SPECTRAL THEORY OF SCHRÖDINGER TYPE OPERATOR ON SPIDER GRAPHS

by

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A dissertation submitted to the faculty of
The University of North Carolina at Charlotte
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in
Applied Mathematics

Charlotte

2023

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ABSTRACT

MADHUMITA PAUL. Spectral Theory of Schrödinger Type Operator On Spider Graphs.
(Under the direction of DR. STANISLAV MOLCHANOV)

The dissertation consists of chapter-1: Introduction, this chapter contains some definitions and examples of quantum graphs, symplectic analysis and its representation on spider graph. Chapter 2-Brownian motion on the spider like quantum graph, this chapter contains the definition of Brownian motion on the N-legged spider graph with infinite legs and Kirchhoff's gluing conditions at the origin and calculation of the transition probability of this process. In addition we study several important Markov moments, for instance the first exit time τ_L from the spider with the length L of all legs. The calculations give not only the moments of τ_L but also the distribution density for τ_L. All results of this section are new ones. Chapter 3- A brief review on the classical spectral theory. This chapter contains the elements of the spectral theory on spider graph. We start from the classical Strum-Liouville theory on the full line $\mathbb{R}^1$ (for the case of the bounded from below potential) and explain how this theory can be generalized to the case of canonical system in $\mathbb{R}^{2d}$:

$$J\overrightarrow{\psi}' = (V + \lambda Q)\overrightarrow{\psi} \qquad\qquad \overrightarrow{\psi} = \begin{bmatrix} \psi \\ \psi' \end{bmatrix}$$

The spectral measure for the canonical system is constructed (like in the Strum-Liouville case) by passing to the limit from the discrete spectral measure on the spider with the finite length of all legs and (say) Dirichlet boundary condition at the outer end points of the legs. The corresponding results (expecting the particular details related to specific case of the spider graphs) are not new. Chapter 4- spectral theory of the Schrödinger operator on the spider like quantum graph, this chapter contains the main results of the dissertation. We start by constructing the spectral analysis on the finite interval of a three-legged spider graph and then pass it to infinity. Spectral analysis is performed for three different types of potentials. The fast-decreasing potentials, the fast-increasing potentials, mixed potentials, and its spectral theory. The details contain, the absolute continuous spectrum of multiplicity 3 and its construction using the reflection-transmission coefficients on each leg for the fast decreasing potential, Bohr's asymptotic formula for $N(\lambda)$ (the negative eigenvalues), instability of the discrete spectrum for the mixed potential on each leg of the spider graph.

DEDICATION

I dedicate this dissertation to my father, Mr. Ratan Chandra Paul, my mother, Mrs. Bapi Paul and my uncle, Mr. Tapan Paul.

ACKNOWLEDGEMENTS

I am extremely grateful to my advisor Dr. Stanislav Molchanov for all the support, guidance and patience that he provided me during my PhD journey and for not giving up on me at my worst. This dissertation would not have been possible without his guidance and constant involvement in every step.

I would also like to thank my committee members Dr. Boris Vainberg, for being an amazing person who I could go to asking for any help, the suggestions that he gave for my dissertation and Dr. Yuri Godin for his comments and feedback on my research statement.

I would like to extend my gratitude to Dr. Shaozhong Deng and Dr. Mohammad A. Kazemi for providing me with the teaching opportunities, Dr. Arindam Roy for helping me with writing software and providing me with other research opportunities, Dr. Alina Reznikova for making the communications smooth through zoom since the pandemic time and Dr. Taufiquar Khan for his guidance and feedback.

I consider myself truly lucky to have wonderful friends around me during the most difficult time of the PhD journey, who took care of me and kept me sane with their positive comments, provided any help that they could. It would have been more difficult to come this far without their help. I want to say thank you to Dr. Manira Akter, Dr. Antardipan Pal, Sayantan Datta, Rittika Mallik and Subhasree Srenevas for their support.

