THE GENERAL NON-STATIONARY ANDERSON PARABOLIC MODEL WITH CORRELATED WHITE NOISE

by

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ABSTRACT

XIAOYUN CHEN. The general non-stationary Anderson Parabolic Model with correlated white noise. (Under the direction of DR. STANISLAV MOLCHANOV)

This dissertation contains the analysis of the general lattice non-stationary Anderson parabolic model with correlated white noise. It starts form the brief description of known results about parabolic problem with local Laplacian and the detailed description of the general non-local Anderson model in the non-stationary random environment (Chapter 2). Chapter 3 is devoted to existence-uniqueness theorems for the parabolic model in the weighted Hilbert space, Feynman-Kac formula representation and moment equations. The chapter 4 contains the results on the first and second moments of the solution and the spectral properties of the Hamiltonian $\mathcal{H}_2$, providing the basic information on the phase transition of the model from the regular to intermittent structure, additional results concerns the other spectral bifurcations of $\mathcal{H}_2$.

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List of Symbols

Z^d	$d-$dimensional Lattice
$L^2(Z^d)$	Hilbert space
$L^2_\mu(Z^d)$	weighted Hilbert space
$\mathfrak{B}$	Banach space
t	time
$x(t)$	random walk at time t
$a(z)$	jumping rate of random walk $x(t)$ at z
$\hat{a}(k)$	$\hat{a}(k) = \sum_{z\neq 0} a(z)e^{i(k,z)}$
$\varkappa$	diffusivity
Δ	Lattice Laplacian $\Delta = \sum_{x':\|x'-x\|=1} f(x') - f(x)$
$\mathcal{L}$	non-local operator $\mathcal{L} = \sum_{z\in Z^d)}[f((x+z) - f(x)]a(z)$
$b(t,x)$	standard Brownian motion
$W(t,x)$	Winner process
$\xi_t(x)$	white noise $\xi_t(x) = \frac{d}{dt}W(t,x)$
$B(x)$	correlation function of Winner process $W(t,x)$
$\hat{B}(k)$	Fourier transform of $B(x)$
$u^{(S)}(t,x)$	solution by Stratonovich integral
$u^{(I)}(t,x)$	solution by Ito integral
$\langle\rangle$	the averaging over random potential
$m_p(t,x_1,\cdots,x_p)$	pth moment $m_p(t,x_1,\cdots,x_p) = \langle u(t,x_1)\cdots u(t,x_p)\rangle$
$p(t,x,y)$	transition probability of random walk $x(t)$
T^d	torus $T^d = [-\pi,\pi]^d$
$S(k)$	$S(k) = \varkappa \sum_{z\neq 0}(1 - cos(k,z))a(z)$
$G_\lambda(x,y)$	Green function: $G_\mu(x,y) = \int_0^\infty e^{-\lambda t}p(t,x,y)dt$
$\mu(x)$	appropriate weight kernel
$\mathcal{H}$	Schrödinger operator
$Sp(\mathcal{H})$	spectrum of operator $\mathcal{H}$
$Sp_{ess}(\mathcal{H})$	essential spectrum of operator $\mathcal{H}$
$Sp_d(\mathcal{H})$	discrete spectrum of operator $\mathcal{H}$

List of Abbreviation

$l.i.m$	limit in mean
CLT	Central Limit Theory
$P-a.c$	absolute continuous in probability
SPDEs	Stochastic partial differential equations

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CHAPTER 1: Introduction

The goal of this work is the generalization of the paper [1]. We'll give the brief review of [1] and the lecture 9 from [2]. Let's consider the following parabolic equation

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \varkappa(\Delta u)(t,x) + \xi_t(x)u(t,x) \\ u(0,x) &= 1 \end{aligned} \tag{1.1}$$

with continuous time $t \geq 0$ and $x \in Z^d,\ d \geq 1$. Operator $\triangle$ is the Lattice Laplacian

$$(\Delta u)(x) = \sum_{x':|x'-x|=1} (\psi(x') - \psi(x)),$$

$\varkappa$ is the diffusion coefficient and time dependent potential $\xi_t(x)$, $(t,x) \in [0,\infty) \times Z^d$ is given formally by the relation

$$\xi_t(x) = \frac{d}{dt} W(t,x) \tag{1.2}$$

where $W(t,x),\ x \in Z^d$ is the family of independent standard $1D$ Wiener process. In the different terms $W(t,x)$ is the Gaussian field on $[0,\infty) \times Z^d$ with the correlation function

$$\langle W(t_1,x_1)W(t_2,x_2)\rangle = min(t_1,t_2)\delta_0(x_1 - x_2). \tag{1.3}$$

Wiener process is not differentiable. However, the expression of Equation (1.2) has the sense as the generalized function, so it means that we'll use $\xi_t(x)$ in the integrals.

For instance,

$$\int_{t_1}^{t_2} \xi_t(x)dt = W(t_2,x) - W(t_1,x)$$

Or in more general setting, for $f(s) \in C_0^\infty$ and $supp(f) \in (t_1,t_2)$ for fixed $x \in Z^d$

$$\int_{t_1}^{t_2} f(s)\xi_s(x)ds = -\int_{t_1}^{t_2} f'(s)W(s,x)ds$$

In the spirit of the general theory of SPDE's (Stochastic equations with partial derivative) we can understand $W(t,\cdot)$ as the Wiener process in the appropriate Hilbert space. Of course, it can't be the standard $L_2(Z^d)$, because

$$\{W(t,\cdot)\} \notin L^2(Z^d), \mathbf{1}(x) \notin L^2(Z^d).$$

The easiest possible way is to work in the weighted Hilbert space $L^2_\alpha(Z^d)$ with the norm

$$\| f(\cdot) \|_\alpha^2 = \sum_{x\in Z^d} |f(x)|\alpha^{|x|}$$

$$|x| = |x_1| + \cdots + |x_d|, \; 0 < \alpha < 1$$

$\alpha^{|x|}$ is the special weight with the following property, to make everything convenient

$$\alpha \| f(\cdot) \|_\alpha^2 \le \| f(\cdot + h) \|_\alpha^2 \le \frac{1}{\alpha} \| f(\cdot) \|_\alpha^2$$

where $f(\cdot + h)$ is a shift of $f(\cdot)$, $|h| = 1$.

It is easy to check $\{W(t,x), x \in Z^d\} = W(t)$ is a Wiener process in $L^2_\alpha(Z^d)$ with the covariance as Equation (1.3). Also by the definition(see equation (1.2)) $\xi_t(x)$ is a "white noise" in $L^2_\alpha(Z^d)$. Now ur probability space $(\Omega_m, \mathcal{F}_m, \mu)$ will have a special structure: $\Omega_m = L^2_\alpha(Z^d)$, filtration $\mathcal{F}_{\le t} \in \mathcal{F}_m$, μ is the distribution of $W(\cdot,\cdot)$.

Thus Equation (1.1) should be thought as stochastic integral equation in $L^2_\alpha(Z^d)$.

In Ito's form

$$u(t,x) = 1 + \varkappa \int_0^t \Delta u(s,x)ds + \int_0^t u(s,x)dw(s,x) \tag{1.4}$$

where $u(t,\cdot) \in L^2_\alpha(Z^d)$ and $\mathcal{F}_{\leq t}$ adapted. Last term of equation (1.4) is the Ito's stochastic integral. Alternatively Stratonovich's integral can be used

$$\int_0^t u(s,x) \circ dW(s,x). \tag{1.5}$$

If both integrals make sense, there exists a trivial relation between them:

$$\begin{aligned} \int_0^t u(s,x) \circ dw(s,x) &= \int_0^t u(s,x)dw(s,x) + \frac{1}{2}[u(\cdot), w(\cdot)]_0^t \\ &= \int_0^t u(s,x)dw(s,x) + \frac{1}{2}\int_0^t u(s,x)ds \end{aligned} \tag{1.6}$$

that's

$$u(t,x) \circ dw(t,x) = u(t,x)dw(t,x) + \frac{1}{2}u(t,x)dt \tag{1.7}$$

Equation (1.7) allow to switch from Ito's SPDE to Stratonovich's SPDE by

$$u^{(S)}(t,x) = u^{(I)}(t,x)e^{t/2}. \tag{1.8}$$

Although Ito's integral $u^{(I)}(t,x)$ is commonly used in applied mathematics, Stratonovich's form $u^{(S)}(t,x)$ is frequently used in physics, due to the fact that it is not only reflect-symmetric in the time but also is a limit of solution with "very short" time correlations. The following result can be found in [1].

Let's consider equation

$$\begin{aligned} &\frac{\partial u^\varepsilon}{\partial t} = \varkappa \Delta u^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\xi(\frac{t}{\varepsilon}, x)u^\varepsilon \\ &u^\varepsilon(t,x) \equiv 1 \end{aligned} \tag{1.9}$$

Here $\xi(t,x)$ is a regular Gaussian field with continuous realization in time and the following first two moments:

$$\langle \xi(t,x) \rangle = 0, \ \langle \xi(t_1,x_1)\xi(t_2,x_2) \rangle = \Gamma(t_1-t_2)\delta(x_1-x_2)$$

Here $\langle \cdot \rangle$ is the expectation with respect to $\xi(\cdot,\cdot)$. After rescaling $t \to \frac{t}{\varepsilon}$, $\xi \to \frac{1}{\sqrt{\varepsilon}}\xi$, the field $\frac{1}{\sqrt{\varepsilon}}\xi(\frac{t}{\varepsilon},x)$ converges (in the sense of Schavartz space) to the white noise with varaince $\sigma^2 = 2\pi\hat{\Gamma}(0) = \int_{R_1}\Gamma(\tau)d\tau$.

Solution $u^\varepsilon(t,x)$ for every $\varepsilon > 0$ exists in the classical sense. If $\varepsilon > 0$, then $u^\varepsilon(t,x) \xrightarrow{Dist} u^0(t,x)$ and $u^0(t,x)$ is the solution of the Stratonovich's SPDE:

$$\begin{aligned}\frac{\partial u^0}{\partial t} &= \varkappa\Delta u^0 + \sigma u(t,x)\circ dw(t,x)\\ u^0(0,x) &\equiv 1\end{aligned}$$

The proof of the following existence uniqueness theorem is simple due to boundness of the operator $\Delta \in L^2_\alpha(Z^d)$.

Theorem 1.0.1. *Equation (1.4) has an unique solution in weighted Hilbert space* $L^2_\alpha(Z^d)$.

One can find detailed proof of this theorem in [1] and [2], but in the future we'll give the proof of more general result.

Now we'll find representation of the solution of equation (1.4) in the Feynman?Kac form. Due to the classical theory solution of the parabolic problem with time dependent potential

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \frac{1}{2}\Delta u(t,x) + V(t,x)u(t,x), \ t \geq 0, \ x \in R^d\\ u(0,x) &\equiv 1\end{aligned} \tag{1.10}$$

can be found as the expectation of the exponential functional of the Brownian motion

$b(t),\ t \geq 0$ associated to Laplacian $\frac{1}{2}\Delta$

$$u(t,x) = E_x e^{\int_0^t V(t-s,b(s))ds}$$

In the Lattice space case, instead of Brownian motion we must use the random walk $x(t)$ on Z^d with continuous time and the generator $\varkappa\Delta$. This random walk has the rate of the jumps $2d\varkappa$. It spends the random time τ_x in each site $x \in Z^d$ with exponential distribution

$$P\{\tau_x > s\} = e^{-2d\varkappa s},\ s \geq 0.$$

At the moment $\tau_x + 0$ it jumps from x to one of the nearest neighbors $x',\ |x'-x| = 1$ with the probability $\frac{1}{2d}$. Transition probability $p(t,x,y) = p(t,0,y-x) = P\{x(t) = y|x(0) = x\}$ is the solution of the problem

$$\frac{\partial p}{\partial t} = \varkappa\Delta_x p,\ p(0,x,y) = \delta_y(x)$$

Using Fourier transform we can present $p(t,x,y)$ in the form (see any text book or [1], [2])

$$p(t,x,0) = p(t,0,x) = p(t,0,-x) = \frac{1}{(2\pi)^d}\int_{T^d} e^{tS(k)+i(k,x)}dk,$$

$T^d = [-\pi,\pi]^d$ is the d-dimensional torus and

$$S(k) = 2\varkappa(\sum_{j=1}^{d}[cos(k_j) - 1])$$

is Fourier symbol of the operator $\varkappa\Delta$.

The analysis of the transition probabilities $p(t,x,0) = p(t,0,x) = p(t,0,-x)$, i.e., CLT, large deviation etc see [1] and [2]. In particular $p(t,0,0) \sim \frac{C}{t^{d/2}},\ t \to \infty$ and this fact means that random walk $x(t)$ associated to $\varkappa\Delta$ is recurrent if $d = 1,2$ and transient for $d \geq 3$, i.e., $\int_0^\infty p(t,x,x,)dt = \infty\ if\ d = 1,2$; $\int_0^\infty p(t,x,x,)dt < \infty\ if\ d \geq$

3. We'll study the transition probabilities of the more general random walk in the Chapter I. Finally we can formulate the following theorem

Theorem 1.0.2. *Solution $u^{(S)}(t,x)$ is given by the Feynman?Kac formula*

$$u^{(S)}(t,x) = E_x \left[exp\{ \int_0^t dw(s, x_{t-s}\} \right]. \tag{1.11}$$

$$u^{(I)}(t,x) = u^{(S)}(t,x)e^{-t/2} \tag{1.12}$$

Here x_{t-s} is a trajectory of the random walk x_s (in inverse time) and Stochastic integral in the exponent has a trivial sense: if $0 < s_1 < s_2 < \cdots < s_v < t$ are the moments of the jumps for x_s and $x_s \equiv x_v$, $s \in [0, s_1),\ x_s = x_{v-1},\ s \in [s_1, s_2), \cdots x_s \equiv x,\ s \in [s_v, t]$, then

$$\int_0^t dw(s, x_{t-s}) = (w_{s_1} - w_{s_0})(x_v) + (w_{s_2} - w_{s_1})(x_{v-1}) + \cdots + (w_{s_t} - w_{s_s v})(x) \tag{1.13}$$

The proof of this theorem is built upon the construction of Markov process: $u(t + \triangle t, x) = E_x[exp\{\int_t^{t+\triangle t} dw(s, x_{t+\triangle t-s})\}u(t, x_\triangle)]$.

Equation (1.11) implies that the solution $u(t,x,\omega)$, being ergodic and homogeneous on Z^d for given t, has the all the moments as :

$$\begin{aligned} \langle [u^{(S)}(t,x)]^p \rangle &= \left\langle \left[E_x exp\{ \int_0^t dw(s, X_{t-s})\} \right]^p \right\rangle \\ &\le E_x exp\{p \int_0^t dw(s, X_{t-s})\} \\ &= E_x exp\left\{ \frac{p^2 t}{2} \right\} = exp\left\{ \frac{p^2 t}{2} \right\} \end{aligned} \tag{1.14}$$

In special case $k = 0$, when $x_s \equiv x_0 = x$, the estimation (1.14) is precise, because the ordinary SDE $du_1 = u_1 \circ dw_t$ has solution $u_t = exp(w_t)$. In this case,

$$\langle [u_t]^p \rangle = exp\left\{ \frac{p^2 t}{2} \right\}$$

Solution $u^{(I)}(t,x)$ or $u^{(S)}(t,x)$ are homogeneous in random field and their properties can be expressed in term of the statistical moment, say ,

$$m_p^{(I)}(t,\vec{x}) = m_p^{(I)}(t,x_1,\cdots,x_p) = \left\langle u^{(I)}(t,x_1)\cdots u^{(I)}(t,x_p)\right\rangle .$$

Notation "$\langle\rangle$" means the averaging over the random potential, i.e., white noise $\xi_t(x),\ x\in Z^d$. We call such non-random function of $(t,\vec{x})$ the annealed moments. Ito and Stratonovich's annealed moments are closely related:

$$m_p^{(I)}(t,\vec{x}) = e^{-\frac{pt}{2}} m_p^{(S)}(t,\vec{x})$$

and we can study over $m_p^{(I)}(t,\vec{x})$ and exclude upper index (I, Ito form).

Ito form is convenient because one can use powerful Ito formula for the Ito stochastic integrals. Using this formula one can prove the following theorem(see [1]).

Theorem 1.0.3. *For each integer $p\geq 1$, each $t\geq 0$ and $x=(x_1,\cdots,x_p)\in Z^{pd}$ let us set*

$$m_p^{(I)}(t,\vec{x}) = m_p^{(I)}(t,x_1,\cdots,x_p) = \left\langle u^{(I)}(t,x_1)\cdots u^{(I)}(t,x_p)\right\rangle .$$

Then these moments(Correlation functions) satisfy the following "p-particle" parabolic equation

$$\frac{\partial m_p}{\partial t} = \varkappa(\Delta_{x_1}+\cdots+\Delta_{x_p})m_p + \left(\sum_{i<j}\delta(x_i-x_j)\right)m_p \qquad (1.15)$$

$$m_p(0,x)\equiv 1.$$

Equation (1.15) can be also written as

$$\begin{aligned} &\frac{\partial m_p}{\partial t} = H_p m_p, \ \ H_p = \varkappa(\Delta_{x_1} + \cdots + \Delta_{x_p}) + V_p(x) \\ &V_p(x) = \sum_{i<j} \delta(x_i - x_j), \ \ p > 1; \ \ V_1(x) \equiv 0 \end{aligned} \tag{1.16}$$

and Hamiltonian H_p is a classical "p-partial " Schrödinger operator on the lattice Z^{pd} with the binary intersection $\delta(x-y)$.

Equation (1.16) can be studied in weighted Hilbert space $L^2_\alpha(Z^d)$, but now it is more suitable for us to work in Hilbert space $L^2(Z^d)$ where operator H_p is self-joint and has "nice" spectral properties. The spectral analysis of the "p-partial " Schrödinger operator in $L^2_\alpha(Z^d)$ or $L^2_\alpha(R^d)$ plays a critical role in modern mathematical physics.

In the future Ito's formula will be used mostly and $u(t,x)$, $m_p(t,x)$ means $u^{(I)}(t,x)$, $m_p^{(I)}(t,x)$.Proof of theorem (1.0.3) see [1] or [2].

The following result reduced the problem of the moments Lyapunov exponents to the spectral theory.

Theorem 1.0.4. *For any $p \geq 1$and the scale $A(t) \equiv t$ there exists*

$$\gamma_p^{(I)}(\varkappa) = \lim_{t\to\infty} \frac{ln\langle [u^{(I)}(t,x)]^p \rangle}{t} \tag{1.17}$$

and

$$\gamma_p^{(I)}(\varkappa) = max\{\lambda : \lambda \in Sp(H_p)\} \tag{1.18}$$

Where $Sp(H_p)$ is a spectrum of the self-adjoint operator H_p. Remark $Sp(H_p)$ is the same for $L^2_\alpha(Z^d)$ and $L^2(Z^d)$.

