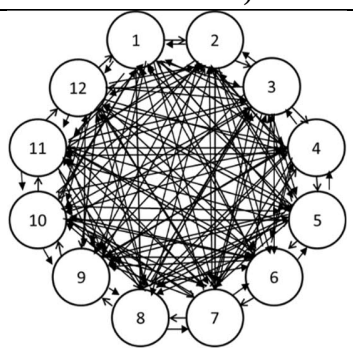
Connecting Acemoglu et al.'s results with our research, we note that a higher expected number of defaults should correspond to a higher cost of preventing those defaults, although we will show through evaluation of our simulation results that intervention cost depends on network structure only indirectly through $\boldsymbol{c^{min}}$.

5.2 Simulations – Network Structure and Intervention Cost

To evaluate the impact of network structure to the cost of preventing defaults, we designed a simulation to apply randomly generated shocks within a particular network ($n = 12$) whose debts are arranged in three different structures of varying connectivity: a ring network structure, a complete network structure, and a structure of intermediate connectedness (three connections for each member). Parameters of the network structures we evaluate are presented in the table below.

Example 5A.

Table 12: Simulation – Network Structure

Parameter	Value	
c	[1, 0.1, 0.3, 0.6, 1.3, 0.3, 0.4, 0.2, 0.6, 0.2, 1.1, 0.4]	
b	[1, 0.7, 0.5, 0.9, 1.5, 0.4, 0.4, 0.6, 0.2, 0.4, 0.75, 0.2]	
Q (ring, intermediate, and complete networks, respectively)	Q_1 (One connection)	
	Q_2 (Three connections)	 Each member i owes member $i+1$, $i+5$, and $i+9$ (mod 12) in equal amounts ($\frac{1}{3}$ of member i's total debt)
	Q_3= (Eleven connections)	 Each member i owes each other member in equal amounts ($\frac{1}{11}$ of member i's total debt)

For simplicity, we evaluate the impact of shocks in a network consisting solely of banks connected by debts. However, as noted in **Section 1.1**, financial networks may also involve

financial interactions and obligations between banks, industries, and consumers. We note that the networks evaluated in this simulation could easily be modified to include non-bank members, either through inclusion of nodes for each individual member (e.g., consumer) connected by obligations to one or more bank, or by inclusion of a single node representing the additional members in aggregate, whose connections represent the aggregate obligations of consumers to or from the other members.

To simulate the effect of first-order shocks directly impacting the liquidity of banks in our network, we applied randomly generated shocks to the initial cash positions of each bank. The magnitude of the shock to each bank was determined through sampling from a normal distribution with mean 0 and a particular standard deviation. To simultaneously force shocks to impact cash positions negatively and randomize which banks were impacted, positive random values were subtracted from the banks' initial cash positions, while negative random values resulted in no change to the initial cash position. In this way, shocks were applied to half the banks on average, with the magnitude Y of a shock (given that a bank experienced a shock) following a half-normal distribution.

$$Y = |X|, \; X \sim N(0, a)$$

To prevent shocks from bringing cash positions to nonsensical negative values, we imposed a floor at 0 for shocked cash positions.[21] Thus, the final shock amount for a bank i experiencing a shock is

[21] Sonin and Sonin (2017, 2020) allow negative cash positions in their model, interpreting them as cash positions reflecting debts external to the network. This approach allows for comparison to values c_i^{min} that may be negative, denoting a net inflow from in-network banks that counteracts the shortfall on external obligations, and allows for situations in which a bank with $c_i^{min} < 0$ may still not have sufficient cash to clear its in-network obligations. However, we note that this approach implies a different priority or maturity for these external debts, and for simplicity, we instead assume all previous or higher-priority external debts have been cleared and any defaults recorded, and any

$$Z_i = \min(Y, c_i).$$

We evaluated shocks of different magnitudes by varying the standard deviation parameter for the normal distribution from which shocked values were generated. For each standard deviation value, our simulation evaluates 5,000 iterations of randomly shocked cash positions within each of the three network structures to determine which banks would default and calculate the cost of intervention to prevent defaults.[22] In this way, we evaluated which network structure offers the most stability in our network, with stability measured by the lowest average intervention cost given the random shocks. **Figure 17** presents the average intervention cost (to prevent all defaults) across 5,000 simulations for each of the three network structures under shocks of increasing severities.