Last, but not the least, I could not have made it this far without my family's support and unconditional love, especially my father, who always believed in me even when I did not and has been the most supportive all my life.

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CHAPTER 1: INTRODUCTION TO THE QUANTUM GRAPH

1.1 Introduction

This dissertation is devoted to the spectral theory of the Schrödinger operators on the special class of the quantum graphs, so called spider graphs. This is the natural generalization usual one dimensional theory, which is widely presented in mathematical literature under the name "Spectral theory of Strum-Liouville operators". We will use for references the monograph [11]. The extension of the theory on the quantum graph is closely related to the transition from the group of 2×2 unimodular matrices $SL(2, \mathbb{R})$ which is behind all the constructions in the Strum-Liouville theory to the symplectic group $SP(2d, \mathbb{R})$.

In the Introduction we will give the definitions of the basic concepts (quantum graph, Kirchhoff's gluing conditions, symplectic group, functional spaces on the spider graphs, Schrödinger operator on such graphs etc.) In the end of introduction, will give the brief review of the results.

1.2 Quantum graphs and spider graphs as the particular case

In this section we will introduce the concept of the quantum graphs, their special case, the spider graph and the Schrödinger type operators on such graphs. Together with usual continuous potentials we will systematically use the generalized (singular) potentials $v(x) = \sum_{i=1}^{N} \sigma_i \delta(x - x_i)$, which we will understand as the "limit" as $\epsilon \to 0$ of the regular potentials $v_\epsilon(x)$ (See figure 1.1 for single δ-function).

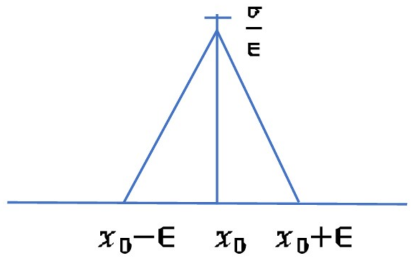

Figure 1.1: Single δ function approximation

$$v_\epsilon(x) = \begin{cases} 0 & |x-x_0| > \epsilon \\ \frac{(\epsilon-|x-x_0|)\sigma}{\epsilon^2} & |x-x_0| \le \epsilon \end{cases} \tag{1.1}$$

$v_\epsilon(x) \to \sigma\delta(x-x_0)$, weakly in $\mathbb{C}(\mathbb{R}^1)$ as $\epsilon \to 0$ or in the distribution sense. The solution of

$$H_\epsilon\psi = -\frac{d^2\psi}{dx^2} + v_\epsilon(x)\psi = \lambda\psi \tag{1.2}$$

for $\epsilon \to 0$, converges to the continuous function $\tilde{\psi}(x)$ on $\mathbb{R}$, which has left and right derivatives at $x = x_0$, satisfying the gluing condition

$$\frac{d\tilde{\psi}}{dx}(x_0^-) - \frac{d\tilde{\psi}}{dx}(x_0^+) = \sigma\tilde{\psi}(x_0) \tag{1.3}$$

This is the simplest example of quantum graph. It consists of the finitely (or countably many) finite or infinite interval, which are gluing together at the branching points. Outside the branching points they have the standard 1-D structure(including the Lebesgue measure).At the branching points we have the Kirchhoff conditions similar to 1.3. We will discuss in the future several particular examples in details.

Definition 1.2.1. *Quantum (or metric) graph is the system of vertices connected by one-dimensional intervals(edges) with euclidean metric (and corresponding Lebesgue measure).*

Example 1. *Spider - This graph contains the single vertex $O \in \mathbb{R}^2$ and finitely many* *infinite legs. It is denoted by $sp(d)$ and on the figure below $d = 3$.*

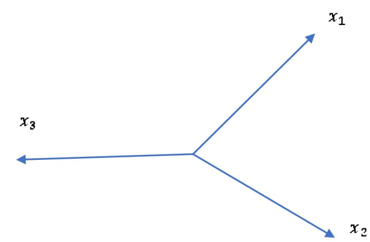

Figure 1.2: A spider quantum graph with three legs.