Let's remark that $\gamma_p^{(I)}(\varkappa) \geq 0$ because $max\{\lambda : \lambda \in Sp(\varkappa\Delta)\} = max\{\lambda : \lambda \in [-2d\varkappa, 0]\} = 0$ and $V_p(x) \geq 0$.

Theorem (1.0.4) is a version of the Perron's Theorem about the asymtotics of the positive semigroups, but in non-compact case. Abstract form and details of of Perron's Theorem see [3], more details see [4].

Corollary 1.0.5. *For every* $p \geq 1$

$$\gamma_p^{(I)}(\varkappa) = max\{\lambda : \lambda \in Sp(H_p)\} = \sup_{\psi:\|\psi\|=1} (H_p\psi, \psi)$$
$$= \sup_{\psi:\|\psi\|=1} \left(-\varkappa \sum_{|h|=1, x\in Z^{pd}} (\psi(x+h) - \psi(x))^2 + \sum_{x\in Z^{pd}} V_p(X)\psi^2(x) \right). \tag{1.19}$$

Corollary (1.0.5) is classical alternative form for the upper boundary of the spectrum of the self-adjoint operator.

Because of $\varkappa \geq 0$, for given ψ the expression in the big bracket of equation (1.19) is linear non-increasing function of $\varkappa$. So it can result in

Corollary 1.0.6. *For every* $p \geq 1$*, the moments Lyapunov exponent* $\gamma_p^{(I)}(\varkappa)$ *is a convex non-decreasing function of diffusivity* $\varkappa \geq 0$

Theorem 1.0.7. *(Ito form)*

1.$\gamma_1^{(I)}(\varkappa) \equiv 0$.

2.$\gamma_2^{(I)}(\varkappa)$ *is equal to the upper boundary of the spectrum of two-body Schrödinger operator*

$$\tilde{H} = 2\varkappa\Delta + \delta_0(x) \tag{1.20}$$

which is a result of the removing of the center of mass for the operator

$$H_2 = \varkappa(\Delta_{x_1} + \Delta_{x_2}) + \delta(x_1 - x_2).$$

2.a If $d = 1, 2$*, then* $\gamma_2(\varkappa) > 0$ *and it is a unique positive solution of the equation*

$$\frac{1}{(2\pi)^2} \int_{S^d} \frac{dk}{2\varkappa\Phi(k) + \gamma} = 1,\ \ \Phi(k) = 2\sum_{j=1}^{d}(1 - cos(k_j)) \tag{1.21}$$

2.b If $d \geq 3$ *and*

$$\varkappa \geq \varkappa_r = \frac{1}{(2\pi)^d} \int_{S^d} \frac{dk}{2\Phi(k)} \tag{1.22}$$

then $\gamma_2(\varkappa) = 0$. *If* $\varkappa < \varkappa_r$, *then* $\gamma_2(\varkappa) > 0$ *and is a positive root of equation (1.19).*

Corollary 1.0.8. *In the low dimensions* $d = 1, \ 2$ *for arbitrary* $k > 0$, *the family of the field* $u(t, x)$ *(i.e., solution of equation (1.1)), which can be understood in both senses(Ito's or Stratonovich's), has a property of full asymptomatic intermittency. In other words, inequalities*

$$\gamma_1(\varkappa) < \frac{\gamma_2}{2}(\varkappa) < \cdots < \frac{\gamma_p}{p}(\varkappa) < \cdots \tag{1.23}$$

hold for all $\varkappa \geq 0$.

If $d \geq 3$, *then inequalities (1.23) has a place only for* $\varkappa < \varkappa_r = \frac{1}{(2\pi^d)} \int_{S^d} \frac{d\phi}{2\Phi(\phi)}$.

If $\varkappa > \varkappa_r$, *then for* $t \to \infty$

$$m_2(t, x) = (u^2(t, x)) \to constant$$

$$m_1(t, x) \equiv 1.$$

that's, the family of distributions of $u(t, x)$ *is tight for* $t \to \infty$.

The proof of the latter statement of corollary (1.23) is based on the Fourier analysis of equation 1.20. Here we do not want to give the details about this calculation.

The most important property of solution $u(t, x)$ is its intermittency. The fundamental property of intermittency (i.e. very strong fluctuations) can be described in the following form: $u(t, x)$ of the equation (1.1) for fixed moment t is ergodic and homogeneous in space random field. This field is defined on the probability space $(\Omega_m, \mathcal{F}_m, \mathrm{P}_m)$, in which m means medium or environment, and it can be identified with $\omega = \omega_m(\cdot) = \{\xi(x, \cdot), x \in Z^d\}$.

Consider the moments

$$m_k(t) = E\left[u^k(t,x,\omega)\right] = \int_{\Omega_m} u^k(t,x,\omega_m)\mathrm{P}_m(d\omega_m)$$

Because of homogeneity, the last integral is independent on x. In typical cases, say, for the Gaussian $N(0,1)$ r.v., there exists the Lyapunov exponents with appropriate normalization $\mathcal{L}(t)$,

$$\gamma_k = \lim_{t\to\infty} \frac{ln(m_k(t))}{\mathcal{L}(t)},\ \ k = 1, 2, \cdots$$

and

$$\gamma_1 < \frac{\gamma_2}{2} < \frac{\gamma_3}{3} < \cdots$$

It means that the moments $m_k(t)$ are growing progressively as the function of t, $t \longrightarrow \infty$:

$$m_1(t) \ll m_2^{\frac{1}{2}}(t) \ll m_3^{\frac{1}{3}}(t) \ll \cdots$$

At the physical level the intermittency, which mathematically is an equivalent relation $\gamma_1 < \frac{\gamma_2}{2} < \frac{\gamma_3}{3} < \cdots$, means that the field has very high local maxima(peaks).

The following example illustrate the concept of the intermittency. Consider the random field $\eta_\varepsilon(x),\ x \in Z^d$ given by the Bernoulli representation. $\eta_\varepsilon(x)$ are i.i.d. *r.v.* for different $x \in Z^d$ and

$$P\{\eta_\varepsilon(x) = 0\} = 1 - \varepsilon,\ P\{\eta_\varepsilon(x) = \frac{1}{\varepsilon}\} = 1 - \varepsilon.$$

Then

$$E[\eta_\varepsilon(x)] = 1,\ Var(\eta_\varepsilon(x)) = \left(\frac{1}{\varepsilon}\right)^2 - 1 = \frac{1}{\varepsilon} - 1 \gg 1.$$

This field contains very space high peaks separated by large area where $\eta_\varepsilon(\cdot) = 0$.

Note that $E[\eta_\varepsilon^k(x)] \sim \frac{1}{\varepsilon^{k-1}},\ k \geq 1$, and

$$(E[\eta_\varepsilon^2(x)])^{1/2} \ll (E[\eta_\varepsilon^3(x)])^{1/3} \ll \cdots \ll (E[\eta_\varepsilon^k(x)])^{1/k} \ll \cdots .$$

This introduction will be closed with a short summary of the contents of this dissertation. Chapter 2 provides the detailed description of our model in non-stationary environment and technical tools. Chapter 3 is devoted to proving the uniqueness and existence of solution in the weighted Hilbert space and Feynman-Kac representation of the solution. Last chapter gives spectral analysis of basic Hamiltonian and the study of the phase transition from the regular to the intermittent structure of the solution.

CHAPTER 2: Description of the model and the technical tools

This section consists of two subsections. First one is to describe the technical tool and limiting theorem that will be used in future, the second one will list out the problems to be solve in this work.

2.1 Description of model and limiting theorem

Our model is the generalization of the model presented in the introduction or paper [1], [2]. It has the following form

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} = \varkappa \mathcal{L} u(t,x) + \xi_t(x) u(t,x) \\ &u(0,x) \equiv 1,\ \ (t,x) \in [0,\infty) \times Z^d \end{aligned} \tag{2.1}$$

Here

$$\varkappa(\mathcal{L}\psi)(t,x) = \varkappa \sum_{z \neq 0} [\psi(t,x+z) - \psi(t,x)] a(z) \tag{2.2}$$

is the generator of symmetric random walk $x(t), t \geq 0$ with continuous time. The rate of the jumps $a(z),\ z \in Z^d$ satisfies the regularity conditions:

1) symmetry (i.e., $\mathcal{L} = \mathcal{L}^*\ in\ Z^d$):

$$a(z) = a(-z); \tag{2.3}$$

2) $a(z) > 0$ if $|z| = 1$ to avoid the periodicity;

3) normalization: the total intensity of jumps is $\varkappa$, it means that $\sum_{z \neq 0} a(z) = -a(0) = 1$.

Like in the introduction we'll call $\varkappa \geq 0$ the diffusion coefficient. The random walk has the following structure. It spends in each site $x \in Z^d$ the exponential distributed time τ_x, i.e., $P(\tau_x > t) = e^{-\varkappa t}$ and at the moment $\tau_x + 0$ it jumps from site x to site $x + z$ with probability $a(z)$. We'll discuss the following three different cases.

I) If $a(z) \leq ce^{-\eta|z|}$, $c, \eta > 0$ (so-called Cramér's condition), then we'll say that the random walk has light tails.

II) If

$$a(z) \sim \frac{C(\dot{z})}{|z|^{d+\alpha}}, \ \dot{z} = \frac{z}{|z|}(direction\ of\ z) \in S^{d-1}(Z^d), \ 0 < \alpha < 2 \tag{2.4}$$

plus regularity conditions[11] [12], then the tails are heavy. Such random walk is related to the symmetric stable process in R^d.

III) Moderate tails case: if

$$a(z) \sim \frac{C(\dot{z})}{|z|^{d+\alpha}}, \ \dot{z} = \frac{z}{|z|}, \ \alpha > 2, \tag{2.5}$$

in particular $\sum |z|^2 a(z) < \infty$ then the process $x(t)$ satisfies the Central Limit Theorem(CLT), i.e., it has asymptotic Gaussian distribution. The additional conditions for this case see in [10].

Now let's introduce the transition probabilities

$$p(t, x, y) = P\{X(t) = y | X(0) = x\}. \tag{2.6}$$

It is the solution of the parabolic problem

$$\begin{aligned} \frac{\partial p(t, x, y)}{\partial t} &= \varkappa \mathcal{L}_x p = \varkappa \mathcal{L}_y p \\ p(0, x, y) &= \delta_y(x) = \delta_0(x - y) \end{aligned} \tag{2.7}$$

Derivation of equation (2.7) is based on the elementary observation

$$
\begin{aligned}
&P\{(X(t+dt)=y|X(t)=x)\}=\varkappa a(x-y)dt\ if\ \ y\neq x\\
&P\{(X(t+dt)=x|X(t)=x)\}=1-\varkappa dt
\end{aligned}
\tag{2.8}
$$

We'll use this relation many times in the future.

Using Fourier analysis (like in the introduction) one can derive the following formula

$$
\begin{aligned}
p(t,x,y)=p(t,y,x)=p(t,0,x-y)=p(t,0,y-x)=p(t,y-x)\\
=\frac{1}{(2\pi)^d}\int_{T^d}e^{-tS(k)+i(x-y,k)}dk
\end{aligned}
\tag{2.9}
$$

Here

$$
S(k)=\varkappa\sum_{z\neq 0}(1-cos(k,z))a(z), k\in T^d=[-\pi,\pi]^d \tag{2.10}
$$

is Fourier symbol of the operator $\varkappa\mathcal{L}$. It is not difficult to prove that

$$
P(t,x,x)\sim\frac{C}{t^{d/\alpha}}
$$

Here α is the index of the stable law, $\alpha=2$ in II, III. Let's give more details. If the distribution of the jump $\{a(z), z\in Z^d\}$ has the second moment $\sigma^2=\sum_{z\neq 0}|z|^2a(z)<\infty$, then the random walk $x(t)$ satisfies the CLT (Gaussian form), i.e.,

$$
p(t,x,y)\sim\frac{e^{-\frac{(x-y)^2}{2\sigma^2t}}}{(2\pi t\sigma^2)^{d/2}},\ \ |x-y|=0(\sqrt{t}).
$$

In particular

$$
p(t,x,x)\sim\frac{C}{t^{d/2}}, i.e.,\ \ \alpha=2.
$$

Of course, second moment is finite for the light tails and for moderate tails.

In the case of the heavy tails (under additional condition of the regularity of

$a(z)$, see details below) the distribution of $p(t,x,y)$ after appropriate normalization asymptotically close to the symmetric stable law with parameter $0 < \alpha < 2$, in particular

$$p(t,x,x) \sim \frac{C}{t^{d/\alpha}}.$$

Let's remember that the Markov chain (random walk with the generator $\varkappa\mathcal{L}$) is recurrent if and only if

$$\int_0^\infty p(t,x,x)dt = \int_0^\infty p(t,0,0)dt = \infty$$

and the transient if

$$\int_0^\infty p(t,x,x)dt = \int_0^\infty p(t,0,0)dt < \infty.$$

It shows that the random walk is recurrent in the case of the finite second moment (say $\sigma^2 = \sum_{z\neq 0} |z|^2 a(z) < \infty$) in dimension $d = 1,\ 2$ and without additional conditions for $d \geq 3$. If $d = 2$ then the random walk is recurrent if only if $\alpha = 2$ (i.e., $\sigma^2 < \infty$) and transient for $\alpha < 2$. If $d = 1$ then it is recurrent for $1 < \alpha \leq 2$ and transient for $0 < \alpha \leq 1$.

In terms of the symbol $S(k),\ k \in T^d$, the process $X(t)$ is recurrent if

$$\int_{T^d} \frac{dk}{S(k)} = \infty$$

and transient if

$$\int_{T^d} \frac{dk}{S(k)} < \infty.$$

Let's give review of the limit theorems for the random walk $X(t)$ in three different cases: heavy, moderate and light tails[8] [11] [12] . In all cases, $a(z)$ is symmetric, non-negative for $z \neq 0$ and $\sum a(z) = 1$. Also second moment exists($\sum_{z\neq 0} |z|^2 a(z) < \infty$)

in light tails case and moderate tails case but does not exist($\sum_{z\neq 0}|z|^2a(z) = \infty$)in heavy tails case. Beside these, additional different stronger assumptions on $a(z)$ for each case can produce distinct results.

The asymtotics of $p(t,x,y)$ and the Green Function

$$G_\mu(x,y) = G_\mu(0,x-y) = \int_0^\infty e^{-\mu t}p(t,x,y)dt,\ \ \mu \geq 0. \tag{2.11}$$

The light tails case was studies in [8]. It that assume that $\forall(\lambda \in R^d)$

$$\sum_{z\neq 0} e^{(\lambda,z)}a(z) = \sum_{z\neq 0} cosh(\lambda,z)a(z) < \infty$$

which is the strongest form of the so-called Cramér's condition, put

$$H(\lambda) = \sum_{z\neq 0} a(z)\left(e^{(\lambda,z)} - 1\right) = \sum_{z\neq 0} a(z)(cosh(\lambda,z) - 1) < \infty.$$

One can prove that the following equation

$$\nabla H(\lambda) = \frac{x}{t}$$

has unique solution $\lambda_*(\frac{x}{t})$, which is a smooth function of x/t, for any $x \in Z^d, t \geq 1$.

Theorem 2.1.1. *For every fixed $A > 0$, as $t \to \infty$, the asymptotic equality*

$$p(t,0,x) \sim \frac{e^{t[H(\lambda_*(x/t))-(\lambda_*(x/t),x/t)]}}{(2\pi t)^{d/2}\sqrt{detB(\lambda_*(x/t))}} \tag{2.12}$$

hold uniformly in $|x| \leq At$, where $|\cdot|$ denotes the Euclidean norm of vectors, $\lambda_*(x/t)$ is the unique root of the equation $\nabla H(\lambda_*) = Z = x/t$, and

$$B(\lambda) = \frac{\partial^2 H}{\partial x_i \partial x_j}(\lambda) = HessH(\lambda) \tag{2.13}$$

is the covariance of the jump distribution $a(z)$. If $|z| = \bar{\bar{o}}(t^{2/3})$ the we have standard Gaussian approximation with covaraince matrix $B(0)$:

$$p(t,0,x) \sim \frac{e^{-\frac{(x,B^{-1}(0)x)}{2t}}}{|2\pi t|^{d/2}\sqrt{detB(0)}},$$

Let $h_\mu(\dot{x}) = (\lambda_*(\dot{x}/s_*), \dot{x})$ where s_* is root of the $\mu = H(\lambda_*(\dot{x}/s))$ for s . Then the Green function has the following symptotics:

Theorem 2.1.2. *For every fixed $\mu > 0$, the Green function $G_\mu(0,x)$ has the following asymptotic representation as $|x| \to \infty$:*

$$G_\mu(0,x) \sim \frac{C_d e^{-|x|h_\mu(\dot{x})}}{|x|^{(d-1)/2}\sqrt{s_*^d}\sqrt{detB(\lambda_*(\dot{x}/s_*))}}$$

where $\dot{x} = \frac{x}{|x|}$ and C_d is a positive constant depending on the dimension of the lattice, matrix B is defined in equation 2.13.

In the transient case for $d \geq 3$ and $|x| \to \infty$, for $\mu = 0$, the Green function still exists and

$$G_0(0,x) \sim \frac{C_d}{(B^{-1}(0)x,x)^{d/2-1}}$$

One can find more details in [8].

The global limit theorem of random walk with moderate tails was established in [10]. Beside the conditions listed before, $a(z)$ has the following asymptotic expansion at infinity

$$a(z) = \sum_{j=0}^{N} \frac{c_j(\dot{z})}{|z|^{d+\alpha_j}} + O(\frac{1}{|z|^{2d+\alpha+\ell}}), z \to \infty, 2 < \alpha_0 = \alpha < \alpha_1 < \cdots < \alpha_N \qquad (2.14)$$

where $\dot{z} = \frac{z}{|z|}$, $c_0(\dot{z}) = c_0(-\dot{z})$, $\ell = 1$ is $\alpha > [\alpha]$, $\ell = 2$ if $\alpha = [\alpha]$, $c_0(\dot{z}) > 0$ and $c_j(\dot{z})$

are sufficiently smooth. It implies that

$$\hat{a}(k) = \sum_{j=0}^{N} b_j(\dot{k})|k|^{\alpha_j} + a_1(k),\ b_j(\dot{k}) \in C^M,\ a_1(k) \in C^M,\ M = M(\alpha) = d + [\alpha] + 1 \tag{2.15}$$

where $b_j(\dot{k})|k|^{\alpha_j}$ are the Fourier series of $\frac{c_j(\dot{z})}{|z|^{d+\alpha_j}}$. Based on these assumptions, [10] gets the following results for random walk in lattice space.