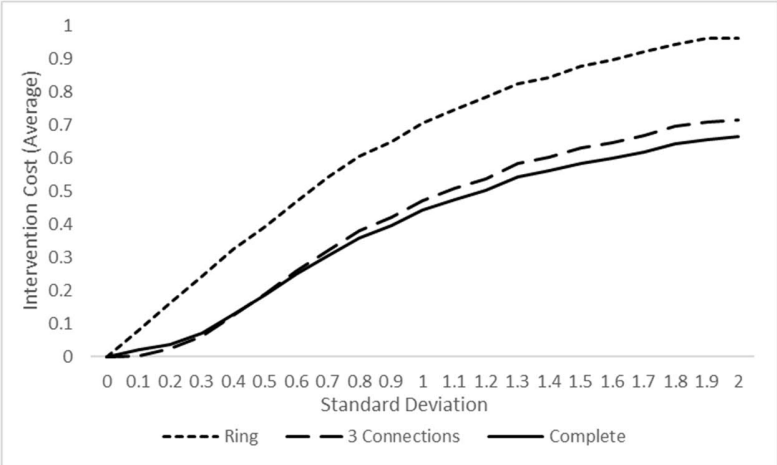


Figure 17: Monte Carlo Simulation – Average Intervention Cost by Shock Size

A half-normal distribution's mean is given by $\frac{\sigma\sqrt{2}}{\sqrt{\pi}}$. Thus, standard deviations from 0 through 2 correspond to mean shocks $E(Y)$ from 0 through 0.7979 for the banks experiencing a shock (half of the 12, on average). We note that the mean realized shock $E(Z)$ for a bank may be a lower value

subsequently due or lower-priority external debts will be addressed in a separate clearing exercise. Alternatively, we may include all external debts in the model with equal priority to in-network debts via connections to a sink node.

[22] This cost can be calculated through comparison of the shocked cash positions to corresponding values c_i^{min}, with positive values $c_i^{min} - c_i$ denoting shortfalls that must be addressed through intervention to allow complete clearing.

as we cap the size of a random shock to bank i at c_i. In **Appendix B** we present a straightforward derivation of the total intervention amount's probability density function from the c_i, the c_i^{min}, and the distribution of shock amounts (if known or estimated), although simulation offers an easy means of producing key statistics (like the mean from **Figure 17**) and quantiles for a given shock distribution and network.

From **Figure 17**, we observe that regardless of shock size, the ring structure always showed the highest average intervention cost, while for shocks above a certain size, the complete network structure showed the lowest average intervention cost. Interestingly, the intermediate network, in which each bank owed and was owed by three other banks, incurred the lowest average intervention cost under the smallest shocks. To further explore these results and develop generalizable conclusions, we refer to some of our earlier findings.

Key Observations and Risk Metrics. In **Section** 4.4, we discussed $\boldsymbol{c^{min}}$, a vector describing the minimum cash positions required by each network member to facilitate complete clearing within the network. Because the random shocks were applied directly to each bank's starting cash position, the cost of intervention in each simulation can be directly observed from whether the shocked cash positions represented a shortfall beneath their corresponding values in $\boldsymbol{c^{min}}$ and the amounts of any such shortfalls. Calculating $c_i^{min} = b_i - \sum q_{j,i} b_j$ for each of the three network structures, we derive the following vector $\boldsymbol{c^{min}}$ for each network structure (positive values of c_i^{min} are in bold):

Structure 1: $\boldsymbol{c^{min}}$ = [**0.8**, -0.3, -0.2, **0.4**, **0.6**, -1.1, 0, **0.2**, -0.4, **0.2**, **0.35**, -0.55]

Structure 2: $\boldsymbol{c^{min}}$ = [**0.43**, -0.2, 0, **0.35**, **0.93**, -0.5, -0.1, **0.05**, -0.37, -0.5, **0.25**, -.35]

Structure 3: $\boldsymbol{c^{min}}$ = [**0.40**, **0.08**, -0.14, **0.30**, **0.95**, -0.25, -0.25, -0.03, -0.47, -0.25, **0.13**, -0.47]

We note that the differing network structures lead to differences in $\boldsymbol{c^{min}}$, which in turn has implications for intervention cost.

Recall that in each network, vector $\boldsymbol{c} = [1, 0.1, 0.3, 0.6, 1.3, 0.3, 0.4, 0.2, 0.6, 0.2, 1.1, 0.4]$. Comparing vector $\boldsymbol{c}$ to each vector $\boldsymbol{c^{min}}$, we note the following:

1) By design, each network structure (prior to shocks) yields a sufficient model, in which each unshocked c_i exceeds the corresponding value c_i^{min} and complete clearing can occur.
2) Banks with $c_i^{min} \leq 0$ can clear without initial cash (i.e., from their receivables alone assuming all are realized, an assumption we may make because we assume intervention will occur to allow all debts to clear), and therefore shocks to these banks' cash positions can drive no increase to intervention cost. Conversely, the values in bold denote banks with $c_i^{min} > 0$, for which a shock will add to intervention cost when its magnitude is large enough to bring $c_i < c_i^{min}$. Under the first network structure, six of the twelve banks have positive values c_i^{min}, indicating that they require cash to clear their debts. Under the second and third network structures, five of the twelve banks have positive values c_i^{min}.
3) We floor shocked cash positions at 0 (that is, no shock to cash will produce a negative cash position). Therefore, in the most extreme case where no cash is available after negative shocks, the largest possible intervention amount is

$$\sum_i \max\left(c_i^{min}, 0\right) \tag{23}$$

the sum of the positive values in $\boldsymbol{c^{min}}$. These values are 2.55, 2.01, and 1.86 for network structures 1, 2, and 3, respectively.