Example 2. *Tree - This is a graph with the index of branching $(d+1)$. (see the figure below with $d = 2$).*

All intervals between vertices have length 1.

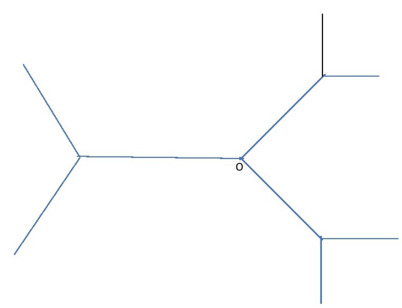

Figure 1.3: A quantum tree

Example 3. *Compact Neumann star graph(truncated spider graph) - This is the simplest non-trivial graph, Star graph, with Neumann boundary condition imposed at the outer vertices, Laplacian defined along the edges and Kirchhoff's gluing condition at the origin (branching point).*

The application of quantum graphs started appearing since at least the 1930s in various areas of chemistry, physics, and mathematics. However,it started picking the growth in the last couple of decades. Quantum graphs appear as simplified models in mathematics, physics, chemistry, and engineering where propagation of waves of various nature can be considered through a quasi-one-dimensional system that looks like

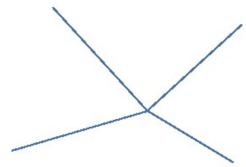

Figure 1.4: A compact star graph

a thin optical channel. For example, free-electron theory of quantum wires, photonic crystals, carbon nano-structures. Quantum graph also appears in some problems of dynamical systems theory. It plays important role in the study of Anderson localization and quantum chaos. Quantum graphs are related to the older spectral theory of standard or combinatorial graphs and uses tools from graph theory, combinatorics, mathematical physics, PDE and spectral theory (See [4] for detailed discussion on quantum graphs).What makes them interesting to study, is their interdisciplinary applications.

1.3 Functions on the quantum graph

On the quantum graphs Γ *one can consider the standard functional spaces,* $L^2(\Gamma, dx)$ *with respect to Lebesgue measure on the edges,* $\mathbb{C}_0^\infty$*: Class of compactly supported smooth functions whose supports do not contain vertices.* $\mathbb{C}(\Gamma)$ *usual space of continuous function with the norm*

$$\|f\|_\infty = sup_{x\in\Gamma}|f(x)|, f \in \mathbb{C}(\Gamma)$$

We call $f(x) \in \mathbb{C}^1(\Gamma)$ *if* $f(x)$ *is continuous at any vertex* v *and for edges* l_i, $i = 1, 2, ..,$ *directed from* v*, there exist limits (may be different) of first order derivatives* $\frac{\delta f}{\delta l_i}$ *for* $i = 1, 2, ..$*at* v *together with a linear condition on such derivatives.*
Simplest example: Spider graph with origin at 0*.If* $f \in C^1$ *and* $\sum_i \frac{\delta f}{\delta l_i}(0) = 0$ *(Kirchhoff condition) Due to continuity of* f*,* $f_{l_1}(0^+) = f_{l_2}(0^+)$*....*

Let us recall that,

$$L^2(\Gamma) = \{f : \sum_{f\in l} \int |f|^2 = \|f\|^2 < \infty\} \qquad \textit{summation over the edges } l \in \Gamma$$

Now we will introduce the transfer matrix (monodromy operator) on the quantum graph. Let us start from the simple examples. Consider the scalar equation

$$H\psi = -\psi'' + v(x)\psi = \lambda\psi \qquad x \in \mathbb{R}^1 \tag{1.4}$$

where $v(x)$ *is the real continuous and bounded from below potential.* λ *is the real or complex number. Let us present the second order equation by the equivalent system of two first order equation for the vector.*

$$\overrightarrow{\Psi}(x) = \begin{bmatrix} \psi \\ \psi' \end{bmatrix}$$
$$\Rightarrow -J\overrightarrow{\Psi'} = (V + \lambda Q)\overrightarrow{\Psi}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, V = \begin{pmatrix} v & 0 \\ 0 & -1 \end{pmatrix}, Q = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$$