Theorem 2.1.3. *Let the conditions (2.3), (2.14) and (2.15) hold, then there are constants $A,\ \varepsilon > 0$, such that the solution of equation (2.7) the following asymptotic behaviors as $|x|^2 \geq At,\ t \geq 0, |x| \to \infty$ (here for simplicity $B(0) = I$)*

$$p(t,0,x) = \frac{t}{|x|^{d+\alpha}} \left[c_0(\dot{x}) + O\left(\left(\frac{1+t}{|x|^2} \right)^{\varepsilon} \right) \right] + E(t,x) \left(1 + O\left(\frac{t^{1/\alpha}}{|x|} \right) \right), \tag{2.16}$$

where α and $c_0(\dot{x})$ are defined in equation (2.14), and $E(t,x) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}}$.

The first term of equation (2.16) dominates outside of logarithmic neighborhood of paraboloid $t = |x|^2$, the second term is larger inside of this neighborhood.

Theorem 2.1.4. *Let the conditions (2.3), (2.14) and (2.15) hold. Then the Green function $G_\mu(x)$ has the following asymptotic behavior when $0 < \mu \leq \Lambda_0 < \infty$, $\mu|x|^2 \to \infty$ (where ε defined in equation (2.16):*

$$G_\mu(x) = \frac{1}{\mu^2 |x|^{d+\alpha}} \left[c_0(\dot{x}) + O\left(\frac{1}{(\mu|x|^2)^{\varepsilon}} \right) \right]. \tag{2.17}$$

In particular, if $\mu > 0$ is fixed, then

$$G_\mu(x) = \frac{1}{\mu^2 |x|^{d+\alpha}} \left[c_0(\dot{x}) + O(|x|^{-2\varepsilon}) \right] \ \ |x| \to \infty$$

The proof can be found in [10].

The global limit theorem of random walk with heavy tails was studied in [11]. Some stronger assumptions on $a(z)$ are introduced. It assumed that $a(z)$ is symmetric($a(z) = a(-z)$), second moment doesn't exist ($\sum_{z \in Z^d} |z|^2 a(z) = \infty$) and $a(z)$ has regular behavior at infinity:

$$a(z) = \sum_{j=0}^{d+\varsigma} \frac{a_j(\dot{z})}{|z|^{d+\alpha+j}} + O(\frac{1}{|z|^{2d+\alpha+1+\varsigma}}), |z| \to \infty, a \in (0,2), a_j \in S^{d-1}, \tag{2.18}$$

where $\varsigma = 1$ if $\alpha = 1$ and $\varsigma = 0$ otherwise, and $a_0(\dot{z}) > \delta > 0$. Then the Fourier series of $a(z)$ has the such asymptotic behavior at zero :

$$\hat{a}(k) = 1 - \sum_{j=0}^{d} b_j(\dot{k})|k|^{\alpha+j} + f(k), k \in T^d = [-\pi, \pi]^d, \tag{2.19}$$

where $b_j \in S^{d-1}$ and function $f \in R^d$ and

$$b_0(\dot{k}) = -\Gamma(-\alpha)cos(\frac{\alpha\pi}{2}) \int_{S^{d-1}} a_0(\dot{x})|(\dot{x}, \dot{k})|^\alpha dS_{\dot{x}} > 0, \tag{2.20}$$

where Γ is the gamma function. One can immediately get the next two properties of $\hat{a}(k)$:

$$\hat{a}(k) = \hat{a}(-k);\ \hat{a}(k) < 1, k \neq 0 \ and \ k \in T^d = [-\pi, \pi]^d. \tag{2.21}$$

These assumptions allow the random walk have very long jumps with a certain probability which can be found in the following global limit theorem [11]:

Theorem 2.1.5. *Let equations (2.19)-(2.21)hold, then*

$$p(t, 0, x) = \frac{1}{t^{d/\alpha}} S\left(\frac{x}{t^{1/\alpha}}\right)(1 + o(1)),\ |x| + t \to \infty, \tag{2.22}$$

where

$$S(y) = \frac{1}{(2\pi)^d} \int_{R^d} e^{i(k,y) - b_0(\dot{k})|k|^\alpha} dk > 0$$

is the stable density $S = S_{\alpha, a_0}(y)$, which is function of $\alpha \in (0, 2)$.

If $\frac{|x|}{t^{1/\alpha}} \to \infty$, $|x| \geq 1$, *then the previous statement can be specified as*

$$p(t, 0, x) = \frac{a_0(\dot{x})}{t^{d/\alpha}} \left(\frac{t^{1/\alpha}}{|x|}\right)^{d+\alpha} (1 + 0(1)) = \frac{a_0(\dot{x})t}{|x|^{d+\alpha}}(1 + 0(1)).$$

The details is in [11].

Based on these results from [11] for the constant branching rate, the paper [12] proves that intermittency exists in the random walk with heavy tails. More precisely, the front of the population propagates exponentially fast but the number of the particles inside of the front is distributed highly non-uniformly. The exact front will be found in following theorem.

Theorem 2.1.6. *Let eqaution 2.14 holds and*

$$\gamma = \frac{2\alpha + d}{\alpha(\alpha + d)}.$$

Then the ratio $\frac{m_2(t,x)}{m_1^2(t,x)}$ *is uniformly bounded in each ball* $|x| < Bt^{\gamma}$ *when* $t \to \infty$, *i.e, the number of the particles is non-intermittent there.*

For each domain $\Omega_\theta(t) = \{x : |x| > t^{\gamma+\theta}\}, \theta > 0$, *then* $\frac{m_2(t,x)}{m_1^2(t,x)} \to \infty$ *uniformly for* $x \in \Omega_\theta(t)$, *i.e, the number of the particles is intermittent there.*

2.2 The problems to be solved

In this section we'll prove the existence-uniqueness theorem for Stochastic Partial Differential Equations(SPDEs) in the space L^2_μ, give the Faynman-Kac representation of the solution and describe the moment equations.

Now we proceed to describe the potential $\xi_t(x)$. $\xi_t(x)$ is Gaussian white noise as the function of (t, x), i.e.,

$$\xi_t(x) = \dot{W}(t, x) = \frac{d}{dt}W(t, x) \tag{2.23}$$

More precisely, let $W(t, x)$ be the field of the correlated Wiener processes on $[0, \infty) \times$

Z^d. This field is uniquely determined by expectation $\langle W(t,x)\rangle = 0$ and covariance operator $\langle W(s,x)W(t,y)\rangle = min(s,y)B(x-y)$. Such field $W(t,x)$ is better to be introduced as the linear transformation of independent standard Brownian motion $b(t,x)$, for which $\langle b(t,x)\rangle = 0$, $\langle b^2(t,x)\rangle = t$, $\langle b(t,x)b(s,x)\rangle = min(t,s)$ and $\langle b(t,x)b(s,y)\rangle = 0$ if $x \neq y$. Namely $W(t,x) = \sum_{z\in Z^d} \ell(x-z)b(t,z)$ for appropriate kernel $\ell(x-z)$, then $\langle W(t,x)\rangle = 0$ and

$$\begin{aligned}\langle W(t,x)W(s,y)\rangle &= \langle \sum_{z_1\in Z^d} \ell(x,z_1)b(t,z_1) \sum_{z_2\in Z^d} \ell(x,z_2)b(s,z_2)\rangle \\ &= \sum_{z_1=z_2=z\in Z^d} \ell(x,z)\ell(y,z)\langle b(t,z)b(s,z)\rangle \\ &= min(s,t)\sum_{z\in Z^d} \ell(x-z)\ell(y-z) \\ &= min(s,t)\sum_{u\in Z^d} \ell(u)\tilde{\ell}(x-y-u) \quad , (\tilde{\ell}(z) = \ell(-z)) \\ &= min(s,t)(\ell * \tilde{\ell})(x-y).\end{aligned}$$

Here $*$ is the convolution operator. Let's define correlation function $B(x,y)$ as $tB(x,y) = \langle W(t,x)W(t,y)\rangle$, obviously

$$B(0,x) = B(x) = \sum_{y\in Z^d} \ell(x-y)\tilde{\ell}(y) \tag{2.24}$$

Due to Bochner-Khinchin's Theorem the correlation function of any stationary field $\vartheta(x),\ x\in Z^d$ with $E(\vartheta(x)) = 0$, and $E(\vartheta(x)\vartheta(y)) = B(x-y)$ has the representation

$$B(x) = \int_{T^d} e^{i(k,x)} d\mu(k)$$

where $T^d = [-\pi,\pi]^d$ is the d-dimensional torus and $d\mu(k)$ is the positive measure on T^d. We call $d\mu(k)$ is the spectral measure of the field $\vartheta(x)$. Let's stress that in this theorem we do not assume that $\vartheta(x),\ x\in Z^d$ is Gaussian.

Let's return to the formula (2.24). Since $\ell \in L^2(Z^d)$, the Fourier series $\hat{\ell}(k) = \sum_{x\in Z^d} \ell(x)e^{i(k,x)}$ converges in $L^2(T^d, dk)$, i.e., $l.i.m.\hat{\ell}_n(k) = \hat{\ell}(k)$ where $\hat{\ell}_n(k) = \sum_{|x|<n} \ell(x)e^{i(k,x)}$.

Of course

$$\ell(x) = \frac{1}{(2\pi)^d} \int_{T^d} \hat{\ell}(k)e^{-i(k,x)}dk. \tag{2.25}$$

Let's note that equation

$$B(0,x) = B(x) = \sum_{y\in Z^d} \ell(x-y)\tilde{\ell}(y)$$

together with relation $\hat{\tilde{\ell}}(k) = \bar{\hat{\ell}}(k)$ leads to

$$\hat{B}(k) = |\hat{\ell}(k)|^2$$

As result spectral measure of the Gaussian filed $W(t,x),\ x \in Z^d$ equals $d\mu(k) = t\hat{B}(k)dk = t|\hat{\ell}(k)|^2 dk$.

CHAPTER 3: Existence and uniqueness of solution $u(t, x)$

In this section we'll prove the existence-uniqueness theorem for Stochastic Partial Differential Equations(SPDEs) in the space L^2_μ (there exists a unique solution $u(t, x)$ of equation(2.1)), give the Faynman-Kac representation of the solution, describe the moment equations and study the transition Probability $p(t, x, y)$. Let's start with showing the correlated Wiener process $W(t, x)$ is the element of $L^2_\mu(Z^d)$ for appropriate weight μ.

3.1 White noise in weighted Hilbert space $L^2_\mu(Z^d)$

As it mentioned in introduction, $W(t, x)$ is not a Wiener process in $L^2(Z^d)$ because $\sum_{x\in Z^d} W^2(t, x) = \infty$ almost surely in probability. But it is not difficult to show that $W(t, x)$ is a Wiener process on weighted Hilbert space $L^2_\mu(Z^d)$. More precisely, let $L^2_\mu(Z^d)$, where $\mu(x)$ is measurable positive function and $\sum_{x\in Z^d} \mu(x) < \infty$, to be a space of real function $f(x) : Z^d \to R$ for which

$$\|f(x)\|^2_\mu = \sum_{x\in Z^d} f^2(t, x)\mu(x)$$

is finite. Of course $W(t, \cdot) = \{W(t, x),\ x \in Z^d\}$ belongs to Hilbert space with norm $\|\cdot\|^2_\mu$ if

$$\|W(t, \cdot)\|^2_\mu = \sum_{x\in Z^d} W^2(t, x)\mu(x) < \infty.$$

The classical Kolmogorov criterion of continuity for random process $\xi(t, \omega)$, $t \in [0, T]$, which usually formulated for scalar (real value) process, is also applicable for $\mathfrak{B}-$valued process in the following form: assume that $\exists\ \alpha, \beta, c > 0$ such that for $\forall\ t_1,\ t_2 \in [0, T]$ (Here $\mathfrak{B}-$Banach space with norm $\|\cdot\|^2_{\mathfrak{B}}$)

$$\langle \|\xi_{t_1}(\omega) - \xi_{t_2}(\omega)\|_{\mathfrak{B}}^{\alpha} \rangle \leq C|t_1 - t_2|^{1+\delta}, \ P - a.s,$$

then $\mathfrak{B}-$valued process $\xi_t(\omega), \ t \in [0, T]$ is continuous in $\mathfrak{B}$ (i.e., $\xi_t(\omega), \ t \in [0, T]$) is the continuous trajectory in $\mathfrak{B}$ P-a.s.). Even more this trajectory belongs to first Hölder class:

$$\|\xi_{t_1}(\omega) - \xi_{t_2}(\omega)\|_{\mathfrak{B}} \leq C(\omega)|t_1 - t_2|^{\beta}, \ \beta < min(\frac{\delta}{\alpha}, 1).$$

All future analysis will concentrate on the weighted Hilbert space $L^2_\mu(Z^d)$:

$$L^2_\mu(Z^d) = \{f : \|f\| = \sum_{x \in Z^d} f^2(x)\mu(x)\}.$$

We'll consider the finite measure $\mu(x)$ with the condition

$$0 < \mu(x) \leq \frac{c}{1 + |x|^{d+\delta_1}}, \ \delta_1 > 0. \tag{3.1}$$

We'll use such estimation in cases of heavy and moderate tail, but in the case of light tail (like in [1]) we can consider $\mu(x)$ has exponential decreasing such that $\mu(x) = e^{-\eta|x|}$ for approximate $\eta > 0$.

Now we have the following theorem:

Theorem 3.1.1. *Assume that*

$$\ell(x) < \frac{C}{1 + |x|^{\frac{d+\delta_2}{2}}} \tag{3.2}$$

$$\mu(x) < \frac{C}{1 + |x|^{d+\delta_2}}, \tag{3.3}$$

then the Gaussian filed

$$W(t,\cdot)=\sum_{z\in Z^d}\ell(\cdot,z)b(t,z)=\sum_{z\in Z^d}\ell(\cdot-z)b(t,z),\ \forall t\in[0,T]$$

is element in $L^2_\mu(Z^d)$. And $W(t,x)$ is a Wiener process on weighted Hilbert space $L^2_\mu(Z^d)$, of course $\xi_t(x)$ defined in (2.23) is a "white noise" in $L^2_\mu(Z^d)$.

Proof. Here (and in the future) let's also recall that

$$B(x)=\sum_{z\in Z^d}\ell(x-z)\tilde{\ell}(z),\ B(0)=\sum_{z\in Z^d}\ell^2(z),\overline{Var}(\mu)=\sum_{x\in Z^d}\mu(x).$$

Obviously in $L^2_\mu(Z^d)$, $W(t,x)$ preserves the following properties:

(1) $\langle W(t,x)\rangle=0$

(2) $W(t,x)$ has independent increments due to the independent increments of $w(t,z)$, i.e., for $t,t+h\in[0,T]$, $W(t+h,x)-W(t,x)$ is independent of $W(t,x)$.

(3) $W(t,x)$ has Gaussian increments: $W(t+h,x)-W(t,x)\sim N(0,hB(0))$.

The series $W(t,x)$ converges almost surely if $\sum_{z\in Z^d}\ell^2(x-z))<\infty$ by Kolmogorov's three-series theorem. First let's prove that $W(t,x)$ preserves the continuity in time for fixed x by the following moments estimates:

$$\begin{aligned}
&\langle(W(t+h,x)-W(t,x))^2\rangle=h\sum_{z\in Z^d}\ell^2(x-z))=hB(0)\\
&\langle(W(t+h,x)-W(t,x))^4\rangle=\langle(\sum_{z\in Z^d}\ell(x-z))\Delta b(t,x))^4\rangle\\
&=3\sum_{z\in Z^d}\ell^4(x-z)h^2+\sum_{z_1\neq z_2}\ell^2(x-z_1)\ell^2(x-z_2)h^2
\end{aligned}$$

Now we want to prove the continuity of $W(t,x)$ in $L^2_\mu(Z^d)$. Let's consider the increment $\Delta W(\cdot) = \{W(t_1, x) - W(t_2, x),\ x \in Z^d\}$ has the norm

$$\|\Delta W(\cdot)\|_\mu^2 = \|W(t_1,\cdot) - (W(t_2,\cdot)\|_\mu^2 = \sum_{x\in Z^d}\left(\sum_{z\in Z^d}\ell(x-z)\Delta b(t,z)\right)^2\mu(x)$$

and

$$\langle\|\Delta W(\cdot)\|_\mu^2\rangle = \langle\sum_{x\in Z^d}\left(\sum_{z\in Z^d}\ell(x-z)\Delta b(t,z)\right)^2\mu(x)\rangle$$
$$= \sum_{x\in Z^d}\sum_{z\in Z^d}\ell^2(x-z)\langle\Delta^2 b(t,z)\rangle\mu(x)$$
$$+ 2\sum_{x\in Z^d}\sum_{z_1\neq z_2}\ell(x-z_1)\ell(x-z_2)\langle\Delta b(t,z_1)\Delta b(t,z_2)\rangle\mu(x)$$
$$= |t_1 - t_2|B(0)\overline{Var}(\mu)$$

We can not use the Kolmogorov's criterion since the degree of factor $\Delta t = |t_1 - t_2|$ is only "1", but we need degree $1 + \delta > 1$. Thus let's calculated the forth moment:

$$\langle\|W(t_2,\cdot) - (W(t_1,\cdot)\|_\mu^4\rangle = \langle\left(\sum_{x\in Z^d}\mu(x)\sum_{z_1,z_2\in Z^d}\ell(x-z_1)\ell(x-z_2)\Delta b(t,z_1)\Delta b(t,z_2)\right)^2\rangle$$
$$= \sum_{x_1,x_2\in Z^d}\mu(x_1)\mu(x_2)\sum_{z_1,z_2,z_3,z_4\in Z^d}$$
$$\ell(x_1-z_1)\ell(x_1-z_2)\ell(x_2-z_3)\ell(x_2-z_4)\langle\Delta b(t,z_1)\Delta b(t,z_2)\Delta b(t,z_3)\Delta b(t,z_4)\rangle$$

But the increments are independent for the different z and $\langle\Delta b(t,x)\rangle = 0$. It means that $\langle\Delta b(t,z_1)\Delta b(t,z_2)\Delta b(t,z_3)\Delta b(t,z_4)\rangle$ is equal to "0" except in the following cases:

(a) if $z_1 = z_2 = u_1 = u_2 = z$, then $\langle \Delta^4 b(x) \rangle = 3(\Delta t)^2 = 3|t_1 - t_2|^2$ and

$$\begin{aligned}
&\sum_{x_1,x_2} \mu(x_1)\mu(x_2) \sum_{z \in Z^d} \ell^2(x_1 - z)\ell^2(x_2 - z)\langle \Delta^4 b(t,x) \rangle \\
&= 3|t_1 - t_2|^2 \sum_z \left(\sum_{x \in Z^d} \mu(x)\ell^2(x - z) \right)^2 \\
&= 3|t_1 - t_2|^2 \overline{Var}(\mu \times \ell^2(\cdot)) = D_0(\delta_2)|t_1 - t_2|^2
\end{aligned}$$

Note that $\ell^2(\cdot) \in L^1, \ \mu(\cdot) \in L^1 \to \mu \times \ell^2(\cdot) \in L^1$.