We note that the sizes of these worst-case scenario intervention costs are descriptive of how the average intervention costs rank-ordered across the network structures under the largest shocks in **Figure 17**, showing that $\sum_i \max(c_i^{min}, 0)$ is itself a powerful descriptor of risk (measured by intervention cost) between network structures. Normalizing the average intervention costs as a percentage of $\sum_i \max(c_i^{min}, 0)$ for each network structure, we find that in addition to having the largest $\sum_i \max(c_i^{min}, 0)$, the ring structure also consistently incurred costs at higher percentages of its maximum.

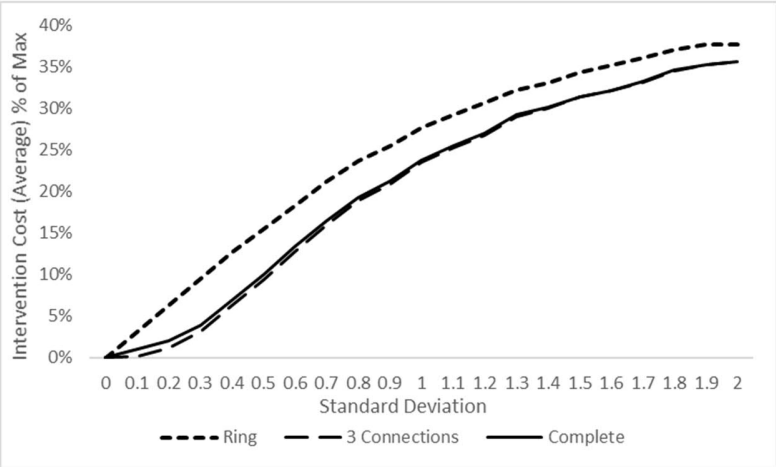


Figure 18: Monte Carlo Simulation – Average Intervention Cost (% of Maximum) by Shock Size

Reviewing other quantiles from our simulations, **Figure 19** presents the maximum intervention cost across the 5,000 simulations for each network structure, which reaches each network structure's $\sum_i \max(c_i^{min}, 0)$ (2.55, 2.01, or 1.86 respectively) under the largest shocks.

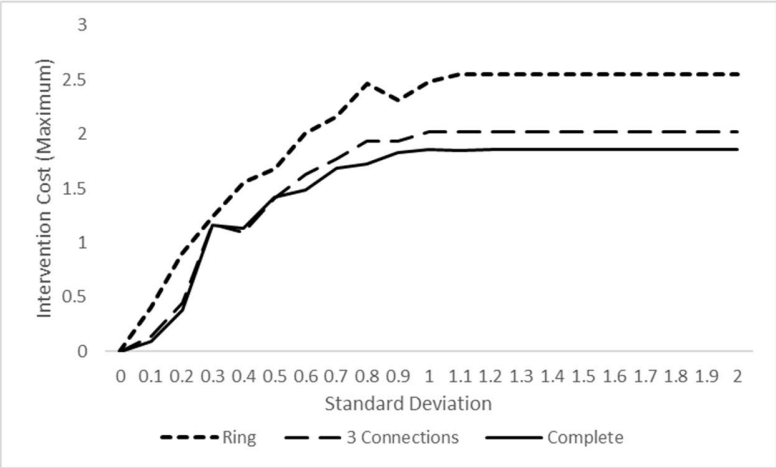


Figure 19: Monte Carlo Simulation – Maximum Intervention Cost by Shock Size

4) Cash positions c_i exceeding the corresponding value c_i^{min} increase the size a shock must be to make cash insufficient and add to the necessary intervention amount. We term any positive $c_i - c_i^{min}$ a *cushion* for bank i, and note that larger cushions decrease intervention costs *ceteris paribus*.

Figure 20 presents the values of c_i and c_i^{min} for each bank i, with any cushion $c_i - c_i^{min}$ indicated by the darker portion of each bar. Given the sufficiency of the unshocked network, we note that $c_i - c_i^{min} \geq 0$ for all i in our network.