In-fact :

$$-J\overrightarrow{\Psi'} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} \psi' \\ \psi'' \end{bmatrix}$$

$$= \begin{bmatrix} \psi'' \\ -\psi' \end{bmatrix}$$

$$(V + \lambda Q)\overrightarrow{\Psi} = \begin{bmatrix} v\psi \\ -\psi' \end{bmatrix} + \begin{bmatrix} \lambda\psi \\ 0 \end{bmatrix}$$

The transfer matrix $T_{[a,b]}(\lambda)$ *between any two points* $a < b$ *on* $\mathbb{R}$ *transform*

$$\begin{bmatrix} \psi \\ \psi' \end{bmatrix}(a) \overset{T_{[a,b]}(\lambda)}{\longrightarrow} \begin{bmatrix} \psi \\ \psi' \end{bmatrix}(b)$$

It has the form

$$T_{[a,b]}(\lambda)(\lambda) = \begin{bmatrix} \psi_1(b) & \psi_2(b) \\ \psi_1'(b) & \psi_2'(b) \end{bmatrix}$$

where, ψ_1*,* ψ_2 *are the solution of equation 1.4 with condition* $\psi_1(a) = 1$*,* $\psi_1'(a) = 0$*,* $\psi_2(a) = 1$*,* $\psi_2'(a) = 1$*.*

Let us now illustrate the continuous theory related to the symplectic group $SP(2d)$ *by the differential equation on* $sp(3)$ *(The spider graph with three legs). The continuous operators along each leg are:*

$$-\frac{d^2}{dx^2} + v_2(x)$$

along x *axis;*

$$-\frac{d^2}{dy^2} + v_2(y)$$

along y *axis;*

$$-\frac{d^2}{dz^2} + v_3(z)$$

along z axis.

At point 0 *one can define gluing condition:*

$$\psi_1(0) = \psi_2(0) = \psi_3(0) = \psi(0)$$
$$\frac{d\psi}{dx} + \frac{d\psi}{dy} + \frac{d\psi}{dz} = \beta\psi(0)$$

This is called generalized Kirchhoff condition. This system of 3 second order equations can be presented(as above) as the system of 6 first order ODE's with gluing condition at the origin. We will present the details of this construction in section 6.

1.4 Symplectic group and self-adjointness

The symplectic group appears in the classical mechanic as essential part of the Hamiltonian formalism [see [2]]. In the spectral theory this concept has the similar origin: the boundary (or gluing) conditions for the linear self-adjoint operator of the second order (Hamiltonian), acting in the space of the vector functions have a symplectic structure. We start from the review of several facts from symplectic analysis and general spectral theory. Let us consider the Hamiltonian defined in (1.4) *acting on the compactly supported smooth vector functions* $\overrightarrow{\psi(x)}$ *with values from* $\mathbb{R}^d$, $d \geq 0$. *Potential* $v(x)$ *is a* $(d \times d)$ *matrix valued function.*

Note that,

$$H\overrightarrow{\psi} = -\overrightarrow{\psi}'' + v(x)\overrightarrow{\psi} = \lambda\overrightarrow{\psi}, \qquad x \in \mathbb{R}^d$$

can be represented (for real λ*) as a canonical system*

$$-J\overrightarrow{\Psi}' = (v + \lambda Q)\overrightarrow{\Psi} \qquad \overrightarrow{\Psi} = [\overrightarrow{\psi}, \overrightarrow{\psi}']' \tag{1.5}$$

where

$$J = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}, v = \begin{pmatrix} v_d & 0 \\ 0 & -I_d \end{pmatrix}, Q = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}$$