(b) if $z_1 = z_2$, $z_3 = z_4$, $z_1 \neq z_3$, then $\langle \Delta^2 b(z_1)\Delta^2 b(z_3) \rangle = |t_1 - t_2|^2$ and

$$\begin{aligned}
&\sum_{x_1,x_2} \mu(x_1)\mu(x_2) \sum_{z_1 \neq z_3} \ell^2(x_1 - z_1)\ell^2(x_2 - z_3)\langle \Delta^2 b(t,z_1)\Delta^2 b(t,z_3) \rangle \\
&= |t_1 - t_2|^2 \sum_{x_1,x_2 \in Z^d} \mu(x_1)\mu(x_2) \sum_{z_1 \neq z_3} \ell^2(x_1 - z_1)\ell^2(x_2 - z_3) \\
&\leq |t_1 - t_2|^2 \sum_{x_1,x_2 \in Z^d} \mu(x_1)\mu(x_2) \left(\sum_{u \in Z^d} \ell^2(u) \right)^2 \\
&= |t_1 - t_2|^2 \overline{Var}(\mu^2)B^2(0)
\end{aligned}$$

The same result can be obtained for last two cases :

(c) $z_1 = z_3$, $z_1 = z_4$, $z_1 \neq z_2$

(d) $z_1 = z_4$, $z_2 = z_3$, $z_1 \neq z_2$

Combing the above all case together can find

$$\langle \|W(t_2,\cdot) - (W(t_1,\cdot)\|_\mu^4 \rangle \leq C(\delta_2)|t_1 - t_2|^2. \tag{3.4}$$

It gives the continuity of function-valued process $W(t,x), \ x \in Z^d, \ t \in [0,T]$ in $L^2_\mu(Z^d)$ under our conditions on $\mu(x)$ and $\ell(x)$.

Remark 1: In fact our result in the space $L^2_\mu(Z^d)$ is similar to the $1 - D$ results.

If $w(t)$ is the standard Wiener process such that

$$\langle b(t)\rangle = 0, \langle b(t_1)b(t_2)\rangle = min(t_1, t_2), \langle |b(t_1) - b(t_2)|^2\rangle = |t_1 - t_2|$$

In fact

$$\langle |b(t_1) - b(t_2)|^4\rangle = 3|t_1 - t_2|^2.$$

Remark 2: Let's consider more general Banach space

$$L^p_\mu(Z^d) = \{f : \sum_{x\in Z^d} |f(x)|^p \mu(x) = \|f\|^p < \infty\},$$

one can prove that under our conditions on $\mu(x)$ and $\ell(x)$ the process $W(t,\cdot) = \sum_{z\in Z^d} \ell(\cdot - z)b(t,z)$ belongs and continues in time t in $L^p_\mu(Z^d)$.

Evidently $W(t,\cdot)$ is a Wiener process in $L^2_\mu(Z^d)$, and $\xi_t(x)$ is a "white noise" by its definition in (2.23). $\square$

Now we have $\Omega_m = L^2_\mu(Z^d)$ in our probability space $\{\Omega_m, \mathcal{F}_m, p\}$.

3.2 Estimation of the operator $\mathcal{L} \in L^2_\mu(Z^d)$

Form spectral viewpoint, we can consider $\mathcal{L}$ as self-joint operator in weight Hilbert space $L^2_\mu(Z^d)$ with dot product

$$(f,g)_\mu = \sum_{x\in Z^d} f(x)\bar{g}(x)\mu(x)$$

where $\mu(x) > 0$ and $\sum_{x\in Z^d} \mu(x) < \infty$. Let $\mu(x) = \mu(|x|)$ (even function) and be monotone concave down function of x such that for $x = 0, 1, 2, \cdots$,

$$1 \le \frac{\mu(0)}{\mu(1)} \le \frac{\mu(1)}{\mu(2)} \le \cdots \le \frac{\mu(n-1)}{\mu(n)} \le \cdots .$$

Then for $k > 0$,

$$\frac{\mu(0)}{\mu(1)} \cdot \frac{\mu(1)}{\mu(2)} \cdots \frac{\mu(n-1)}{\mu(n)} = \frac{\mu(0)}{\mu(n)} \leq \frac{\mu(k)}{\mu(k+1)} \cdot \frac{\mu(k+1)}{\mu(k+2)} \cdots \frac{\mu(k+n-1)}{\mu(k+n)} = \frac{\mu(k)}{\mu(k+n)} \tag{3.5}$$

and

$$\frac{\mu(0)}{\mu(n)} \leq \frac{\mu(0)}{\mu(n)} \cdot \frac{\mu(n)}{\mu(n+1)} \cdot \cdots \cdot \frac{\mu(k+n-1)}{\mu(k+n)} = \frac{\mu(0)}{\mu(n+k)}. \tag{3.6}$$

The general form of (3.6) is

$$\frac{\mu(a)}{\mu(n)} \leq \frac{\mu(a)}{\mu(n+1)} \leq \cdots \leq \frac{\mu(a)}{\mu(n+k)}. \tag{3.7}$$

This leads to the following lemma:

Lemma 3.2.1. *For any $a \neq 0$, we have*

$$\sup_x \frac{\mu(x+a)}{\mu(x)} = \sup_x \frac{\mu(x-a)}{\mu(x)} = \frac{\mu(0)}{\mu(|a|)} = h(a). \tag{3.8}$$

Proof. Let's note that

$$\frac{\mu(x+a)}{\mu(x)} = \frac{\mu(z)}{\mu(z-a)} = \frac{\mu(|z|)}{\mu(|a-z|)}$$

$$\frac{\mu(x-a)}{\mu(x)} = \frac{\mu(z)}{\mu(z+a)} = \frac{\mu(|z|)}{\mu(|a+z|)}$$

By relation shown in (3.6) for fixed $|z|$

$$max \frac{\mu(|z|)}{\mu(|a-z|)} = \frac{\mu(|z|)}{\mu(|a|+|z|)} = max \frac{\mu(|z|)}{\mu(|a+z|)}. \tag{3.9}$$

Now consider (3.9) for $z = 0,\ |z| = 1,\ |z| = 2, \cdots, |z| = k$:

$$
\begin{aligned}
&\frac{\mu(0)}{\mu(|a|)} \\
&\max_{z:|z|=1} \frac{\mu(|z|)}{\mu(|a-z|)} = \max_{z:|z|=1} \frac{\mu(|z|)}{\mu(|a+z|)} = \frac{\mu(1)}{\mu(|a|+1)} \\
&\max_{z:|z|=2} \frac{\mu(|z|)}{\mu(|a-z|)} = \max_{z:|z|=2} \frac{\mu(|z|)}{\mu(|a+z|)} = \frac{\mu(2)}{\mu(|a|+2)} \\
&\cdots \\
&\max_{z:|z|=k} \frac{\mu(|z|)}{\mu(|a-z|)} = \max_{z:|z|=k} \frac{\mu(|z|)}{\mu(|a+z|)} = \frac{\mu(k)}{\mu(|a|+k)}
\end{aligned}
$$

□

This lemma 3.2.1 can guide one to find the estimation of upper boundary for $\mathcal{L} \in L^2_\mu(Z^d)$:

Lemma 3.2.2.

$$\parallel \mathcal{L} \parallel_\mu \leq \sum_{z:z\in Z^d} a(z) \sqrt{\frac{\mu(0)}{\mu(a)}} \tag{3.10}$$

Proof. Let's observe that the normalization of $a(z)$ indicates we can get $\sum_{z\in Z^d} a(z) = -a(0) = 1$, so

$$
\begin{aligned}
(\mathcal{L}f)(x) &= \sum_{z\neq 0} (f(x+z) - f(x))a(z) \\
&= \sum_{z\neq 0} f(x+z)a(z) - \sum_{z\neq 0} f(x)a(z) \\
&= \sum_{z\neq 0} (f(x+z)a(z) + f(x+0)a(0) \\
&= \sum_{x\in Z^d} f(x+z)a(z) \\
&= \sum_{y\in Z^d} f(y)a(x-y) = \sum_{x\in Z^d} f(x-z)a(z)
\end{aligned}
$$

Also in $L^2_\mu(Z^d)$ where $\| f \|^2_\mu = \sum_{x\in Z^d} f^2(x)\mu(x) < \infty$, shift function $f(.+a)$ has such norm

$$\begin{aligned}\| f(.+a) \|^2_\mu &= \sum_{x\in Z^d} f^2(x+a)\mu(x) = \sum_{x\in Z^d} f^2(x+a)\frac{\mu(x)\mu(x+a)}{\mu(x+a)}\\ &= \sum_{z\in Z^d} f^2(z)\mu(z)\frac{\mu(z-a)}{\mu(z)} \le \frac{\mu(0)}{\mu(a)} \| f \|^2_\mu,\end{aligned}$$

this is equivalent to

$$\| f(\cdot+a) \|_\mu \le \sqrt{\frac{\mu(0)}{\mu(a)}} \| f \|_\mu \ .$$

Obviously

$$\| \mathcal{L}f \|_\mu \le \sum_{z\in Z^d} |a(z)| \, \| f(\cdot - z) \|_\mu \le \sum_{z\in Z^d} a(z)\sqrt{\frac{\mu(0)}{\mu(a)}} \| f \|_\mu, \tag{3.11}$$

i.e.,

$$\| \mathcal{L} \|_\mu \le \sum_{z\in Z^d} a(z)\sqrt{\frac{\mu(0)}{\mu(a)}}.$$

□

One can also find the another better estimate of $\| \mathcal{L} \|^2$ by follows :

$$\begin{aligned}\|(\mathcal{L}f)(x)\|^2 &= \left(\sum_{y\in Z^d} a(x-y)f(y)\right)^2 = \left(\sum_{y\in Z^d} \frac{a(x-y)}{\sqrt{\mu(y)}} f(y)\sqrt{\mu(y)}\right)^2\\ &\le \sum_{y\in Z^d} \frac{a^2(x-y)}{\mu(y)} \|f\|^2_\mu\end{aligned}$$

Multiplying two sides by $\mu(x)$ and dividing two sides by $|f\|^2_\mu$ to obtain

$$\|\mathcal{L}\|^2_\mu \le \sum_{y\in Z^d} \frac{a^2(x-y)\mu(x)}{\mu(y)} = \sum_{z\in Z^d} \frac{a^2(z)\mu(y+z)}{\mu(y)} \le \sum_{z\in Z^d} \frac{a^2(z)\mu(0)}{\mu(z)} \tag{3.12}$$

.

Lemma 3.2.3. *We call (3.12) as Lemma 3.2.3.*

Finally we have to select $\mu(x)$ in such a way that $\|\mathcal{L}\|_\mu < \infty$. Let's note that for the light tails where $a(z) \leq e^{-\gamma|z|}$ one can put $\mu(z) = e^{-\gamma|z|}$, then

$$\|\mathcal{L}\|_\mu \leq \sum_{z \in Z^d} e^{-\frac{\gamma}{2}|z|} \leq C(\gamma).$$

In the case of the moderate or heavy tails the weight $\mu(x)$ has not exponential order but power asymptotics. Put $\mu(z) = \frac{1}{1+|z|^\beta}$, then

$$\frac{a^2(z)}{\mu(z)} \leq \frac{C|z|^\beta}{|z|^{2d+2\alpha}} \sim \frac{C_1}{|z|^{2d+2\alpha-\beta}},$$

i.e.,

$$\sqrt{\frac{a^2}{\mu}(z)} \leq \frac{C_2}{|z|^{d+\alpha-\frac{\beta}{2}}}.$$

The series $\sum_{z \in Z^d} \sqrt{\frac{a^2}{\mu}(z)}$ convergences if $\beta < 2\alpha$.

Lemma 3.2.3 is especially important in the case of the heavy tails of $a(\cdot)$, we want to include in the theory case of the transient $1-D$ and $2-D$ random walk.

3.3 Existence and uniqueness of solution

Equation (2.1) is written formally. This equation can be considered either in usual Ito form or in the popular physical literature Stratonovich form. We'll use now Itô's approach where

$$\int_0^t u(s,x)dW(s,x) = \lim_{h\to 0} \sum_{k=0}^{n-1} u(kh,x)(W((k+1)h,x) - W(kh,x)).$$

Here $n = [\frac{t}{h}]$ and we understand "lim" in L^2-sense.

The proof of the existence theorem is based on the standard Picard method of successive approximations. We have

$$
\begin{aligned}
&u_0(t,x) = 1\\
&u_1(t,x) = 1 + \varkappa \int_0^t \mathcal{L}u_0(s,x)ds + \int_0^t u_0(t,x)dW(s,x)\\
&\cdots\cdots\\
&u_{n+1}(t,x) = 1 + \varkappa \int_0^t \mathcal{L}u_n(s,x)ds + \int_0^t u_n(t,x)dW(s,x)\\
&\cdots\cdots
\end{aligned}
$$

Now one can repeat the calculations of [1] using information on the boundness of $\mathcal{L}$ in $L^2_\mu(Z^d)$ and the covariance operator $B(\cdot)$. They will give the exponential convergence of $\mu_n(\cdot)$ to the limit

$$u(t,x) = 1+(u_1(t,x)-u_0(t,x))+(u_2(t,x)-u_1(t,x))+\cdots+(u_{n+1}(t,x)-u_n(t,x))+\cdots .$$

Similarly under our conditions on $\mathcal{L}(3.10)$, $\mu(3.3)$, ℓ (3.2) one can prove the uniqueness of the solution. If $u_1(t,x)$, $u_2(t,x)$ are two solutions of our SDE, then $u_1(t,x) - u_2(t,x)$ is also solution and $u_1(t,x) - u_2(t,x) \equiv 0$. Then the standard calculations (like in the finite-dimensional theory) give

$$E[(u_1(t,x) - u_2(t,x))^2] = v(t,x) \leq C \int_0^t v(s,x)ds$$

and Grönwall lemma gives $v(s,x) \equiv 0$, i.e., $u_1(t,x) = u_2(t,x)(P - a.s.)$.

Theorem 3.3.1. *We call this result (existence-uniqueness theorem)*

Equation(2.1) is written formally.We have to understand it in the sense of Stochastic Partial Differential Equations(SPDEs). In the classical theory of SPDEs, we use the integration in Itô's form, like

$$\text{(I)} \int_0^t W_s dW_s = \frac{W_t^2 - t}{2}$$

$$\text{(I)}\ dy(t) = y(t)dW(t), y(0) = 1 \Longrightarrow y(t) = e^{W_t - t/2}$$

etc.

However in the physical literature the white noise $\dot{W}_t$ is necessarily understood dynamically, as the sequence of the Gaussian process $\xi_\varepsilon(t)$, which correlates function $B_\varepsilon(t)$, converges to $\delta_0(t)$ if $\varepsilon \to 0$, see discussion in [1]). We'll use below Itô form of SPDEs, i.e., we can understand equation (2.1) as

$$\text{(I)}\ u(t,x) = 1 + \int_0^t \varkappa \mathcal{L} u(s,x)ds + \int_0^t u(s,x)\xi_s(x)ds, \tag{3.13}$$

or alternately

$$\text{(I)}\ u(t,x) = 1 + \int_0^t \varkappa \mathcal{L} u(s,x)ds + \int_0^t u(s,x)dW(s,x). \tag{3.14}$$

where $u(t,x) \in L^2_\mu(Z^d)$ is $\mathcal{F}_{\leq t}$ adapted. The solution of (2.1) can be also viewed as stochastic partial differential equation(SPDE) in $L^2_\mu(Z^d)$.

Thus (3.14) can be replaced by the Stratonovich's form

$$u^{(S)}(t,x) = 1 + \int_0^t \varkappa \mathcal{L} u(s,x)ds + \int_0^t u(s,x) \circ dW(s,x), \tag{3.15}$$

By the calssical theory, Itô's form solution can rewritten as

$$u^{(I)}(t,x) = 1 + \int_0^t \left(\varkappa \mathcal{L} u(s,x) + \frac{1}{2}B(0) \right) ds + \int_0^t u(s,x) \circ dW(s,x) \tag{3.16}$$

where $B(x) = \sum_{y \in Z^d} \ell(x-y)\tilde{\ell(y)}$ is convolution operator which associated to the covariance of $W(s,x)$. Note that

$$B(0) = \sum_{x \in Z^d} \ell^2(x) < \infty. \tag{3.17}$$

Then (3.15) and (3.16) provide the evidence to exchange $u^{(I)}(t,x)$ and $u^{(s)}(t,x)$ by

$$u^{(I)}(t,x) = e^{-\frac{B(0)t}{2}} u^{(S)}(t,x). \tag{3.18}$$

The following Theorem 3.3.2 gives the useful Feynman-Kac representation of $u(t,x)$:

Theorem 3.3.2. *Under the condition of the existence-uniqueness theorem solution $u(t,x)$ in Itô's form has the representation*

$$u(t,x) = E_x[e^{\int_0^t dW(t-s,x(s))ds - B(0)t/2}]. \tag{3.19}$$

Here $x(s),\ s \geq 0$ is the random walk with the generator $\varkappa\mathcal{L}$ and $E_x(\cdot)$ is the expectation over the law of $x(s), s \geq 0$ for the fixed random environment.

Proof. Consider more general expression: for $s < t,\ x \in Z^d$, $u(t,t,x) \equiv 1$,

$$\begin{aligned} u(t,s,x) &= E_x[e^{\int_s^t dW(t-s,x(s))}] \\ &= E_x[e^{\int_s^{s+h} dW(t-s,x(s)) + \int_{s+h}^t dW(t-s,x(s))}] \\ &= E_x[e^{\int_s^{s+h} dW(t-s,x(s))} u(s+h,t,x(s-h))] \end{aligned}$$

If $\Phi(x(\cdot), W(\cdot,\cdot))$ is functionally dependent on the trajectory of the random walk $x(s),\ s \in [0,t]$ and Winner field $W(s,x(s)),\ s \in [0,t]$ then we'll use notations $\bar{E}_x\Phi,\ \bar{P}_x\{\Phi > a\}$ for $\langle E_x\Phi(\cdot,\cdot)\rangle = E_x\langle\Phi(\cdot,\cdot)\rangle$, $\langle P_X(\Phi > a)\rangle$ where $\langle\ \rangle$ is the integration over the law of the field $W(\cdot,\cdot)$.