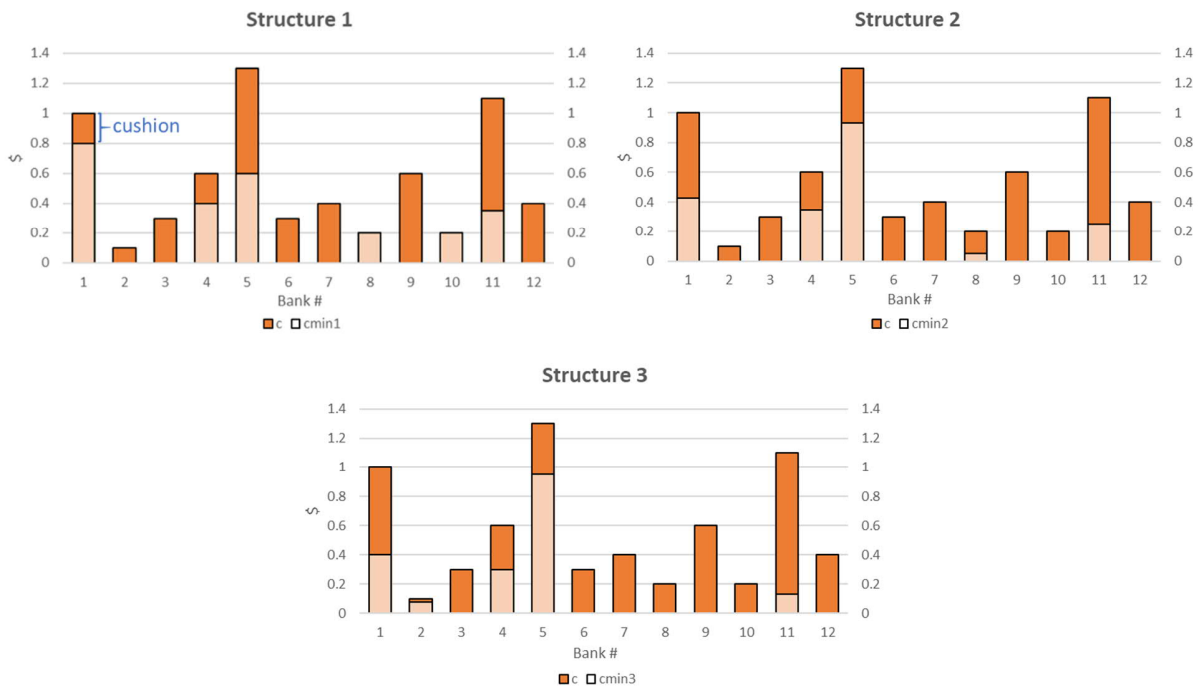


Figure 20: $\boldsymbol{c}$ vs. $\boldsymbol{c^{min}}$ by Network Structure

Given the simulation's design, only half of the banks are expected to experience a negative shock to their cash positions in any simulation. The same proportion holds true for the subset of banks with positive values of c_i^{min}, and therefore, of the 6, 5, and 5 such banks (in network structures 1, 2, and 3, respectively), the expected number that will experience shocks are 3, 2.5, and 2.5. Indeed, as the shock size increases, we see from **Figure 18** that average intervention costs increase asymptotically to values near half of $\sum_i \max\,(c_i^{min}, 0)$ for the three network structures.

Further, we note slight differences in the banks with positive c_i^{min} under network structures 2 and 3, which help explain structure 2's lower intervention cost under small shocks as observed in **Figure 17**. In network structure 2, Bank 8 had positive c_8^{min}, while in network structure 3, Bank 2 had positive c_2^{min}. Both values of c_i^{min} are low, but a higher cash cushion for Bank 8 in network structure 2 explains the lower intervention cost under smaller shocks, as a larger shock would be required before any amount is added to intervention cost.

Although our goal is to evaluate risk measured through potential intervention costs for the broader network, this simulation approach could similarly be used to evaluate the cost of interventions with alternate goals. For instance, in an intervention with the goal of preventing default for a single bank i (potentially letting other banks default), the comparison between shocked c_i and c_i^{min} is similarly in focus, but because the intervention will not ensure that $p_i = b_i$ for every bank i, c_i^{min} for the bank i to rescue will be calculated as $c_i^{min} = b_i - \sum q_{j,i} p_j$, with $\boldsymbol{p}$ calculated assuming bank i will satisfy its debts, and $max\left(c_i^{min} - c_i, 0\right) = max(b_i - \sum q_{j,i} p_j - c_i, 0)$ the shortfall that must be overcome. Through p_j, shocks to other banks that affect the clearing vector and those banks' payments to bank i will affect the required intervention amount $max(c_i^{min} - c_i, 0)$.

Example 5B. To further illustrate the point that network structure matters less than the corresponding vector $\boldsymbol{c^{min}}$, consider two networks with the following debt allocations:

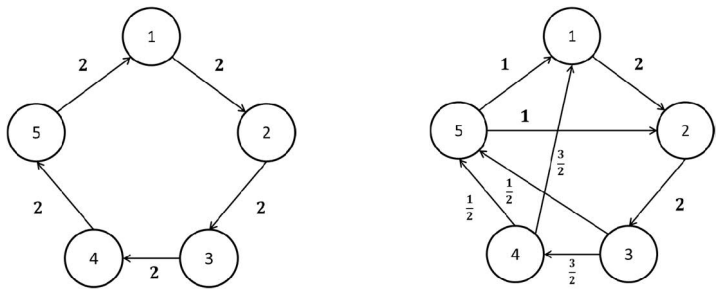


Figure 21: Example 5B – Network Structure vs. $\boldsymbol{c^{min}}$

In both networks, each member owes \$2, although the second network is better-connected. Calculating $\boldsymbol{c^{min}}$ for each, we see that for the first network, $\boldsymbol{c^{min}} = [0,0,0,0,0]$, while for the second network, $\boldsymbol{c^{min}} = [-\frac{1}{2}, -1, 0, \frac{1}{2}, 1]$. The positive values of $\boldsymbol{c^{min}}$ describe the cash amounts needed for complete clearing, a total of 0 in the first network because every member's debts can

be satisfied by their receivables alone (or through restructuring as demonstrated in **Section** 4.5) , but a total of $\frac{3}{2}$ under the second network structure, denoting that at least $1.50 must exist in the network (across specific banks) to allow the network to clear its debts.