All blocks are $d \times d$ and corresponding quadratic form is defined on the Hilbert space of the vector functions $\overrightarrow{\psi}$, with the dot product

$$\begin{aligned} (H\overrightarrow{\psi}, \overrightarrow{\phi}) &= \int_{\mathbb{R}^1} (-\overrightarrow{\psi}'' + v\overrightarrow{\psi}, \overrightarrow{\phi}) \\ &= \int_{\mathbb{R}^1} (-\overrightarrow{\psi}'', \overrightarrow{\phi})dx + \int_{\mathbb{R}^1} (v\overrightarrow{\psi}, \overrightarrow{\phi})dx \\ &= \int_{\mathbb{R}^1} -\left(\sum_{i=1}^{d} \overrightarrow{\psi_i''}\overrightarrow{\phi_i}\right) dx + \int_{\mathbb{R}^1} (v\overrightarrow{\psi}.\overrightarrow{\phi})dx \end{aligned} \tag{1.6}$$

The functions here, in general, are complex valued. The condition of the symmetry of H on the compactly supported smooth functions $\overrightarrow{\psi}, \overrightarrow{\phi}$: $(H\overrightarrow{\psi}, \overrightarrow{\phi}) = (\overrightarrow{\psi}, H\overrightarrow{\phi})$ has the form,

$$(H\overrightarrow{\psi}, \overrightarrow{\phi}) = \int_{\mathbb{R}^1} \left(\overrightarrow{\psi}', \overrightarrow{\phi}'\right) dx + \int_{\mathbb{R}^1} (\overrightarrow{\psi}, v\overrightarrow{\phi})dx = \int_{\mathbb{R}^1} \left(\overrightarrow{\psi}', \overrightarrow{\phi}'\right) dx + \int_{\mathbb{R}^1} (v^*\overrightarrow{\psi}, \overrightarrow{\phi})dx$$

To prove that $(H\psi, \phi) = (\psi, H\psi)$ is we need the symmetry of the matrix potential $v(x) = v^*(x)$. recall,

$$v^* = [v_{ij}]^* = [\bar{v_{ij}}]$$

If v is a real valued matrix potential then $v = v^*$ that means $v = v^T$ where $v \in \mathbb{C}_{loc}$. We also need the homogeneous boundary condition at $x = 0$. It can be, for instance, the Neumann's boundary condition $\overrightarrow{\psi}'(0) = 0$. Now we consider the same problem of self-adjointness for the operator H on the "Spider Graph" and its boundedness from below in the Hilbert space. On this quantum(metric)graph we have a single vertex O and d half-axes, parameterized by the length parameters $s_j \geq 0$, $j = 1, 2..., d$

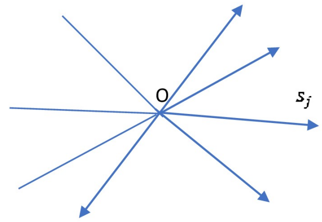

Figure 1.5: A quantum spider graph with d legs

Put ,

$$\overrightarrow{\Psi}(s) = \begin{bmatrix} \overrightarrow{\psi_1}(s_1) \\ \overrightarrow{\psi_2}(s_2) \\ ..... \\ \overrightarrow{\psi_d}(s_d) \end{bmatrix} \in \mathbb{R}^d$$

let us assume that Hamiltonian H for $s > 0$ acts on $\overrightarrow{\Psi}(.)$ as: $H\overrightarrow{\Psi} = -\overrightarrow{\psi}'' + V(s)\overrightarrow{\psi}$

We assume that the following limits exist: $\overrightarrow{\Psi}(0) = \lim_{s\to 0} \overrightarrow{\psi}(s)$; $\overrightarrow{\Psi}'(0) = \lim_{s\to 0} [\frac{d\overrightarrow{\psi_i}}{ds_i}]$.