From the independence of the increments of $W(s,\cdot)$ it is easy to prove from the representation

$$\left\langle e^{\int_0^t dW(t-s,x(s))} \right\rangle = \left\langle e^{\sum_{i=1}^n (W(t-\tau_i,x_i)-W(t-\tau_{i-1},x_{i-1})} \right\rangle$$

where $\tau_1, \tau_2, \cdots$ are the moments that the walk $x(s),\ s \in [0,t]$ jumps. The last term in the sum is $W(t,x) - W(t-\tau_n, x_n)$ where $0 < \tau_1 < \tau_2 < \cdots < \tau_n < t$ that

$$\left\langle e^{a\int_0^t dW(t-s,x(s))} \right\rangle = e^{\frac{B(0)a^2 t}{2}}$$

i.e.,

$$\left\langle e^{a(\int_0^t dW(t-s,x(s)) - \frac{B(0)t}{2})} \right\rangle = e^{\frac{B(0)(a^2-a)t}{2}}.$$

Then due to Cauchy inequality

$$\left(E_x\left[e^{\int_0^t dW(t-s,x(s)) - \frac{B(0)t}{2}}\right]\right)^a \leq \left(\left(E_x\left[e^{a\int_0^t dW(t-s,x(s)) - \frac{t}{2}}\right]\right)^{\frac{1}{a}}\right)^a = e^{\frac{B(0)(a^2-a)t}{2}}$$

Now we'll prove the Feynman?Kac type representation of the solution $u(t,x)$. Let's remember that we used above the Ito form of the SPDEs. The simplest scalar equation for the geometric Brownian motion has the form

$$\text{(I)}\quad dx_t = x_t db(t),\ x(0) = 1$$

and

$$\text{(I) } x(t) = e^{b(t)-t/2} \text{ (Exponential martingale).}$$

The Stratonowich form of the same equation is

$$\text{(S) } dx_t = x_t \circ db(t),$$

has another solution

$$x(t) = e^{b(t)}.$$

If we consider the family of shot correlated Gaussian stationary process ξ_t^ε such that $E[\xi_t^\varepsilon] = 0$, $E[\xi_{t_1}^\varepsilon \xi_{t_2}^\varepsilon] = \frac{1}{\varepsilon} B(\frac{t_1 - t_2}{\varepsilon}) \to \delta_0(x)$ as $\varepsilon \to 0$, then

$$dx_t^\varepsilon = x_t^\varepsilon \xi_t^\varepsilon dt \Longrightarrow x_t^\varepsilon = e^{\int_0^t \xi_s^\varepsilon ds} \underset{\varepsilon \to 0}{\longrightarrow} e^{w(t)} \quad \text{(in the weak sense)}$$

These formal calculation explain why physicists not use Ito but use Stratonovich SPDEs.

Now let's start to prove that (3.19) is finite for $t > 0$, i.e., $E_x \left[e^{\int_0^t dW(s,x_s) - B(0)t/2} \right] < \infty$. I Define $Y_s = e^{\int_0^t dW(s,x_s)}$ which has log-normal distribution, in more specific, $log(Y_s)$ is normally distributed with mean 0 and variance $\int_0^t \sum_{z \in Z^d} \ell^2(x_s - z) ds$. Thus

$$E_x \left[e^{\int_0^t dW(s,X_s)} \right] = e^{\frac{1}{2} \int_0^t \sum_{z \in Z^d} \ell^2(X_s - z) ds} \leq e^{\frac{1}{2} \sum_{z \in Z^d} \ell^2(z) t} < \infty$$

because $\sum_{z \in Z^d} \ell^2(z) < \infty$. So,

$$E_x \left[e^{\int_0^t dW(s,x_s) - B(0)t/2} \right] \leq E_x \left[e^{\int_0^t dW(s,x_s)} \right] < \infty.$$

Now we can prove that (3.19) indeed is the real solution.

$$u(t+h,x) = E_x \left[e^{\int_0^h dW(s,X_s) - B(0)h/2} e^{\int_h^{t+h} dW(s,x_s) - B(0)t/2} \right]$$
$$= E_x \left[e^{\int_0^h dW(s,x_s) - B(0)h/2} E_{x_h} [e^{\int_0^t dW(s,x_s) - B(0)t/2}] \right]$$
$$= E_x \left[\left(1 + \int_0^h dW(s,x_s) + (\int_0^h dW(s,x_s))^2 + \cdots \right) \left(B(0) - B(0)\frac{h}{2} + \bar{\bar{o}}(h^2) \right) u(t,x_h) \right]$$

Let's remember that

$$P\{x(t+h) = x | x(t) = x\} = e^{-\varkappa h} = 1 - \varkappa h + 0(h^2)$$
$$P\{x(t+h) = x + z | x(t) = x\} = a(z)h + \bar{\bar{o}}(h^2),$$

thus

$$\begin{aligned} u(t+h,x) &= \left[(1-\varkappa h)u(t,x) + \sum_{z\neq 0, z\in Z^d} a(z)hu(t,x+z)\right]\left(1+\xi_t(x)h + o(h^2)\right) \\ &= u(t,x) + \xi_t(x)hu(t,x) + \varkappa h \sum_{\neq 0, z\in Z^d} (u(t,x+z) - u(t,x))a(z) \\ &= u(t,x) + \xi_t(x)hu(t,x) + h\varkappa\mathcal{L}u(t,x) \end{aligned}$$

Consequently

$$\frac{u(t+h,x) - u(t,x)}{h} = \xi_t(x)u(t,x) + \varkappa\mathcal{L}u(t,x)$$

As $h \to 0$,

$$\frac{\partial u(t,x)}{\partial t} = \varkappa\mathcal{L}u(t,x) + \xi_t(x)u(t,x).$$

Evidently (3.19) is indeed the solution to our model (2.1). □

3.4 Moments equation

The purpose of this section is to prove existence and equations of the moments of all order of $u(t,x)$ which is solution of Ito's equation (3.16). For positive integer $p \geq 1$, $t \geq 0$, let $x_1, x_2, x_3, \cdots$ be fixed points in Z^d and denote p^{th} moment by $m_p(t, x_1, x_2, \cdots, x_p)$, $x = (x_1, \cdots, x_p)$ such that $m_p(t, x_1, x_2, \cdots, x_p) =$

$\langle u(t,x_1)u(t,x_2)\cdots u(t,x_p)\rangle$. By Feynman-Kac formula(3.19) we have

$$\begin{aligned} m_p(t,x_1,x_2,\cdots,x_p) &= \langle u(t,x_1)u(t,x_2)\cdots u(t,x_p)\rangle \\ &= \left\langle E_{x_1}[e^{\int_0^t dW_1(t-s,x_s))-B(0)t/2}]\cdots E_{x_p}[e^{\int_0^t dW_p(t-s,x_s))-Bb(0)t/2}]\right\rangle \\ &= \left\langle E_{\{x_1,\cdots,x_p\}}[e^{\int_0^t dW_1(t-s,x_s))+\cdots+\int_0^t dW_p(t-s,x_s))-\frac{B(0pt}{2}}]\right\rangle \\ &= E_x\left[\langle e^{\int_0^t dW_1(t-s,x_s))+\cdots+\int_0^t dW_p(t-s,x_s)}\rangle\right]e^{-\frac{B(0)pt}{2}} \end{aligned}$$

Here $\int_0^t dW_1(t-s,x_s)) + \cdots + \int_0^t dW_p(t-s,x_s)$ is Gaussian process with mean 0 and covariance

$$\left\langle \left(\int_0^t dW_1(t-s,x_s)) + \cdots + \int_0^t dW_p(t-s,x_s)\right)^2 \right\rangle = t\sum_{1\le i<j\le p} B(x_i-x_j).$$

And conditions (3.2) shows that the right side of above equation is finite for each $t\geq 0$, thus $m_p(t,x_1,x_2,\cdots,x_p)<\infty$.

Now we can solve the equation of the moments by using Ito's stochastic calculus and the true that $u(t,x_1),\cdots,u(t,x_p)$ are semimartigales. It's clear that function $u(t,x_1)\cdots u(t,x_p)$ is smooth function with p-variables, Ito's formula provides

$$\begin{aligned} u(t,x_1)\cdots u(t,x_p) &= u(0,x_1)\cdots u(0,x_p) \\ &\quad + \varkappa\sum_{i=1}^p\int_0^t u(s,x_1)\cdots u(s,x_{i-1})\mathcal{L}_{x_i}u(s,x_i)\cdots u(s,x_p)ds \\ &\quad + \sum_{i=1}^p\int_0^t u(s,x_1)\cdots u(s,x_p)dW(s,x_i) \\ &\quad + \sum_{1\le i<j\le p}\int_0^t W(s,x_i)W(s,x_j)u(s,x_i)\cdots u(,x_j)ds. \end{aligned}$$

Now we can derive the equations for $m_p(t,x_1,x_2,\cdots,x_p)$ using the Ito formula.

Let $f_p(t,w) = u(t,x_1)\cdots u(t,x_p)$. Then

$$
\begin{aligned}
df_p(t,w) &= d(u(t,x_1)\cdots u(t,x_p)) \\
&= d(u(t,x_1))u(t,x_2)\cdots u(t,x_p) + u(t,x_1)\cdots u(t,x_{n-1})du(t,x_p) \\
&+ \sum_{i<j} du(t,x_i)du(t,x_j)
\end{aligned}
$$

But $du(t,x_i) = \varkappa\mathcal{L}u(t,x_i)dt + u(t,x_i)dW(t,x_i)$. Since $dW(t,x_i)dW(t,x_j) = B(x_i - x_j)dt$, it gives the equation

$$
\begin{aligned}
&d\langle u(t,x_1)\cdots u(t,x_p)\rangle = dm_p(t,x_1,\cdots,x_p) \\
&= \sum_{i=1}^{p} \varkappa\mathcal{L}_{x_i} m_p(t,x_1,\cdots,x_p)dt + m_p(t,x_1,\cdots,x_p)B_p(x_1,\cdots,x_p)dt \\
&B_p(x_1,\cdots,x_p) = \sum_{i<j} B(x_i - x_j)
\end{aligned}
$$

Finally,

$$
\frac{dm_p(t,x_1,\cdots,x_p)}{dt} = \varkappa\left(\sum_{i=1}^{p} \mathcal{L}_{x_i} m_p(t,x_1,\cdots,x_p)\right) + \left(\sum_{i<j} B(x_i - x_j)\right) m_p(t,x_1,\cdots,x_p) \tag{3.20}
$$

with the initial condittion $m_p(0,x_1,\cdots,x_p) = 1$.

In more details, the equation for first moment $m_1(t,x) = \langle u(t,x)\rangle$ is

$$
\begin{aligned}
&\frac{\partial m_1(t,x)}{\partial t} = \varkappa\mathcal{L}m_1(t,x),\ (t,x) \in [0,\infty) \times Z^d \\
&m_1(0,x) = 1.
\end{aligned} \tag{3.21}
$$

By Feynman-Kac formula(3.19), $m_1(t,x) \equiv 1$.

For 2nd moment $m_2(t,x_1,x_2) = \langle u(t,x_1)u(t,x_2)\rangle$, we have

$$\begin{aligned} \frac{\partial m_2(t,x_1,x_2)}{\partial t} &= \varkappa(\mathcal{L}_{x_1}+\mathcal{L}_{x_2})m_2(t,x,y) + B(x-y)m_2(t,x_1,x_2) \\ m_2(0,x_1,x_2) &= \langle u(0,x_1)u(0,x_2)\rangle = 1. \end{aligned} \tag{3.22}$$

And due to the fact for fixed t, $u(t,x)$ is homogeneous in space, that is, $m_2(t,x_1,x_2) = m_2(t,x_1-x_2) = m_2(t,x_1-x_2) = m_2(t,v)$, thus preceding equation is equivalent to

$$\begin{aligned} \frac{\partial m_2(t,v)}{\partial t} &= \mathcal{H}_2 m_2(t,v), \mathcal{H}_2 = 2\varkappa\mathcal{L}_v + B(v) \\ m_2(0,v) &= 1 \end{aligned} \tag{3.23}$$

For general case $p \geq 3$, (3.20) can be rewritten as

$$\frac{\partial m_p}{\partial t} = \mathcal{H}_p m_p,\ \mathcal{H}_p = \varkappa(\mathcal{L}_{x_1} + \cdots \mathcal{L}_{x_p}) + \sum_{1\leq i<j\leq p} B(x_i - x_j). \tag{3.24}$$

Here Hamilton $\mathcal{H}_p$ is classical "p-particle" Schrodinger operator on Z^d. The spectral analysis of $\mathcal{H}_p$ on $L^2_\mu(Z^d)$ is critical in modern mathematical physics. One main purpose of this work is to investigate this asymptotic behavior.

3.5 Transition Probability $p(t,x,y)$

Let us find out the transition probability $p(t,x,y) = P(x(t) = y|x(0) = x)$ of the random walk $x(t)$ associated with $2\varkappa\mathcal{L}$. It is well known that $p(t,x,y)$ is the fundamental solution of the following parabolic problem

$$\begin{aligned} \frac{dp}{dt} &= 2\varkappa\mathcal{L}p, t > 0 \\ p(0,x,y) &= \delta_y(x) \end{aligned} \tag{3.25}$$

Applying the Fourier transform $\hat{p}(t,x,k) = \sum_{y\in Z^d} p(t,x,y)e^{iky}$ to both sides of equation group (3.25),then taking the inverse Fourier transform, we will get the solution to (3.25):

$$p(t,x,y) = \frac{1}{(2\pi)^d} \int_{T^d} e^{2\varkappa\hat{\mathcal{L}}(k)t} e^{ik(x-y)} dk, \;\; k \in T^d = [-\pi,\pi]^d \tag{3.26}$$

where $\hat{\mathcal{L}}(k) = \sum_{z\neq 0}(cos(k,z)-1)a(z) = \hat{a}(k)-1$ is the Fourier symbol of the operator $\mathcal{L}$ and $\hat{a}(k) = \sum_{z\neq 0} cos(k,z)$.

Thus

$$p(t,x,x) = p(t,0,0) = \frac{1}{(2\pi)^d} \int_{T^d} e^{2\varkappa\hat{\mathcal{L}}(k)t} dk, \;\; k \in T^d. \tag{3.27}$$

In the future we'll study the phase transition with represent to $\varkappa$ of the top eigenvalue for the Hamiltonian $\mathcal{H} = 2\varkappa\mathcal{L} + V(x), \;\; V(x) \geq 0, \;\; x \in Z^d$. In this situation it is convenient to introduce the standard diffusivity $\varkappa_0 = \frac{1}{2}$, i.e., $2\varkappa_0 = 1$. Let's denote in this case

$$p_0(t,x,y) = \frac{1}{(2\pi)^d} \int_{T^d} e^{t\hat{\mathcal{L}}(k)+ik(x-y)} dk.$$

Then (for arbitrary $\varkappa$)

$$p(t,x,y) = p_0(2\varkappa t, x, y).$$

Due to CLT in the case of finite second moment $\sigma^2 = \sum_{z\in Z^d} |z|^2 a(z)$

$$p_0(t,x,y) \sim \frac{exp\left[-\frac{\Pi^{-1}\{x-y,x-y\}}{2}\right]}{(2\pi t)^{d/2}\sqrt{det\Pi}}$$

if $t \to \infty$, $|x-y| = \underline{o}(\sqrt{t})$. Here Π is the correlation matrix of the random walk $x(t)$ in the case $2\varkappa_0 = 1$:

$$\Pi = -\left[-\frac{\partial^2 \hat{\mathcal{L}}}{\partial k_i \partial k_j}\right]$$

Due to our general assumption of the connectivity $det(\Pi) > 0$. It means that for appropriate constant C^+ depending only on $\hat{a}(k)$ and all $t \geq 1$ we have estimate

$$p_0(t,x,x) \leq \frac{C^+}{t^{d/2}}, \;\; t \geq 1. \tag{3.28}$$

The important and non-trivial questions on the estimation of C^+ will be discussed in the future. The following well known result is corollary of (3.28) .

Theorem 3.5.1. *If* $\sigma^2 = \sum_{z \in Z^d} |z|^2 a(z) < \infty$, *then random walk* $x(t)$ *is transient for* $d \geq 3$ *and recurrent for* $d = 1,\ 2$.

In the heavy tailed jump distribution under regularity conditions (2.4) similarly for $t \geq 1$

$$p_0(t,x,x) = p_0(t,0,0) \leq \frac{C^+}{t^{d/\alpha}},\ \ 0 < \alpha < 2.$$

Corollary 3.5.2. *Under regularity condition of* $a(z)$, *the random walk* $x(t)$ *is recurrent if* $d = 1$ *and* $\alpha \in (1,2)$, *or* $d = 2$ *and* $\alpha = 2$. *Otherwise it is transient (in particular it always transient* $d \geq 3$*).*

CHAPTER 4: Spectral analysis of the operator $\mathcal{H}_2 = 2\varkappa\mathcal{L} + B(x)$ and the problem of intermittency

This chapter is devoted to the spectral analysis of the operator $\mathcal{H}_2 = 2\varkappa\mathcal{L} + B(x)$ and the problem of intermittency. The first moment of the field $u(t,x)$ equals to 1 identically. The fluctuations of this field for $t \to \infty$ depends on the second moment $m_2(t, 0, x_1 - x_2) = m_2(t, z)$. If this moment is bounded then the fluctuations are moderate, this is the regular case. If $m_2(t,z)$ tends to ∞ exponentially it is the manifestation of intermittency. In this situation the main contribution to $m_2(\cdot,\cdot)$ give the very high and very spare peaks, see details in [14].

To find out the asymptotic for $m_2(\cdot,\cdot)$, it is important to know the top point of the spectrum for the operator $\mathcal{H}_2$, i.e., $\lambda_0(\mathcal{H}_2)$. Exponential growth of $m_2(\cdot,\cdot)$ corresponds to the existence of positive top eigenvalue $\lambda_0(\mathcal{H}_2) > 0$.

For the better orientation in this big chapter let's start from the review of these results: new effects in the spectral theory of non-local Schrödinger operator with positively definite potential.

Let $\varkappa\mathcal{L}\psi(x) = \varkappa\sum_{x\in Z^d}(\psi(x+z) - \psi(x))a(z)$ be the non-local Laplacian. It can be also considered as the generator of the Markov Chain(random walk) $x(t)$. We know that $x(t)$ is transient iff

$$I = \int_{T^d} \frac{dk}{1 - \hat{a}(k)} < \infty$$

and recurrent iff

$$I = \int_{T^d} \frac{dk}{1 - \hat{a}(k)} = \infty.$$

In particular for $d \geq 3$ (with connectivity and non-periodicity conditions, it is sufficiently to suppose that $a(z) > 0$ for $|z| = 1$, $a(z) = a(-z)$) any random walk is transient. If $d = 1$ it is transient under the following regularity condition: distribution of $x(t)$ belongs to the domain of attraction of the symmetric stable law with parameter $\alpha < 1$. If $d = 2$ and $\sum_{z \in Z^d} |z|^2 a(z) < \infty$ then random walk is recurrent. Again under some regularity conditions $x(t)$ belongs to the domain of attraction of the symmetric stable law with parameter $\alpha < 2$ and then $x(t)$ is transient.