We again note that this summation $\sum_i \max\,(c_i^{min}, 0)$ provides one effective measure of risk in the network, with larger values denoting larger amounts of required cash. In addition, we noted above that this summation provides an upper bound for the intervention amount required to facilitate complete clearing. If all members of the network have 0 cash, providing this amount (to the right members) would allow clearing. Despite the higher density of connections in the second network structure, the need for cash to facilitate clearing means that losses have the potential to drive defaults and require intervention, a risk not present in the first network where all debts may be paid from receivables alone (or satisfied through multilateral debt cancellation as described in **Section** 4.5).

In practice, network members will have some cushion above the corresponding value c_i^{min}, but a negative shock driving cash losses would diminish these cushions, potentially leaving cash positions below the corresponding values c_i^{min}. As discussed in the simulation results above, the risk of defaults and the cost of intervention are further diminished by cushions of cash above corresponding values c_i^{min}. In two networks with equal $\sum_i \max\,(c_i^{min}, 0)$, the network with larger cash cushions c_i^{min} will weather negative shocks better and drive lower intervention costs. Thus, the ratio or difference between $\sum_i \max\,(c_i^{min}, 0)$ and total cushion may be even more descriptive of risk in a network than $\sum_i \max\,(c_i^{min}, 0)$ alone.

Example 5C. Consider our $n = 19$ network from Examples 1C and 4B. Initial parameters of this network are:

$$\boldsymbol{c} = [3,8,0,0,9,5,0,0,0,0,12,2,1,3,8,0,0,0,0]$$

$$\boldsymbol{b} = [3,9,6,6,5,10,8,4,3,8,2,2,4,0,3,3,6,6]$$

We may calculate $c_i^{min} = b_i - \sum q_{j,i} b_j$, and comparing $\boldsymbol{c^{min}}$ to $\boldsymbol{c}$, we have the following:

Table 13: Example 5C – **c** vs. $\mathbf{c^{min}}$ Comparison

Bank	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
c_i^{min}	-4	3.4	-0.6	0	3	-1.6	4.5	-1.2	1.5	1	6	-7.5	-1.5	1	-4	-3	0	1	2
c_i	3	8	0	0	9	5	0	0	0	0	12	2	1	3	8	0	0	0	0

From this comparison, we note that important distinctions may be made between the network's banks:

Green cells in **Table 13** denote banks with $c_i^{min} \leq 0$. These banks, assuming intervention to allow all banks to clear, will be able to satisfy their debts from their receivables alone without need for intervention, even if shocks deplete their cash positions.

Yellow cells in **Table 13** denote vulnerable banks with $c_i \geq c_i^{min} > 0$. Given no shocks affecting their cash positions, these banks have sufficient cash to satisfy their debts, although large enough shocks may make these banks' shocked cash positions insufficient, contributing to the cost of intervention or resulting in defaults. Summing c_i^{min} for these banks, we find that in the worst case, shocks completely depleting their cash can add up to \$13.4 to intervention costs.

Red cells in **Table 13** denote *illiquid* banks with $c_i^{min} > c_i = 0$. These banks already have insufficient cash positions before any shocks. We note that in our example, each of these banks already has a cash position of zero, beneath the corresponding value c_i^{min}, and each will thus contribute to intervention cost given no shocks to the network's cash positions. With $c_i = 0$, these

banks cannot be affected further by shocks to their cash position. Summing the shortfall $c_i^{min} - c_i$ for these banks, we obtain a baseline intervention cost of $10 given no shocks.

A fourth case not present in our network would contain banks with $c_i^{min} > c_i > 0$. These banks would already have insufficient cash positions that could become even more insufficient given shocks, contributing additional amounts to intervention costs.

Through simulation, we confirmed that intervention costs for this network range from $10 given no shocks to a maximum of $23.4 under extreme shocks depleting the entire cash reserves of the vulnerable banks.

Potential Applications of Simulation. Addressing our fourth key question from **Section** 1.2, such a simulation approach may be useful to a regulator wishing to impose and evaluate such credit exposure limits or reserve requirements. After the financial crisis, the Dodd-Frank Wall Street Reform and Consumer Protection Act implemented regulation to improve stability and mitigate risk within the U.S financial system (see Smith, 2016). Key provisions of this act implement limits on the amount of credit exposure Systemically Important Financial Institutions (SIFIs), banks of certain asset sizes, may have to a single other bank (affecting network structure) and ensure banks maintain particular amounts of capital as a percentage of their risk-weighted assets (a liquidity/reserving requirement). Projected capital ratios are measured annually during the Comprehensive Capital Analysis and Review (CCAR) and Dodd-Frank Act Stress Test (DFAST), which evaluate whether banks' capital positions are adequate to weather shocks and ensure the banks can continue operations. Using $\boldsymbol{b}$ and Q to derive $\boldsymbol{c^{min}}$, a regulator could simulate a large number of random shocks to $\boldsymbol{c}$ and require in-network banks to maintain capital reserves such that each bank's shocked cash position will exceed the corresponding value c_i^{min} in some adequately

large percentage of simulations. A simulation approach can also yield useful information about what magnitudes of network connections lead to adverse outcomes or excessive intervention costs in an unacceptably large percentage of simulations.