One can put,

$$\overrightarrow{\Psi}(0) = \begin{bmatrix} \overrightarrow{\psi}(0) \\ \overrightarrow{\psi}'(0) \end{bmatrix} \in \mathbb{R}^{2d}$$

after integration by the parts the condition $(H\overrightarrow{\Psi}, \overrightarrow{\Phi}) = (\overrightarrow{\Psi}, H\overrightarrow{\Phi})$ *gives two restrictions on* $\overrightarrow{\Psi}, \overrightarrow{\Phi}$ *and* v:

a) $v(s) = v^*(s)$, *symmetry of the potential as before.*

b)

$$\overrightarrow{\psi}(0)\overrightarrow{\phi}'(0) - \overrightarrow{\psi}'(0)\overrightarrow{\phi}(0) = 0$$

for the vectors from $\mathbb{R}^{2d}$

$$\overrightarrow{\Psi} = \begin{bmatrix} \overrightarrow{\psi}(0) \\ \overrightarrow{\psi}'(0) \end{bmatrix}, \overrightarrow{\Phi} = \begin{bmatrix} \overrightarrow{\phi}(0) \\ \overrightarrow{\phi}'(0) \end{bmatrix}$$

This relation can be presented as :

$$(J\overrightarrow{\Psi},\overrightarrow{\Phi}) = -(\overrightarrow{\Psi},J\overrightarrow{\Phi}) = 0$$

where,

$$J = \begin{bmatrix} 0 & -I_d \\ I_d & 0 \end{bmatrix}$$

1.5 Elements of symplectic analysis

Properties of the fundamental solution of system 1.5(propagator) and related objects will be described in terms of symplectic group $SP(2d,\mathbb{R})$ *We can now consider the* $\mathbb{R}^{2d}$ *with vectors:* $\xi = \begin{bmatrix} \overrightarrow{\xi} \\ \overrightarrow{\eta} \end{bmatrix}$, $\overrightarrow{\xi}$, $\overrightarrow{\eta} \in \mathbb{R}^d$ *equipped with the usual dot-product*

$$(\overrightarrow{\xi_1},\overrightarrow{\xi_2}) = \begin{bmatrix} \overrightarrow{\xi_1} \\ \overrightarrow{\eta_1} \end{bmatrix} . \begin{bmatrix} \overrightarrow{\xi_2} \\ \overrightarrow{\eta_2} \end{bmatrix} = \overrightarrow{\xi_1}.\overrightarrow{\xi_2} + \overrightarrow{\eta_1}\overrightarrow{\eta_2}$$

and the skew-product

$$\left[\overrightarrow{\xi_1}.\overrightarrow{\xi_2}\right] = (J\overrightarrow{\xi_1},\overrightarrow{\xi_2}) = -(\overrightarrow{\xi_1},\overrightarrow{\eta_2}) + (\overrightarrow{\eta_1},\overrightarrow{\xi_2})$$

We call such space the symplectic space $S\mathbb{R}^{2d}$

Definition 1.5.1. *We call d-dimensional linear subspace* $\mathbb{L} \subset \mathbb{SR}^{2d}$ *the Lagrangian plane if* $[\mathbb{L},\mathbb{L}] = 0$*, that is* $\forall\ \overrightarrow{\xi_1},\overrightarrow{\xi_2} \in \mathbb{L}$*,*

$$[\overrightarrow{\xi_1},\overrightarrow{\xi_2}] = (J\overrightarrow{\xi_1},\overrightarrow{\xi_2}) = 0$$

The condition of the symmetry of H *can be presented now in the following form: all functions* $\overrightarrow{\Psi}$, $\overrightarrow{\Phi}$, ... *from the domain of definition of* H *belong to the fixed Lagrangian plane* $\mathbb{L} \in \mathbb{R}^{2d}$.

Examples of Lagrangian planes and corresponding "gluing conditions" (G.C.)