The Schrödinger operator $\mathcal{H} = \varkappa\mathcal{L} + V(x)$ with positive potential $V(x)$, particularly for $V(x) = \delta_{x_0}(x)$, has positive eigenvalue $\lambda_0(\mathcal{H}) > 0$ for arbitrary diffusivity $\varkappa$ iff $x(t)$ is recurrent. For transient $x(t)$ the situation is different (spectral bifurcation): there exists $\varkappa_{cr} > 0$ such that for $\varkappa < \varkappa_{cr}$ the positive eigenvalue $\lambda_0(\mathcal{H}) > 0$ exists, while for $\varkappa > \varkappa_{cr}$ the discrete spectrum of $\mathcal{H}$ is empty.

The spectrum of $\varkappa\mathcal{L}$ (as a closed set, the support of the spectral measure) equals to $[-\iota\varkappa, 0]$ where $-\iota = \min_{k \in T^d}[\hat{a}(k) - 1]$, the spectrum of $\mathcal{H} = \varkappa\mathcal{L} + V(x)$ for $V(x) \geq 0,\ V(x) \to 0,\ |x| \to \infty$ contains two parts: $Sp_{ess}(\mathcal{H}) = [-\iota\varkappa, 0]$ and $Sp_d(\mathcal{H})$. The discrete spectrum is at most countable with possible accumulation point 0. Of course (in transient case) it can be empty.

What can be say about spectral measure of the selfjoint operator $\mathcal{H} = \varkappa\mathcal{L} + V(x)$? Let's start from Laplacian $\varkappa\mathcal{L}$. The Fourier representation of $\varkappa\mathcal{L}$ is simply operator of multiplication on $\varkappa\hat{\mathcal{L}} = \varkappa(\hat{a}(k) - 1),\ \hat{a}(k) = \sum_{n \in Z^d} a(n) e^{i(k,n)},\ k \in T^d$ in the Hilbert space $L^2(T^d, dk)$. It is known that for analytic function $\hat{a}(k)$ the spectral measure of the operator of multiplication by $\varkappa(1 - \hat{a}(k))$ is absolutely continuous (a.c.)(proposition 4.2.1). If $\hat{a}(k)$ is only infinitely differentiable ($C^\infty-$class) then spectral measure of $\varkappa\mathcal{L}$ can contain discrete component(corresponding example see below)(proposition 4.2.2)) .

However, if the function $\hat{a}(k),\ k \in T^d$ belongs to class $C^2(T^d)$ and has only finitely many critical point $K_m,\ m = 1, 2, \cdots, N$ where $\nabla\hat{a}(K_m) = 0$ and $Hess\hat{a}(K_m)$ is non-degenerated $(d \times d)$ matrix:

$$det(Hess\ \hat{a}(K_m)) = det \left[\frac{\partial^2 \hat{a}(k)}{\partial k_i \partial k_j} |_{\substack{k=K_m,\ m=1,\cdots,N \\ i,j=1,\cdots,d}} \right] \neq 0$$

then again spectral measure is a.c., such function are called generic or Morse type function.

Now we'll present several results which will demonstrate effects of the positively definite potential $B(x)$ on the discrete spectrum of the Hamiltonian $\mathcal{H} = \varkappa\mathcal{L} + B(x)$ (or $\varkappa\mathcal{L} + V(x)$, general $V(x),\ V(x) \to 0,\ x \to \infty$).

a) Transient case

If $\sum_{z\in Z^d} |z|^2 a(z) < \infty$ and $d \geq 3$ then using CLT asymptotics $p_0(t,x,x) \sim \frac{C}{t^{d/2}}$ and the general theorem form [10] one can prove the Cwikel-Lieb-Rozenblum type estimate

$$\sharp\{\lambda_i > 0\} = N_0(V) \leq C(d,\hat{a}) \sum_{z\in Z^d} |\frac{V(x)}{\varkappa}|^{d/2}.$$

It means that for $C(d,\hat{a})\sum_{z\in Z^d} |\frac{V(x)}{\varkappa}|^{d/2} < 1$, the inequality

$$\varkappa > \left(C(d,\hat{a}) \sum_{z\in Z^d} |V(x)|^{d/2} \right)^{2/d}$$

implies that $N_0(V) = 0$, i.e., $Sp_d(\mathcal{H})$ is empty.

If $\sum_{z\in Z^d} |z|^2 a(z) = \infty$ but there are strong additional assumption that after normalization distribution of $x(t)$ converges to the stable law with parameter $0 < \alpha < 2$ if $d = 2$ or $0 < \alpha < 1$ if $d = 1$ (see above and references [8] [9] [11] [12]), then

$$N_0(\mathcal{H}) \leq C(d,\hat{a}) \sum_{z\in Z^d} |\frac{V(x)}{\varkappa}|^{d/\alpha}.$$

i.e., for

$$\varkappa > \left(C(d, \hat{a}) \sum_{z \in Z^d} |V(x)|^{d/\alpha} \right)^{\alpha/d}$$

again there is no positive eigenvalue.

If $\sum_{z\in Z^d} |z|^2 a(z) < \infty$ but $\sum_{x\in Z^d} |V(x)|^{d/2} = \infty$ the situation is more complicated. it can be explained bt the following theorems:

Theorem 4.0.1. *Consider the lattice operator* $\Delta f(t,x) = \sum\limits_{\substack{z:|z-x|=1 \\ z\in Z^d}} (f(x+z) - f(x))$ *and the Hamiltonian* $\mathcal{H} f = \varkappa \Delta f + B(x) f$. *Assume that for multidimensional case* $d \geq 3$, $B(x) \geq \frac{C_0}{1+|x|^2}$, C_0 *is large enough constant. Then* $\sum |B(x)|^{d/2} = \infty$, *then* $N_0(B) = \infty$.

Theorem 4.0.2. *Consider again for* $\mathcal{L} = \Delta,\ d \geq 3$, *and the sparse potential*

$$V(x) = \sum_n \sigma \delta(x - x_n)$$

where $|x - x_n| \to \infty$ *sufficiently fast,* σ_{cr} *corresponds to the one-point potential* $V(x) = \sigma\delta(x - x_0)$. *If* $\sigma < \sigma_{cr}$ *operator* $\mathcal{H}_{\prime} = \Delta + \sigma\delta(x)$ *has no positive eigenvalue.*

See proof below. **b)Recurrent case**

We already formulated the following result: if $V(x) \geq 0$ but $V(x)$ is not identical to 0, and the random walk is recurrent then $\lambda_0(\mathcal{H}) > 0$ exists for any $\varkappa > 0$.

Now we consider the case when $V(x)$ contains the negative part. Note that if $V(x) = B(x)$ is positively definite then $\sum_{x\in Z^d} B(x) = \left(\sum_{x\in Z^d} \ell(x)\right)^2 \geq 0$. We have the following:

Conjecture: If $x(t)$ is recurrent and $B(x)$ is positively definite then $\lambda_0(H) > 0$. Now we have only particular results in this direction. Let's present two of them.

Theorem 4.0.3. *Consider the spectral problem*

$$H\psi(x) = \varkappa\Delta\psi + V(x)\psi = \lambda\psi(x), x \in Z^1, V(x) \in L^1(Z^1), \sum_{z\in Z^1} V(x) = \delta > 0$$

then $\forall(\varkappa > 0),\ \exists(\lambda_0(\mathcal{H})) > 0.$

Proposition 4.0.4. *Assume that* $b_0(x)$ *equals to* 1 *if* $x = 0$ *and to* -1 *if* $x = 1$*, then*

$$B_0(x) = \begin{cases} 2,\ x = 0 \\ -1,\ x \pm 1 \\ 0,\ |x| > 1 \end{cases},\ i.e., \sum_{x\in Z^d} B_0(x) = 0.$$

We state that the maxi eigenvalue $\lambda_0(\mathcal{H})$ *of* $\mathcal{H} = \Delta + \beta V(x)$ *is positive and* $\lambda_0(\mathcal{H}) \sim \beta^4$ *if* $\beta \ll 1$.

See proof below.

4.1 Spectral proposition of non-local operator $\varkappa\mathcal{L}$

Let's recall that the lattice Laplacian $\varkappa\Delta\psi(x) = \varkappa\sum_{x':|x'-x|=1}(\psi(x') - \psi(x))$ in Fourier representation equals to the operator of multiplication by

$$\varkappa\hat{\Delta}(k) = \varkappa\left(\sum_{i=1}^{d}(cos(k_j) - 1)\right).$$

The spectrum of $\varkappa\Delta$ (as a set) is

$$Range(\varkappa\hat{\Delta}) = [-4d\varkappa, 0],\ -4d = \min_{k\in T^d} \hat{\Delta}(k). \tag{4.1}$$

For the non-local Laplacian $\varkappa\mathcal{L}$ and Hamiltonian $\mathcal{H} = \varkappa\mathcal{L} + V(x)$ the following theorem gives the description of the spectrum.

Theorem 4.1.1. *Assume* $\mathcal{H} = \varkappa\mathcal{L} + V(x)$ *where* $V(x) \geq 0$ *and* $V(x) \to 0$ *as* $|x| \to \infty$, *then* $Sp(\mathcal{H}) = Sp_{ess}(\mathcal{H}) \cup Sp_d(\mathcal{H})$ *where* $Sp_{ess}(\mathcal{H}) = Sp_{ess}(\varkappa\mathcal{L}) = [\alpha, 0]$ *and*

$$\alpha = \min_{k \in T^d} \varkappa\hat{\mathcal{L}}(k) = \min_{k \in T^d} \varkappa \sum_{z \neq 0} a(z)(cos(k,z) - 1) < 0.$$

$Sp_d(\mathcal{H})$ *is most countable set of eigenvalues with possible accumulation point* 0 *or* α. *The top eigenvalue* $\lambda_0 > 0$ *(if it exists) is simple and corresponding eigenfunction* $\psi_0(\lambda)$ *is strictly positive. If the random walk* $x(t)$, $t \geq 0$ *on* Z^d *is recurrent then* $N_0(V) > 0$. *If* $x(t)$ *is transient and the potential is small enough in the appropriated sense,* $N_0(V) = 0$.

Let's recall that $x(t)$ is recurrent if $\int_{T^d} \frac{dk}{1-\hat{a}(k)} = \infty$ and transient if this integral is finite. If $d \geq 3$ then $x(t)$ is transient for arbitrary $\mathcal{L}$ (i.e., arbitrary distribution of the jump $a(\cdot)$).

If $d = 2$ then the random walk is recurrent if $\sigma^2 = \sum_{z \in Z^2, z \neq 0} |z|^2 a(z) < \infty$. If $\sigma^2 = \infty$ then under minimal additional assumption the walk is transeint. In particular it is transient if $a(z) > \frac{c}{1+|z|^{2+\alpha}}, 0 < \alpha < 2$. For $1-D$ case and $a(z) \sim \frac{c}{|z|^{1+\alpha}}, |z| \to \infty$, recurrent case corresponds condition $\alpha > 1$ and transient to condition $\alpha \leq 1$. The proof can be found in [8] [9].

Now Let's illustrate theorem 4.1.1 by following example.

Example 4.1.2. If $V(x) = \sigma\delta_0(x),\ \sigma > 0$, let's consider the spectral problem for

$$\mathcal{H}\psi(x) = \varkappa(\mathcal{L}\psi)(x) + \sigma\delta_0(x)\psi(x).$$

Assume that $\mathcal{H}\psi = \lambda_0\psi$, i.e., $\varkappa\mathcal{L}\psi(x) + \sigma\psi(0) = \lambda_0\psi(x)$. Then Fourier transformation results in

$$-\varkappa\hat{\mathcal{L}}(k)\hat{\psi}(k) + \sigma\psi(0) = \lambda_0\hat{\psi}(k),\ k \in T^d = [-\pi, \pi]^d \tag{4.2}$$

where $\hat{\psi}(k) = \frac{\sigma\psi(0)}{\lambda_0 + \varkappa\hat{\mathcal{L}}(k)}$, $\hat{\mathcal{L}}(k) = \sum a(z)(1 - cos(k,z))$ or

$$\frac{\psi(0)}{\sigma} = \frac{\psi(0)}{(2\pi)^d}\int_{T^d}\frac{dk}{\lambda_0 + \varkappa\hat{\mathcal{L}}(k)}.$$

If $\sum |z|^2 a(z) < \infty$, then

$$\hat{\mathcal{L}}(k) \sim C_0|k|^2, as\ k \to 0,$$

i.e.,

$$\psi(0) = \frac{\sigma\psi(0)}{(2\pi)^d}\int_{T^d}\frac{dk}{\lambda + \varkappa k^2(1 + \bar{\bar{o}}(k^4))}. \tag{4.3}$$

and we have final equation for $\lambda_0 = \max Sp(\mathcal{H})$

$$\frac{1}{\sigma} = \frac{1}{(2\pi)^d}\int_{T^d}\frac{dk}{\lambda + \varkappa k^2(1 + \bar{\bar{o}}(k^4))} = I(\lambda).$$

$I(\lambda)$is bounded if $\lambda > 0$ and $d \geq 3$ but unbounded if $d = 1, 2$ (see figure (4.1)).

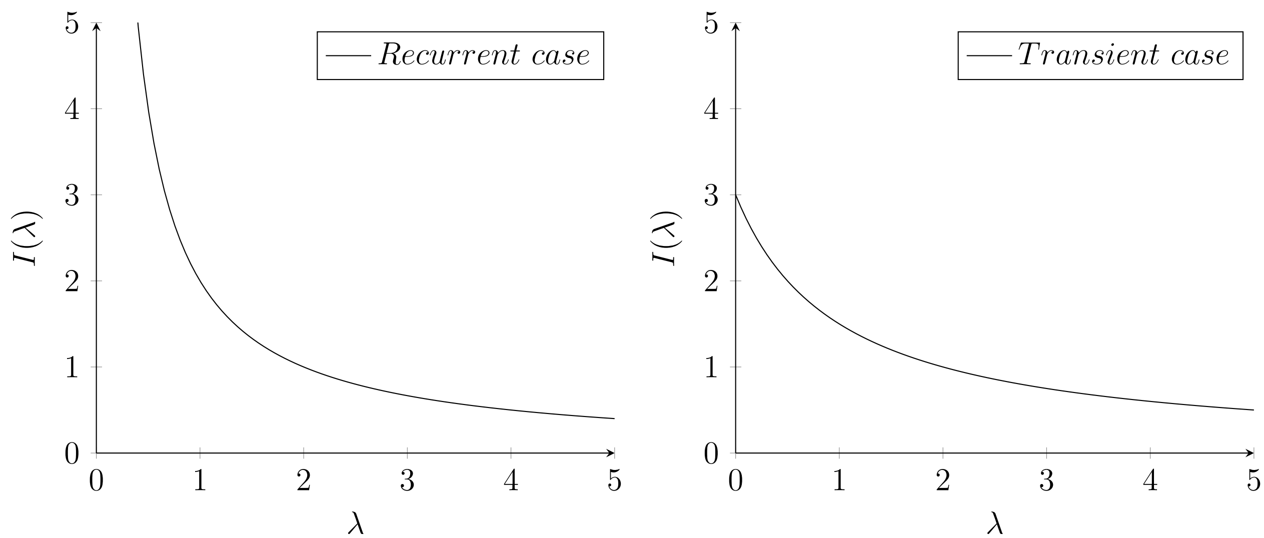

Figure 4.1: The graph of $I(\lambda)$ in recurrent case (top) and in transient case (bottom).

If $a(z) \sim \frac{C(\dot{z})}{|z|^{d+\alpha}}$ where $0 < \alpha < 2$, $\dot{z} = \frac{z}{|z|} \in S^{d-1}$ and $C(\dot{z}) \subset C(T^d)$ is positive and continuous function. then

$$1-\hat{a}(k)\sim C(\dot{k})|k|^{\alpha}. \tag{4.4}$$

In the cases where $d=1$ and $\alpha\in(1,2)$ or $d=2$ and $\alpha=2$, random walk is recurrent due to $\int_{T^d}\frac{dk}{\hat{\mathcal{L}}(k)}=\infty$ and $N_0(V)=1$ for any $\varkappa,\sigma$, otherwise the random walk is transient and $N_0(V)=0$ for sufficient large $\varkappa$.

To prove theorem 4.1.1, consider the operator

$$(\mathcal{H}\psi)(x)=\varkappa(\mathcal{L}\psi)(x)+V(x)\psi(x),\ x\in Z^d \tag{4.5}$$

acting in the Hilbert space $L^2(Z^d)$. Here

$$(\mathcal{L}\psi)(x)=\sum_{z\in Z^d}(\psi(x+z)-\psi(x))a(z),\ a(z)\geq 0 \tag{4.6}$$

In the Fourier space

$$\hat{\psi}(k)=\sum_{x\in Z^d}\psi(x)e^{i(k,x)},\ k\in T^d=[-\pi,\pi]^d,\ \psi(x)\in L^2(Z^d), \tag{4.7}$$

we have $\hat{\psi}(k)\in L^2(T^d,dk)$ and $(\widehat{\mathcal{L}\psi})(k)=\hat{\mathcal{L}}(k)\hat{\psi}(k)$, $\hat{\mathcal{L}}(k)=\sum_{z\neq 0}(cos(k,z)-1)a(z)$. But because of Fourier isomorphism, we have

$$Sp(\varkappa\hat{\mathcal{L}})=Sp(\varkappa\mathcal{L})=Range(\varkappa\hat{\mathcal{L}}(k))=[-\iota\varkappa,0] \tag{4.8}$$

where $-\iota=\min_{k\in T^d}[\sum_{z\neq 0}(cos(kz)-1)a(z)]$ and $-\iota\varkappa\geq -2\varkappa$.

Now consider the general potential $V(x)$ such that $V(x)\geq 0$ and $V(x)\to 0$ as $|x|\to\infty$. Now let's present $V(x)$ in the form

$$V(x)=V_{\delta}^{(1)}(x)+V_{\delta}^{(2)}(x)$$

where

$$V_\delta^{(1)}(x) = \begin{cases} 0, & \text{if } V(x) > \delta \\ V(x), & \text{if } 0 \le V(x) \le \delta \end{cases}$$

$$V_\delta^{(2)}(x) = \begin{cases} V(x), & \text{if } V(x) > \delta \\ 0, & \text{if } 0 \le V(x) \le \delta \end{cases}$$

Since $|V_\delta^{(1)}(x)| \le \delta$ and $Sp(\varkappa\Delta) = [-\iota\varkappa, 0]$, the spectrum of $\mathcal{H}^{(1)} = \varkappa\mathcal{L} + V_\delta^{(1)}$ belongs to the interval $[-\iota\varkappa, \delta]$, i.e., resolvent of $\mathcal{H}^{(1)}$ is analytic for $Re\lambda > \delta$. As result, $\mathcal{H}^{(1)}$ has no eigenvalue λ_i which is greater than δ.

The operator

$$\mathcal{H} = \varkappa\mathcal{L} + V(x) = \mathcal{H}^{(1)} + V_\delta^{(2)}$$

is the finite rank perturbation of $\mathcal{H}^{(1)}$. This rank $R(\delta) = \sharp\{x : V(x) > \delta\}$. It means that $\mathcal{H}$ has at most $R(\delta)$ eigenvalue $\lambda_i > \delta$.