To summarize, while the expected number of defaults may be influenced by network structure and size of negative shocks, the cost of intervention to prevent those defaults is a function of $\boldsymbol{c^{min}}$ (and specifically, any shortfall denoted by shocked cash positions beneath the corresponding value c_i^{min}), which is influenced by network structure but also by the particular magnitudes of debts between the members (consistent with Glasserman and Young, 2016, who note that the number and size of linkages and other factors should be considered when measuring a network's connectivity). Thus, regardless of network structure, review of $\boldsymbol{c}$ and $\boldsymbol{c^{min}}$ provides the most direct information about sufficiency/insufficiency and the risk of insufficiency given shocks.

CHAPTER 6: OPPORTUNITIES FOR FURTHER RESEARCH

Multi-Period Clearing Problems. Our analysis focused on clearing and intervention in networks involving liabilities due at a single point in time, although we note that the BaT model and our intervention framework may also be applied to multi-period clearing problems. As noted by Kusnetsov and Veraart (2018), real-world clearing problems involve payments due at multiple points in time, and real-world bankruptcy law apportions the assets of bankrupt institutions among all creditors, whether the debts are due presently or at an agreed-upon future date. Thus, with a schedule of payments due at multiple points in time (and expected operating income in between), we may apply the BaT model within each payment date, removing institutions going bankrupt during a particular period without being rescued and apportioning their assets among all claimants of current and future debts. Such a framework may also incorporate the addition of expected operating income between periods by incrementing each surviving institution's cash reserves before repeating the exercise for the next payment date.

In such a multi-period framework, it becomes the lender of last resort's role to evaluate whether intervening on behalf of a defaulting institution by providing a timely loan will protect the stability of the network *over time*, and whether the recipient's schedule of expected cash flows (both operating income and payments received) make likely the satisfactory repayment of such a loan.

Networks Involving Debts of Differing Seniorities. Our research has adhered to Eisenberg and Noe's original assumption of proportionality, which requires that when a bank cannot satisfy the entirety of its debts, it pays out the entirety of its value paying an equal proportion of each of them. This is accomplished by paying each creditor a portion of the total assets equal to the

creditor's proportion of the bank's total debt. In real-world applications however, debts often have differing levels of seniority, meaning that certain debts must be paid in full before others are paid even in part. For an example, one need only look as far as preferred stock (i.e., debt that has been securitized) to see that differing series have differing seniorities (e.g., in the event of a default, owners of Series A preferred stock are paid before owners of Series B).

Suzuki (2002), Elsinger (2009), and Fischer (2014) review clearing in networks where differing liability classes (e.g., debt and equity) have differing seniorities or where liabilities within the same class may have differing seniorities. In such a network, differing seniorities would impact the cost of an intervention. To prevent a default, all remaining debts must be paid, regardless of their seniority. However, differing levels of seniority may indirectly affect the cost of an intervention by altering how contagion affects other banks during organic clearing, and therefore affecting which banks default and the magnitude of their remaining debts—in the event of a default, claimants of senior debts may receive payments while claimants of more junior debts receive no payments, more deeply affecting their ability to pay their own debts. Also, differing levels of seniority would similarly change the post-organic clearing payment dynamics governing how invested funds would be paid to other banks. Banks that are owed more junior debts would benefit less from interventions because any provided funds would first be paid towards more senior debts.

Simplification of Complex Financial Networks for Anti-Money Laundering (AML). A well-known tactic of money launderers is layering, in which a money launderer attempts to add distance between funds and their illicit source and intentionally complicate the audit trail by initiating a

complex series of legitimate-looking transactions.[23] Often, a goal is to also obscure the final recipient of the funds. Given that a set of completed transactions represents a sufficient model in which all debts are payable (because the transactions under review have already occurred), the obligation cascade model described in **Section** 4.5 could be used to simplify the complex series of transactions and reveal the true cash movement and beneficiaries.

Clearing Houses. We also note the potential for applicability within clearing houses, which also may have comprehensive information about liabilities among their members. Peery (2012) notes that separately-negotiated over-the-counter (OTC) derivatives that were poorly-regulated and unstandardized were a driver of the financial crisis, and central clearing of derivatives contracts was a key mandate of the Dodd-Frank Wall Street Reform and Consumer Protection Act of 2010. Amini, Filipovic, and Minca (2015) note that clearing houses act as the counterparty to both sides of a transaction (i.e., as the recipient of funds owed and as the payer of funds due), and therefore assume the risk of losses from defaults. Clearing houses generally insure themselves against such losses through collection of a margin from each member that will be applied to any shortfall in that member's payments and a pooled guarantee fund (also consisting of member contributions) that the clearing house can draw upon if needed after a defaulting member's margin. Foresight about the likelihood and magnitude of potential defaults can help inform the necessary size of such margins and guarantee funds.