Example 4. *a)* $\overrightarrow{\psi}(0) = 0$, $\overrightarrow{\psi}'(0)$ *is arbitrary . It is the classical Dirichlet G.C.*

b) $\overrightarrow{\psi}'(0) = 0$, $\overrightarrow{\psi}(0)$ *is arbitrary. It is the Neumann's G.C.*

c) $\overrightarrow{\psi}(s)$ *is continuous at* $s = 0$ *that is* $\overrightarrow{\psi_1}(0) = \overrightarrow{\psi_2}(0) = ... = \overrightarrow{\psi_d}(0)$, *((d − 1) equations)*

and $\sum_{i=1}^{d} \overrightarrow{\psi_i}'(0) = (\overrightarrow{\psi}'.\overrightarrow{1}) = 0$ *(one equation). This is called Kirchhoff's G.C.*

The most general equation of the Lagrangian plane, that is corresponding gluing condition has a form

$$A\overrightarrow{\psi}(0) + B\overrightarrow{\psi}'(0) = 0$$

Where A, B *are* $(d \times d)$ *matrices and* $rank[A, B] = d$. *Assume that* $detB \neq 0$.

That is we can present this condition in the simpler form,

$$\overrightarrow{\psi}'(0) = c\overrightarrow{\psi}(0) \tag{1.7}$$

Proposition 1.5.2. *The relation* (1.7) *defines the Lagrangian iff* $c = c^*$.

That is,

$$\left(J \begin{bmatrix} \overrightarrow{\psi}(0) \\ c\overrightarrow{\psi}(0) \end{bmatrix}, \begin{bmatrix} \overrightarrow{\phi}(0) \\ c\overrightarrow{\phi}(0) \end{bmatrix} \right) = 0$$
$$\Rightarrow \left(\overrightarrow{\psi}(0), c\overrightarrow{\phi}(0)\right) = \left(\overrightarrow{\phi}(0), c\overrightarrow{\psi}(0)\right)$$
$$\Rightarrow c = c^*$$

We will now give the most general equation for $\mathbb{L}$.

Definition 1.5.3. *The group of the non-degenerated linear transformation* $S : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^{2d}$ *preserving the skew product,*$[S\overrightarrow{x}, S\overrightarrow{y}] = [\overrightarrow{x}, \overrightarrow{y}] \, \forall \, (\overrightarrow{x}, \overrightarrow{y})$ *is the symplectic group* $SP(2d)$.

Proposition 1.5.4. *Each symplectic matrix $S \in SP(2d)$ maps Lagrangian plane into Lagrangian plane. In fact, $dim(S\mathbb{L}) = d$ since $detS \neq 0$ and*

$$[S\mathbb{L}, S\mathbb{L}] = [\mathbb{L}, \mathbb{L}] = 0$$

Proposition 1.5.5. *The symplectic matrices satisfy the equation*

$$S^* J S = J \tag{1.8}$$

or taking into account that $J^2 = -I_{2d} = \begin{bmatrix} -I_d & 0 \\ 0 & -I_d \end{bmatrix}$

$$S^{-1} = -JS^*J$$

Proof. The condition $[S\overrightarrow{\xi}, S\overrightarrow{\eta}] = [\overrightarrow{\xi}, \overrightarrow{\eta}]$ gives (since $\overrightarrow{\xi}, \overrightarrow{\eta}$ are arbitrary)

$$\left(JS\overrightarrow{\xi}, S\overrightarrow{\eta}\right) = \left(S^*JS\overrightarrow{\xi}, \overrightarrow{\eta}\right) = \left(J\overrightarrow{\xi}, \overrightarrow{\eta}\right)$$
$$\Rightarrow S^*JS = J$$

Proposition 1.5.6. *$SP(2d)$ forms a group.*

Proof. 1) symplectic matrices are closed under multiplication.

let, s_1, $s_2 \in SP(2d)$ then $(s_1.s_2) \in SP(2d)$ as

$$(s_1.s_2)^* J (s_1.s_2) = (s_2^*.s_1^*) J (s_1.s_2) = s_2^*(s_1^* J s_1) s_2 = s_2^* J s_2 = 2 = J$$