It shows that the spectrum of $\mathcal{H}$ in the region $\lambda > 0$ is discrete with possible accumulation point $\lambda_0 = 0$.

If the discrete positive spectrum is empty then the Dirichlet form

$$(\mathcal{H}\psi, \psi) \le 0, \forall(\psi \in L^2(Z^d)). \tag{4.9}$$

If this spectrum is non-empty then the top-eigenvalue is

$$\lambda_0 = \max_{\substack{\psi : \|\psi\| = 1, \\ \psi \in L^2(Z^d)}} (\mathcal{H}\psi, \psi) > 0 \tag{4.10}$$

and in addition, λ_0 is a simple eigenvalue and corresponding eigenfunction $\psi_0(x)$ is strictly positive. In fact equation $\mathcal{H}\psi_0 = (\varkappa\mathcal{L} + V(x))\psi_0$ can be presented in the form

$$\varkappa\mathcal{L}\psi_0 + (V(x) - \mu)\psi_0 = (\lambda_0 - \mu)\psi_0. \tag{4.11}$$

The operator $\varkappa\mathcal{L} + (V - \mu) = \tilde{\mathcal{H}}$ is strictlt negatively defined and the Green function, i.e., kernel of

$$\tilde{G} = \int_0^\infty e^{t\tilde{\mathcal{H}}} dt = -\tilde{\mathcal{H}}^{-1}$$

is strictly positive. Namely,

$$\tilde{G}(x, y) = \int_0^\infty q(t, x, y) dt$$

where $q(t, x, y)$ is the transition density of the sub-process which can be constructed by the killing of the chain $x(t)$ with generator $\varkappa\mathcal{L}$ by the negative potential $V - \mu$, (Feynman?Kac formula), then

$$\frac{1}{\mu - \lambda_0} \psi_0 = \tilde{G} \psi_0$$

and as result

$$\frac{1}{\mu - \lambda_0} = \max_{\psi : \|\psi\| = 1} (\tilde{G}\psi, \psi) = (\tilde{G}\psi_0, \psi_0).$$

Since $\tilde{G}(x, y) > 0$, we can apply the Perron-Frobenius theorem. It gives the simplicity of $\frac{1}{\mu - \lambda_0}$, i.e., λ_0, and the positivity of ψ_0.

The variation principle gives that

$$\begin{aligned} \mathcal{H}_\delta &= Span\{\psi_i(x) : (\varkappa\mathcal{L} + V)\psi_i = \lambda_i \psi_i; \lambda_i \geq \delta > 0\} \\ &= Span\{\psi_i(x) \in L^2(Z^d) : (\mathcal{H}\psi, \psi) \geq \delta(\psi, \psi)\} \end{aligned}$$

and $dim(l^2_\delta) = N(\delta) = \sharp\{\lambda_i \geq \delta > 0\}$. Now let's explore the new effect from $\hat{a}(k)$.

4.2 Effect of $\hat{a}(k)$ on $Sp(\varkappa\mathcal{L})$

One of most important goal in this work is to study the effect of $\hat{a}(k)$ on the random walk with non-local operator

$$\varkappa\mathcal{L}\varphi(x) = \varkappa\sum_{z\neq 0}(\varphi(x+z)-\varphi(x))a(z), \varphi(x)\in L^2(Z^d).$$

The Fourier transform of $\varphi(x)$ is

$$\hat{\varphi}(k) = \sum_{x\in Z^d} e^{i(k,x)}\varphi(x),\ \ k\in T^d = [-\pi,\pi]^d.$$

If $\varphi(x)\in L^2$, then the series above converges in L^2-sense and (Parseval's identity)

$$(\varphi_1,\varphi_2)(x) = \sum_{x\in Z^d}\varphi_1(x)\bar{\varphi}_2(x) = \frac{1}{(2\pi)^d}\int_{T^d}\hat{\varphi}_1(k)\bar{\hat{\varphi}}_2(k)dk$$

Now the Fourier symbol of $\varkappa\mathcal{L}\varphi(x)$ is

$$(\widehat{\varkappa\mathcal{L}\varphi})(x) = \varkappa(\hat{a}(k)-1)\hat{\varphi}(k),\ \ \hat{a}(k) = \sum_{z\neq 0} a(z)(cos(k,z).$$

Obviously $Sp(\varkappa\mathcal{L}) = Sp(\varkappa(\hat{a}(k)-1)\hat{\varphi}(k))$. The properties of $\hat{a}(k)$ can result in the following important propositions.

Proposition 4.2.1. *If $\hat{a}(k)$ is analytic, i.e., $a(z)\leqslant e^{-\gamma|z|}$, the law of jumping in random walk $a(z)$ decades exponentially, i.e., random walk has light-tailed jump distribution, then $Sp(\varkappa\mathcal{L})$ is pure absolutely continuous and $Sp(\varkappa\mathcal{L}) = [-\iota\varkappa, 0]$ where $-\iota = \min_{k\in T^d}[\hat{a}(k)-1]$.*

Proof. This fact is well-know but we present the sketch of the proof. Consider $F(\lambda) = mes\{k\in T^d : \hat{a} < \lambda\}$, i.e., the distribution function of $\hat{a}$ in the probability space

$\{T^d,\ m(dk) = dk/(2\pi)^d\}$, then

$$F(\lambda + d\lambda) - F(\lambda) = m\{k : \lambda < \hat{a}(k) < \lambda + d\lambda\}.$$

Assume that the surface $\hat{a} = \lambda$ has no singular points, i.e., $\nabla\hat{a}(k) = \{\frac{\partial\hat{a}}{\partial k_1}, \cdots, \frac{\partial\hat{a}}{\partial k_d}\} \neq 0$ if $\hat{a} = \lambda$, then

$$F(\lambda + d\lambda) - F(\lambda) = d\lambda \int_{k:\hat{a}(k)=\lambda} \frac{dS(k)}{|\nabla\hat{a}(k)|}, (integral\ over\ the\ surface\ measure)).$$

and for all $\lambda \in R$ except the critical points where $\{\nabla\hat{a}(k) = 0, \hat{a}(k) = \lambda\}$ the function $F(\lambda)$ has bounded derivative, i.e., $\frac{dF}{d\lambda} = p(\lambda)$ is well defeined outside finite number of critical points. The distribution of $\hat{a}(k),\ k \in T^d$ is absolutely continuous. □

But in the heavy-tailed jump case where $a(z)$ is decreasing slowly, $-\varkappa\hat{a}(k)$ can be constant on some intervals [13], then $\hat{\mathcal{L}}(k)$ has infinite dimensional eigenspace, that is, $\varkappa\mathcal{L}$ has eigenvalues inside $[-\iota\varkappa, 0]$.

Proposition 4.2.2. *We call this effect as proposition 4.2.2.*

We can prove this proposition by following particular examples.

Example 4.2.3. Suppose $d = 1$, $k \in [-\pi, \pi] = T^1$, there exists $\hat{a}(k) = \sum_{x\in Z^1} cos(kx)a(x) \in C^\infty$ such that $\hat{a}(k)$ is constant when $k \in (\alpha, \beta)$(see [21]). Obviously for $k \in [\alpha, \beta] \subset [0, \pi]$, $\hat{a}(k) = 1 - h$, $1 - \hat{a}(k) = h$ and for function $\hat{\psi}(k)$ supported on (α, β)

$$\varkappa\hat{\mathcal{L}}\hat{\varphi}(k) = \varkappa h\hat{\varphi}(k).$$

In other words if $\hat{a}(k)$ is a constant on some intervals, then operator $\hat{\mathcal{L}}(k)$ has infinite dimensional eigenspace.

Function $\hat{a}(k)$ on figure 4.2 (a) is not smooth but we can improve situation if consider the convolution $\hat{a}(k) * b_0(k)$, where $b_0(k)$ is C^∞ compactly supported on $[-\varepsilon, \varepsilon]$

and $\varepsilon < \frac{|\beta-\alpha|}{3}$ (figure 4.2 (b))

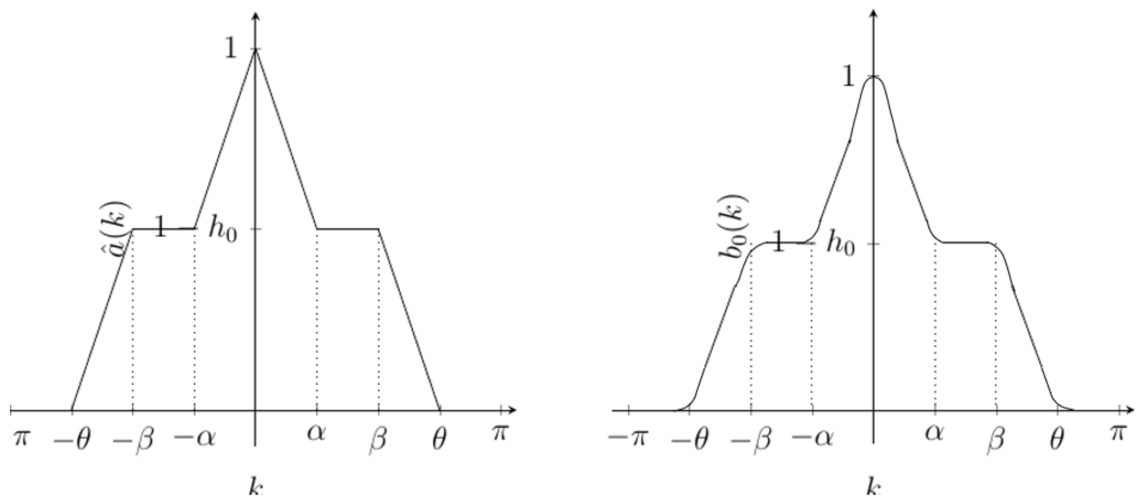

Figure 4.2: The graph of $\hat{a}(k)$, $b_0(k)$ when d=1

Remark: In multidimension Z^d where $d \geq 2$, $\hat{a}(k_1, k_2, \cdots, k_d) = \sum_{x \neq 0} a(x)cos(k,x) = \hat{a}(k_1)\hat{a}(k_2)\cdots\hat{a}(k_d)$, similarly $\varkappa\mathcal{L}$ also has positive eigenvalue if $\hat{a}(k)$ is constant on some parallelepiped (product of intervals).

4.3 Discrete spectrum of $\mathcal{H}_2$ outside $Sp(\varkappa\mathcal{L})$ in transient case

Let's formulate one more time the basic results in the transient case. In the transient situation $\int_0^\infty p(t,x,x)dt = \int_0^\infty p(t,0,0)dt < \infty$ and for the number $N_0(V)$ of positive eigenvalues for the general potential $V(x)$(not necessarily $B(x)$ which is positively definite potential) there is the following general formula [15]:

$$N_0(V) \leq \frac{1}{C(\sigma)} \sum_{x \in Z^d} |V(x)| \int_{\frac{\sigma}{V(x)}} p(t,0,0)dt \tag{4.12}$$

where $C(\sigma) = e^{-\sigma} \int_0^\infty \frac{te^{-2}}{z+\sigma} dz$.

In the formula (3.28) the diffusivity $\varkappa$ is present only implicitly in the transition function $p(t,x,y)$. It is convenient now introduce the standard $\varkappa_0 = \frac{1}{2}$,i.e., $2\varkappa_0 = 1$.

Let's denote corresponding transition probability

$$p_0(t,x,y) = p_0\{x(t)=y|x(0)=x\} = \frac{1}{(2\pi)^d}\int_{T^d} e^{\varkappa(\hat{a}(k)-1)+i(x-y,k)}dk.$$

It is clear that

$$p(t,x,y) = p_0(2\varkappa t,x,y);\; p(t,x,x) = p_0(2\varkappa t,0,0).$$

Then generalized Cwikel-Lieb-Rozenblum (CLR) estimate [15] for $V(x)\geq 0$ can be formulated differently:

$$\begin{aligned} N_0(V) &= \frac{1}{C(\sigma)}\sum_{z\in Z^d} V(x)\left(\int_{\frac{\sigma}{V(x)}} p_0(2\varkappa t,0,0)dt\right) \\ &= \frac{1}{2\varkappa C(\sigma)}\sum_{z\in Z^d} V(x)\left(\int_{\frac{2\varkappa\sigma}{V(x)}} p_0(s,0,0)ds\right) \end{aligned} \tag{4.13}$$

If $\sum_{z\in Z^d}|Z|^2 a(z)<\infty$ then due to CLT for appropriate constant C_+ (depending on $\hat{a}(\cdot)$ but not $\varkappa$) for $s\geq 1$

$$p_0(s,0,0)\leq \frac{C_+}{s^{d/2}},\; s\geq 1. \tag{4.14}$$

If $V(x)\to 0,\; x\to\infty$(in our case $V(x)=B(x)\to 0$ since $B(x)\in L^2(Z^d)$) then for appropriate σ_0 and all $x\in Z^d$

$$\frac{2\varkappa\sigma_0}{V(x)}\geq 1$$

(If $V(x)=B(x)$ and $\max_{x\in Z^d}B(x)=B(0)$ one can take $\sigma_0=\frac{B(0)}{2\varkappa}$, i.e.,

$$N_0(V)\leq \frac{1}{2\varkappa C(\sigma_0)}\sum_{x\in Z^d}|B(x)|\int_{\frac{2\varkappa\sigma_0}{B(x)}}\frac{C_+}{s^{d/2}}ds = C_1\sum_{x\in Z^d}\frac{|V(x)|^{d/2}}{\varkappa^{d/2}} \tag{4.15}$$

Let's stress that constant C_1 in the Bargmann estimate (4.15) depends only on dimension d and the function $\hat{a}(k)$ (i.e., the distribution of jumps $a(\cdot)$ and $B(0)$).

This estimate (4.15) proves the existence of the phase transition: $\exists(\varkappa_{cr})$ such that $N_0(B) = 0$ for $\varkappa > \varkappa_{cr}$ and $N_0(B) \geq 1$ for $\varkappa < \varkappa_{cr}$.

We can formulate these results as following theorem:

Theorem 4.3.1. *If $d \geq 3$, $\sum_{z\in Z^d} |z|^2 a(z) < \infty$, $\sum_{z\in Z^d} |B(z)|^{d/2} < \infty$, then there exists $\varkappa_{cr}$ such that $N_0(B,\varkappa) = \sharp\{\lambda_i > 0$ of the Hamiltonian $\mathcal{H}_2 = 2\varkappa\mathcal{L} + B\} = 0$ if $\varkappa > \varkappa_{cr}$ and $N_0(B,\varkappa) \geq 1$ if $\varkappa < \varkappa_{cr}$.*

Let's stress that for $d \geq 3$ any random walk on Z^d is transient independently on the existence of the second moment. In low dimensions $d = 1, 2$ the random walk can be also transient but corresponding limiting distribution is necessarily stable and symmetric. Stable limiting distribution with $\alpha < 2$ exists also for $d \geq 3$ under some technically restrictions [8][12] [16]. In this case the Bargmann type estimate has the following form

$$N_0(\varkappa, B) \leq C_1 \sum_{x\in Z^d} \left| \frac{B(x)}{\varkappa} \right|^{d/\alpha}. \tag{4.16}$$

It gives

Theorem 4.3.2. *If the random walk $x(t)$ has the limiting stable law with parameter $0 < \alpha < 1$ for $d = 1$ and $0 < \alpha < 2$ for $d \geq 2$ (i.e., $\frac{x(t)}{t^{d/\alpha}} \xrightarrow{law} St_{\alpha()}$) and $\sum_{x\in Z^d} |B(x)|^{d/\alpha} < \infty$ then again $\exists(\varkappa_{cr} > 0)$ such that for small $\varkappa < \varkappa_{cr}$, $N_0(\varkappa, B) \geq 1$ and for $\varkappa > \varkappa_{cr}$ there is no positive eigenvalues.*

Theorem 4.3.1, 4.3.2 demonstrate the transition from the regular behavior of the field $u(t,x)$ to intermittent behavior if $\varkappa$ goes through $\varkappa_{cr}$. see details in [16].

The spectral-bifurcation above is the transition from recurrent to transient walks. To prove this fact we used the Central Limit Theorem for the stable laws (in dimension $d = 1, 2$). The limit theorem required the strong regularity assumptions on the tails of $x(t)$. In fact this is true for any jumps law $a(z)$.

Theorem 4.3.3. *Assume that for $d = 1$ or $d = 2$,*

$$\frac{1}{(2\pi)^d} \int_{T^d} \frac{dk}{1 - 2\varkappa \hat{a}(k)} = \infty, \;\; \hat{a}(k) = \sum_{\substack{x \in Z^d \\ x \neq 0}} a(x) cos(k, x)$$

then the operator $\mathcal{H} = 2\varkappa\mathcal{L} + V(x)$, $V(x) \geq 0$ ($V(x)$ is not identically 0) has the positive eigenvalues.

Proof. Due to variational principle it is sufficiently to prove theorem for the potential $V(x) = \sigma\delta_{x_0}(x)$ or Due to translation invariance for $V(x) = \sigma\delta_0(x)$. Then (see [1] [8]) for $\lambda_0(x)$ we have equation

$$\frac{1}{\sigma} = \frac{1}{(2\pi)^d} \int_{T^d} \frac{dk}{\lambda + 2\varkappa(1 - \hat{a}(k))} = I(\lambda) \tag{4.17}$$

In the recurrent case $I(\lambda) \uparrow \infty$ as $\lambda \downarrow 0$ and equation has unique solution. □

What will happens if $V(x)$ has negative part? This is the subject of the future discussion.

Let's formulate technical proposition and conjectures.

Proposition 4.3.4. *For any $\hat{a}(k)$ and dimension $d \geq 1$ the $\lambda_0(\mathcal{H}) > 0$ exists for small enough $\varkappa$ if potential $V(x)$ is positive at least in one points x_0: $V(x_0) > 0$. In particular if $V(x) = B(x)$ and $B(0) = \sum_{x \in Z^d} \ell^2(x) > 0$, then $\lambda_0(\varkappa\mathcal{L} + B) > 0$ for small $\varkappa$.*

Proof. Assume that $B(x_0) > 0$, let's consider the test function $\psi(x) = \delta_0(x)$ then

$$(\mathcal{H}\psi, \psi) = \varkappa(\mathcal{L}\psi, \psi) + (B\psi, \psi) = -\varkappa + V(x_0) > 0$$

for $\varkappa < B(0)$. It means that $\lambda_0(\mathcal{H}) > 0$ due to variational principle. The central question is what we can say about discrete positive spectrum for small $\varkappa$.