[23] See Federal Financial Institutions Examination Council (2014)

CHAPTER 7: CONCLUSION

We have explored several methodologies that may be useful to a well-informed regulatory body seeking to intervene in a struggling financial network to limit the adverse effects of a bankruptcy within the network or a broader shock affecting many members. In particular, we have explored the network dynamics—following from the framework of the BaT model proposed by Sonin and Sonin (2017, 2020)—affecting the benefit of an intervention to any given bank or to the network at large. While still requiring manual interpretation of results and decisions about intervention locations, a process using multipliers $m_{i,k}$, m_i^{total}, or m_i^A is useful when a regulator's goal is to prevent default of particular banks, or to optimally deploy a fixed amount of cash not sufficient to allow for clearing of all debts.

By contrast, problems in which the regulator's goal is to prevent all defaults or defaults of a particular group of banks can be more simply solved through a modification of the BaT model that allows the rescued banks' cash positions to become negative. Negative cash amounts from the final cash vector denote the additional amounts the corresponding banks would require in order to facilitate complete clearing of the network's debts. Similarly, calculation of the minimum sufficient cash positions in the vector $\boldsymbol{c^{min}}$ and comparison to corresponding pre-clearing cash positions also allows for identification of the shortfall amounts that must be addressed to facilitate complete clearing, and facilitates assessments of risk within the pre-clearing network.

We have explored methods for modeling "payments" (i.e., multilateral cancellation of debts) in swamps or other ergodic networks, resulting in a simplified "obligation cascade" diagram describing the simplified debts. This diagram can also be used in intervention planning, as its hierarchical structure provides useful details of which banks are most upstream in the hierarchy of

payments; these banks may be larger sources of financial contagion within the network, and also prime candidates for an intervention with maximal benefit.

Lastly, we have explored the factors within networks affecting intervention cost and identified key metrics describing risk through application of simulated shocks to the network's vector $\boldsymbol{c}$ of pre-clearing cash positions. Such simulation can inform policy by identifying factors that contribute to the risk that large interventions might be needed. These factors can then be monitored by a regulator or mitigated through regulation.

Ideally, this research provides a framework that may be used in further research of dynamic clearing and intervention planning.

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APPENDIX A: R PROGRAMS AND CODE

Note: All R code was written and executed using R version 3.2.3.

Table 14: R Programs and Descriptions

Program	Function
BaT Clearing Code.R	Implements the Banks as Tanks model introduced by Sonin and Sonin (2017, 2020) to calculate a clearing vector and final statuses, cash positions, and total debt positions, given initial cash positions, debt positions, and debt allocation matrix.
Benefit Multipliers.R	Uses initial debt allocation matrix and network parameters after clearing to calculate a matrix M of benefit multipliers, with $m_{i,j}$ describing the benefit to bank j relative to a $\$w$ investment in bank i.
BaT Negative Cash Clearing.R	Implements a modified Banks as Tanks model that allows cash positions to become negative and defines only positive and absorbing statuses. Negative values in the final cash positions denote the minimal additional cash required by the corresponding banks to facilitate complete clearing.
Obligation Cascade.R	For a swamp or other ergodic sub-network, takes inputs describing the total debt positions and debt allocation to produce a table describing the obligation cascade, a simplified view of the debt structure after multilateral netting that identifies the bank or banks from the sub-network that will have no remaining debt after cancellation.
Monte Carlo Simulation – Intervention Cost.R	Executes a Monte Carlo simulation to compare the cost of an intervention to allow complete clearing in multiple network structures under many iterations of simulated shocks to banks' cash positions.

These R programs are available upon request.

APPENDIX B: EXPECTED INTERVENTION COST

In our Monte Carlo simulation, the expected intervention amount $E(Int)$ equals $\sum_i[P(shock > 0) \times E_i(shortfall|shock > 0)]$. In our simulation with 12 banks and sampling from a normal distribution, $P(shock > 0) = \frac{1}{2}$.Thus, for each bank i, $E_i(shortfall|shock > 0)$ for each bank i is the quantity of interest to calculate.

For banks i with $c_i^{min} \leq 0$ that need no cash to clear, $E_i(shortfall|shock) = 0$ because we do not allow shocks to bring cash positions below 0.

For banks i with $c_i^{min} > 0$, $E_i(shortfall|shock > 0)$ depends on the distribution of shocks and the corresponding expected value, but also on $(c_i - c_i^{min})$, the amount of bank i's cash reserves beyond c_i^{min}. A shock to a bank with $c_i = c_i^{min} + 1$ must exceed \$1 before any intervention cost is added, and because we don't allow shocks to make a cash position negative, the intervention cost is capped at c_i^{min}. For these banks with $c_i^{min} > 0$,

$$(shortfall|shock > 0) = \begin{cases} (shock|shock > 0) - \left(c_i - c_i^{min}\right), shock \leq c_i \\ c_i^{min}, shock > c_i \end{cases}$$

When the distribution of shocks is known, the distribution of the shortfall (a shock minus a bank's reserves beyond c_i^{min}) is easily derived by shifting the distribution's mean. The probability density function can then be modified by attributing the probability of any value beyond a shortfall value of c_i^{min} instead to the value c_i^{min}, and the expected value of $(shortfall|shock > 0)$ can be calculated for bank i from this density function.

APPENDIX C: DETAILED RESULTS

Table 15: BaT Example 1C Parameters by Interval

Interval	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{j}$	length t_k
1	[3,9,6,6,6,5,10,8,4,3,8,2,2,4,0]	[3,8, ε, ε, 9,5, ε, ε, ε, ε, 12,2,1,3,8]	[+, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +]	$tiny$
2	[3,9,6,6,6,5,10,8,4,3,8,2,2,4,0]	[3,8, ε, ε, 9,5, ε, ε, 0, ε, 12,2,1,3,8]	[+, +, +, +, +, +, +, +, +,0, +, +, +, +, +,*]	$tiny$
3	[3,9,6,6,6,5,10,8,4,3,8,2,2,4,0]	[3,8, ε, ε, 9,5, ε, ε, 0,0,12,2,1,3,8]	[+, +, +, +, +, +, +, +, +,0,0, +, +, +, +,*]	2
4	[1,7,4,4,4,3,8,6,3.4,2.7,6,0,0,2,0]	[3,8.2,0.2, ε, 7.9,4.7, ε, ε, 0,0,11.5,2.5,0.5,2.5,10]	[+, +, +, +, +, +, +, +, +,0,0, +,*,*, +,*]	1
5	[0,6,3,3,3,2,7,5,3.1,2.6,5,0,0,1,0]	[3,8.3,0.3, ε, 7.4,4.6, ε, ε, 0,0,10.8,3.5,1,1.8,10.5]	[*, +, +, +, +, +, +, +, +,0,0, +,*,*, +,*]	$tiny$
6	[0,6,3,3,3,2,7,5,3.1,2.6,5,0,0,1,0]	[3,8.3,0.3,0,7.4,4.6, ε, ε, 0,0,10.8,3.5,1,1.8,10.5]	[*, +, +,0, +, +, +, +,0,0, +,*,*, +,*]	0.87
7	[0,5.1,2.1,2.4,2.1,1.1,6.1,4.1,2.8,2.4,4.1,0,0,0.1,0]	[3.8,8,0,0,6.9,4.4, ε, ε, 0,0,10.1,4.4,1.4,1.1,10.9]	[*, +,0,0, +, +, +, +,0,0, +,*,*, +,*]	0.13
8	[0,5,2,2.4,2,1,6,4,2.8,2.4,4,0,0,0,0]	[3.9,7.9,0,0,6.8,4.4, ε, ε, 0,0,10,4.5,1.5,1,11]	[*, +,0,0, +, +, +, +,0,0, +,*,*,*,*]	1
9	[0,4,1.4,1.8,1,0,5,3,2.5,2.3,3,0,0,0,0]	[4.6,7.4,0,0,6.3,4.3, ε, ε, 0,0,9,5.3,1.8,1.3,11.3]	[*, +,0,0, +,*, +, +,0,0, +,*,*,*,*]	$tiny$
10	[0,4,1.4,1.8,1,0,5,3,2.5,2.3,3,0,0,0,0]	[4.6,7.4,0,0,6.3,4.3,0, ε, 0,0,9,5.3,1.8,1.3,11.3]	[*, +,0,0, +,* ,0, +,0,0, +,*,*,*,*]	$tiny$
11	[0,4,1.4,1.8,1,0,5,3,2.5,2.3,3,0,0,0,0]	[4.6,7.4,0,0,6.3,4.3,0,0,0,0,9,5.3,1.8,1.3,11.3]	[*, +,0,0, +,* ,0,0,0,0, +,*,*,*,*]	1
12	[0,3,0.8,1.3,0,0,4.5,2.4,2.4,2.2,2,0,0,0,0]	[5.3,6.9,0,0,5.4,4.8,0,0,0,0,8,5.6,2,1.5,11.5]	[*, +,0,0,*,* ,0,0,0,0, +,*,*,*,*]	1.65
13	[0,1.4,0,0.5,0,0,4.5,2.4,2.4,2.2,0.4,0,0,0,0]	[6.4,5.8,0,0,5.4,4.8,0,0,0,0,6.4,6,2.4,1.9,11.9]	[*, +,* ,0,*,* ,0,0,0,0, +,*,*,*,*]	0.7
14	[0,0.7,0,0.2,0,0,4.5,2.4,2.4,2.2,0,0,0,0,0]	[6.7,5.2,0.3,0,5.4,4.8,0,0,0,0,6,6.1,2.5,2,12]	[*, +,* ,0,*,* ,0,0,0,0,*,*,*,*,*]	0.65
FINAL	[0,0,0,0,0,0,4.5,2.4,2.4,2.2,0,0,0,0,0]	[7,4.6,0.6,0,5.4,4.8,0,0,0,0,6,6.1,2.5,2,12]	[*,*,*,*,*,* ,0,0,0,0,*,*,*,*,*]	-