We know that $\lambda_0(\mathcal{H}) > 0$ for any $\varkappa > 0$ if $B(x) \geq 0$ in dimension $d = 1, 2$. But $B(x)$ is only positively definite function and it can be negative. We know that in the transient case the positive eigenvalues are absent for large $\varkappa$, $\sum_{z \in Z^d} |z|^2 a(z) < \infty$ and $\sum_{x \in Z^d} |B(x)|^{d/2} < \infty$ or in the case of the limiting stable law with parameter α $\sum_{x \in Z^d} |B(x)|^{d/\alpha} < \infty$ (see Theorem 4.3.1, 4.3.2). □

What happens if $\sum_{x \in Z^d} |B(x)|^{d/2} = \infty$ or $\sum_{x \in Z^d} |B(x)|^{d/\alpha} = \infty$? The answer of this question is not unique and depends on the structure of the correlation function $B(x),\ x \in Z$. There are two important special case concerning the transient case.

Theorem 4.3.5. *If we have local lattice operator* $\Delta f(t, x) = \sum_{\substack{z : |z-x|=1 \\ z \in Z^d}} (f(x+z) - f(x))$ *and define* $\mathcal{L} f(t, v) = \varkappa \Delta f(t, v) + B(v) f(t, v)$. *Assume that for multidimensional case* $d \geq 3$, $B(x) \geq \frac{C_0}{1+|x|^2}$ *and* C_0 *is fixed and large enough , and* $\sum |B(x)|^{d/2} = \infty$, *then* $N_0(B) = \infty$.

Proof. Suppose that for $x = (x_1, x_2, \cdots, x_d)$ where $2^m < x_1 < 2^{m+1}$, $0 < x_i < 2^m$ for $i = 2, 3, \cdots, d$. The boundary of $x_i, i = 1, 2, \cdots, d$ makes the cube Q_m whose volume size is 2^{md}. And the potential $B(x)$ has such lower bound

$$B(x) \geq \frac{C_0}{1+|x|^2} \geq \frac{C_0}{1 + 2^{2(m+1)} + 2^{2m} + \cdots + 2^{2m}} \geq \frac{C_0}{1 + (d+3)2^{2m}} \geq \frac{C_0}{(d+3)2^{2m}}.$$

Then consider the following function for $x = (x_1, x_2, \cdots, x_d) \in Q_m$,

$$\psi_m(x_1, x_2, \cdots, x_d) = sin(k(x_1 - 2^m)) sin(kx_2) \cdots sin(kx_d)$$

Assume $\psi_m = 0$ on boundary of Q_m (∂Q_m), i.e., $sin(k2^m) = 0$ and $k = \frac{\pi}{2^m}$.

$$\begin{aligned}\varkappa\Delta\psi_m(x) = &\varkappa\Delta_{x_1}\sin(k(x_1 - 2^m))\sin(kx_2)\cdots\sin(kx_d)\\ &+ \varkappa\Delta_{x_2}\sin(k(x_1 - 2^m))\sin(kx_2)\cdots\sin(kx_d) + \cdots\\ &+ \varkappa\Delta_{x_d}\sin(k(x_1 - 2^m)\sin(kx_2)\cdots\sin(kx_d)\end{aligned}$$

Since $\varkappa\Delta\sin(\alpha x) = 2\varkappa(\cos(\alpha) - 1))\sin(\alpha x)$ for arbitrary α,

$$\lambda_m = 2\varkappa(cos(k) - 1) \sim -\frac{\varkappa\pi^2}{2^{2m}}$$

Then

$$\mathcal{L}\psi_m(x) \geq \lambda_m\psi_m + \frac{C_0}{(d+3)2^{2m}}\psi_m \sim [-\frac{\varkappa\pi^2}{2^{2m}} + \frac{C_0}{(d+3)2^{2m}}]\psi_m$$

Let the left hand side of preceding equation be positive, say $C_0 > 2\varkappa(d+3)\pi^2$, then we will have

$$(\mathcal{L}\psi_m(x), \psi_m) \geq 0,$$

and we have infinitely many such cubes Q_m and compactly supported test functions ψ_m on Q_m, which means there are infinitely many positive eigenvalues. It is equivalently that if potential $B(x)$ decreases quadratically, there exists infinity many positive eigenvalues. $\square$

For the very sparse potentials the situation can be apposite (there is no positive eigenvalue). Consider the following spectral problem in $L^2(Z^d),\ d \geq 3$

$$\mathcal{H}\psi(x) = \Delta\psi(x) + V(x)\psi(x) = \lambda\psi(x) \tag{4.18}$$

where $V(x) = \sum_{n=1}^{\infty}\sigma_n\delta(x - x_n)$. Here $\sigma_n \to 0$ very slowly and $\{x_n,\ n \geq 1\}$ is a very sparse sequence of the potential on Z^d. It means that $dis(x_n, \{m : m \neq n\}) = l_n \to \infty$

very fast. without of generality, one can take, for $d \geq 3$,

$$\sigma_n = \frac{1}{ln(1+n)},\ x_n = (0, \cdots, 0, 2^n, 0, \cdots, 0),\ n \geq 1. \tag{4.19}$$

Let's note that for any very large $\kappa > 0$

$$\sum_{x \in Z^d} V^{\kappa}(x) = \infty.$$

Proposition 4.3.6. *Operator (4.18) has no positive eigenvalue.*

Proof. We'll give the proof for particular case presented above, but it will be clear that the result is very general.

Since $V(x) \geq 0$ and $V(x) \to 0$ ther spectrum of $\mathcal{H}$ for $\lambda > 0$ is discrete. Assume that it is non-empty and $\lambda_0(\lambda)$ is the maximum postitive eigenvalue with positive eigenvalue with positive eigenfunction $\psi_0(x)$, then Fourier transform for $\psi_0(x_n),\ n \geq 1$ is

$$-\hat{\Delta}(k)\hat{\psi}(k) + \sum_{m=1}^{\infty} \sigma_m \psi_0(x_m) e^{i(k,m)} = \lambda \hat{\psi}_0(k)$$

i.e.,

$$\hat{\psi}_0(k) = \sum_{m=1}^{\infty} \sigma_m \psi_0(x_m) \frac{e^{i(k,m)}}{\lambda + \hat{\Delta}(k)}$$

and (for inverse Fourier transform)

$$\psi_0(x_n) = \sum_{m=1}^{\infty} \sigma_m \psi_0(x_m) G_{\lambda_0}(x_n, x_m),\ G_{\lambda_0}(0, z) = \frac{1}{(2\pi)^d} \int_0^{\infty} \frac{e^{i(k,z)} dk}{\lambda + \hat{\Delta}(k)}$$

Finally,

$$\psi_0(x_n) = \sum_{m: m \neq n} \frac{\sigma_m \psi_0(x_m) G_{\lambda_0}(x_n, x_m)}{1 - \sigma_n G_{\lambda}(x_n, x_n)} \tag{4.20}$$

But for $d \geq 3$

$$G_{\lambda_0}(x,y) = \int_0^\infty e^{-\lambda_0 t} p(t,x,y)dt \leq \frac{C(d)}{1+|x-y|^{d-2}}.$$

Let's consider now the homogeneous system (4.20) in the Banach space $L^\infty(Z^d)$ with the norm

$$\|f(x)\|_\infty = \max_{x \in Z^d} |f(x)|.$$

Let

$$\tau_{m,n} = \begin{cases} \frac{\sigma_m G_{\lambda_0}(x_n,x_m)}{1-\sigma_n G_\lambda(x_0,x_0)}, \ m \neq n \\ 0, \ m = n \end{cases}$$

be the matrix of (4.20). Assume that $|x_n - x_m| \geq |2^n - 2^m|$ for $m \neq n$ (like in particular example above), then one can check that

$$\|\tau\|_\infty \leq \max_n \sum_{m:m\neq n} |\tau_{m,n}| \leq C_1(d) \max_m \sigma_m$$

and for sufficiently small $\max_m \sigma_m$ we'll get $\|\tau\|_\infty < 1$, i.e., system (4.20) has no solution. □

4.4 Discrete spectrum in the general recurrent case: the spectral conjecture

If process $x(t)$ with generator $\varkappa\mathcal{L}$ is recurrent and $V(x) \geq 0$ and $\exists(x_0) : V(x_0) > 0$ then $\lambda_0(\mathcal{H}_2) = maxSp(\mathcal{H}_2) > 0$. Since $B(x) \in L^2(Z^d)$, i.e., $B(x) \to 0$ as $|x| \to \infty$, the spectrum of $\mathcal{H}_2$ outside $Sp(\mathcal{L})$ is discrete with possible accumulation 0 and maybe empty. Then the $\max_i \lambda_i(\mathcal{H}_2) = \lambda_0(\mathcal{H}_2) > 0$ is the simple eigenvalue with positive eigenfunction $\psi_0(x) > 0$. Due to variation principle $\lambda_0(\mathcal{H}_2) = \max_{\psi \in L^2(Z^d) \|\psi\|=1} (\mathcal{H}_2\psi, \psi) = (\mathcal{H}_2\psi_0, \psi_0)$. One can estimate $\lambda_0(\mathcal{H}_2)$ from below by the eigenvalue of $\tilde{H} = \varkappa\mathcal{L} + V(x_0)\delta_{x_0}(x)$ which satisfies the equation

$$\frac{1}{V_0(x)} = \frac{1}{(2\pi)^d}\int_{T^d}\frac{dk}{\lambda + 2\varkappa(1-\hat{a}(k))} = I(\lambda). \qquad (4.21)$$

In the recurrent case $I(\lambda) \to \infty$ as $\lambda \downarrow 0$ and equation (4.21) has unique solution.

Assume now that $V(x)$ has negative values. We'll discuss the **conjecture**: if the potential $V(x)$ is positively definite (i.e., $V(x) = B(x)$) then $\lambda_0(\mathcal{H}) > 0$.

The following two resutls support this conjecture.

Theorem 4.4.1. *Consider the spectral problem*

$$H\psi(x) = \varkappa\Delta\psi + V(x)\psi = \lambda\psi(x), x \in Z^1, V(x) \in L^1(Z^1), \sum_{z\in Z^1} V(x) = \delta > 0$$

then $\forall(\varkappa > 0),\ \exists(\lambda_0(\mathcal{H})) > 0$.

Proof. Let's define the following even function $\psi_1(x) = \psi_1(-x)$

$$\psi_1(x) = \begin{cases} 1,\ x \in [-L, L] \\ \frac{L_1 - x}{L_1 - L},\ |x| \in [L, L_1] \\ 0,\ |x| \geq L_1 \end{cases}$$

Here $0 < L < L_1$ are two large parameters: $L \gg 1,\ L_1 - L \gg 1$.

Let's calculate the Dirichlet quadratic functional $(\mathcal{H}\psi_1, \psi_1) = \varkappa(\Delta\psi_1, \psi_1) + (V(x)\psi_1, \psi_1)$. Note that

$$(V(x)\psi_1, \psi_1) = \sum_{x\in[-L,L]} V(x) + R_{L,L_1},\ |R_{L,L_1}| \leq \sum_{|x|>L} |V(x)| (Since\ |\psi_1| \leq 1).$$

For any $0 < \delta_1 < \delta$, say $\delta_1 = \frac{1}{2}\delta$, one can find large enough L such that $(V(x)\psi_1, \psi_1) \geq$

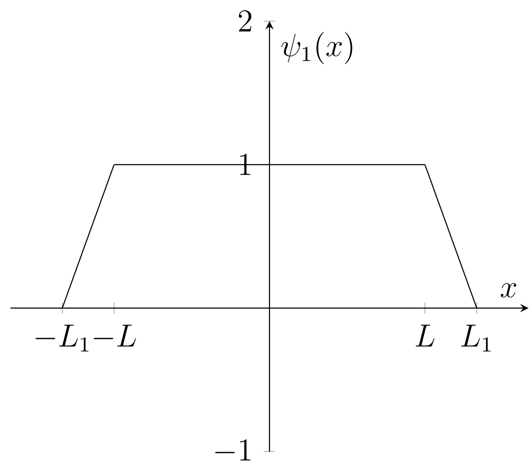

Figure 4.3: The graph of $\psi_1(x)$

δ_1. But $\Delta\psi_1(x) = 0$ if $x \neq \pm L$, $x \neq \pm L_1$. And $|\Delta\psi_1(L)| = |\Delta\psi_1(L_1)| = \frac{1}{L_1 - L}$. Finally

$$(\mathcal{H}\psi_1, \psi_1) \geq \frac{4\varkappa}{L_1 - L} + \frac{1}{2}\sum_{x\in Z^d} V(x) = \frac{4\varkappa}{L_1 - L} + \frac{1}{2}\delta \geq \frac{1}{3}\delta$$

if only for given $\varkappa$ and L the second parameter L_1 is the large enough. $\square$

Remark: more accurate calculations show that $\lambda_0(\mathcal{H}) > 0$ under conditions $V(x) \in L^1(Z^1), \sum_{x\in Z^1} V(x) = \delta > 0,\ \sum_{x\in Z^1} |z|^2 a(z) < \infty$.

Let's remember that under condition $\ell(x) \in L^1(Z^1)$,

$$B(x) \in L^1(Z^1),\ \sum_{x\in Z^1} B(x) = \left(\sum_{x\in Z^1} \ell(x)\right)^2 \geq 0.$$

Due to our conjecture for $d = 1, 2$ and recurrent $x(t)$ we still have $\lambda_0(\mathcal{H}) > 0$, even if $\sum_{x\in Z^1} B(x) = 0$. In $1-D$ case we can prove this conjecture only in several particular cases, $2-D$ is still open. Let's present the simplest example. It is convenient to study the spectral problem in the form $\mathcal{H}\psi = \Delta + \beta B_0(x)$ where $\beta = \frac{1}{\varkappa}$ is coupling number.

Proposition 4.4.2. *Assume that $b_0(x)$ equals to 1 if $x = 0$ and to -1 if $x = 1$, then*

$$B_0(x) = \begin{cases} 2, \ x = 0 \\ -1, \ x \pm 1 \\ 0, \ |x| > 1 \end{cases}, \ i.e., \sum_{x \in Z^d} B_0(x) = 0.$$

We state that the maximal eigenvalue $\lambda_0(\mathcal{H})$ of $\mathcal{H} = \Delta + \beta V(x)$ is positive and $\lambda_0(\mathcal{H}) \sim \beta^4$ if $\beta \ll 1$.

Proof. We'll construct eigenfunction $\psi_0(x)$ such that $\Delta\psi_0 + \beta B_0(x)\psi_0 = \lambda\psi_0$ in the form

$$\psi_0(x) = \psi_0(-x), \psi_0(0) = h, \ \psi_0(x) = e^{-\mu(|x|-1)} if|x| \geq 1 (see \ figure(4.4)).$$

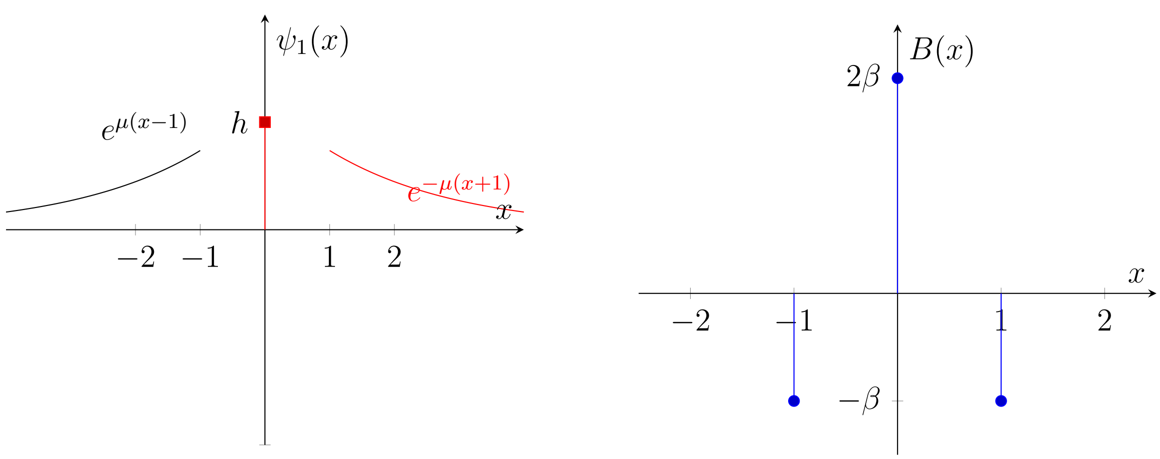

Figure 4.4: The graph of $\psi_1(x)$, $B(x)$ with negative value for $d = 1$.

Put(see figure(4.4))

$$B(x) = \begin{cases} 2\beta, \ x = 0 \\ -\beta, \ x = \pm 1 \\ 0, |x| > 1 \end{cases}, \tag{4.22}$$

then equation $\Delta\psi_0 + \beta B(x)\psi_0 = \lambda\psi_0$ gives

a) for $|x| > 1$: $e^{-\mu} + e^{\mu} = 2 + \lambda \Rightarrow cosh\mu = 1 + \frac{\lambda}{2}$ or

$$cosh\mu - 1 = \frac{\lambda}{2}. \tag{4.23}$$

i.e., for any small μ we have $\lambda = \mu^2 + \underline{\underline{o}}(\mu^4)$

b) for $x = 0$:

$$1 + 2\beta h - 2h = \lambda h \Rightarrow h = \frac{2}{2 - 2\beta + \lambda} \tag{4.24}$$

c) for $x = 1$(the same for $x = -1$):

$$h + e^{-\mu} - 2 - \beta = \lambda \Rightarrow h + e^{-\mu} = 2 + \lambda + \beta \tag{4.25}$$

It follows from (4.23) and (4.25) :

$$h = e^{\mu} + \beta.$$

Then (4.24) gives

$$\frac{2}{2 - 2\beta + \lambda} = \frac{1}{1 - (\beta - \lambda/2)} = e^{\mu} + \beta.$$

i.e.,

$$1 + (\beta - \lambda/2) + (\beta - \lambda/2)^2 + \cdots = e^{\mu} + \beta$$

$$1 - \lambda/2 + (\beta - \lambda/2)^2 = 1 + \mu + \underline{\underline{o}}(\mu^2)$$

$$\beta^2 + \underline{\underline{o}}(\lambda\beta) = \mu + \underline{\underline{o}}(\mu^2)$$

Then $\mu = \beta^2 + \underline{\underline{o}}(\beta^2)$, $\lambda = 2\mu^2 + \underline{\underline{o}}(\mu^2) = 2\beta^4 + \underline{\underline{o}}(\beta^4)$.

It is interesting that the same answer can be proven for $\lambda_0(\mathcal{H})$ of the operator

$\mathcal{H} = \Delta + \beta V_1(x)$ where

$$V_1(x) = -B_0(x) = \begin{cases} -2, \ x = 0 \\ 1, \ x \pm 1 \\ 0, \ |x| > 1 \end{cases} .$$

Probably, at least in dimension $d = 1$ the positivity of $\lambda_0(\mathcal{H})$ correpsonds to all potential with $\sum_{x \in Z^1} V(x) \geq 0$. $\square$

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