2) Associativity holds as well. let s_1, s_2, $s_3 \in SP(2d)$ then

$$\{s_1.(s_2.s_3)\}^* J \{s_1(s_2.s_3)\} = (s_2.s_3)^* s_1^* J s_1.(s_2.s_3) = s_3^*.s_2^* s_1^* J s_1.s_2.s_3 = s_3^* s_2^* (s_1^* J s_1)(s_2.s_3)$$
$$= s_3^*(s_2^* J s_2).s_3 = s_3^* J s_3 = J$$

also,

$$\{(s_1.s_2).s_3\}^* J\{(s_1.s_2).s_3\} = (s_3)^*(s_1.s_2)^* J\{(s_1.s_2).s_3\} = s_3^*\{(s_1.s_2)^* J(s_1.s_2)\}s_3 = s_3^* J s_3 = J$$

3) Identity exists. That is, $s.e = e.s = s$ where e is the identity element.

$$(s.e)^* J(s.e) = e^*.s^*.J.s.e = e^*.(s^* Js).e = e^* Je = J$$

also,

$$(e.s)^* J(e.s) = s^*.e^*.J.e.s = s^*.(e^* Je).s = s^* Js = J$$

4) Inverse exist with respect to matrix multiplication. since, $s^* Js = J$ and $det(s) = 1$ then

$$s^* Js = J$$
$$s^* J = Js^{-1}$$
$$J^{-1} s^* J = s^{-1}$$

but $J^{-1} = J^*$, this means

$$s^{-1} = J^* s^* J \Rightarrow (s^{-1})^* = J^* SJ$$

hence,

$$(s^{-1})^* J(s^{-1}) = (J^*.s.J).J.(J^*.s^*.J) = (J^*.s.J)(JJ^*).s^*.J$$
$$= (J^*.s.J)(JJ^{-1}).s^*.J = J^*.s.Js^*.J = J^*(s.Js^*)J = J^* JJ = J$$

Note that the class of all matrices O preserving the dot-product

$$(O\overrightarrow{x}, O\overrightarrow{y}) = (\overrightarrow{x}, \overrightarrow{y})$$

is the usual orthogonal group.

Corollary 1.5.7. *Note that,* $det(s^*Js) = (dets)^2.1 = 1$

$$\Rightarrow (dets)^2 = 1$$

We will consider in future only that connected component of $SP(2d)$ *where* $det(S) = 1$

Corollary 1.5.8. *If* $d = 1$ *then* $SP(2)$ *is simply the group of* 2×2 *uni-modular matrices. We consider here only the case of real matrices.*
In-fact,

$$\begin{bmatrix} a & c \\ b & -d \end{bmatrix} . \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & -a \\ d & -b \end{bmatrix} . \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ac - ac & bc - ad \\ ad - bc & bd - bd \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = J$$

$$\Rightarrow detA.J = J \Leftrightarrow detA = 1$$

Let us now consider the general symplectic system of first order and its fundamental solution (monodromy operator)

$$J\frac{dY}{dt} = AY \tag{1.9}$$
$$Y(0) = I$$

where $A = A^*$ is Lipschitz class in $t \in [0,T]$, $A = A(t, \lambda, \mu, ...)$, this $2d \times 2d$ matrix can depend analytically over some parameters (say spectral parameter λ).

Theorem 1.5.9. *Let Y(t)(2d×2d matrix) be the fundamental solution of the boundary problem (1.9). Then $Y(t) :\in \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is symplectic.*

Proof. Since $Y(t)$ is the solution of (1.9) then

$$\frac{dY}{dt} = J^{-1}AY = J^*AY = -JAY \tag{1.10}$$

and

$$\frac{dY^*}{dt} = Y^*AJ$$

Now

$$\begin{aligned}
\frac{dY}{dt}(Y^*JY) &= \frac{dY^*}{dt}JY + Y^*J\frac{dY}{dt} \\
&= (Y^*AJ)JY + Y^*J(-JAY) \\
&= Y^*AJJY - Y^*JJAY \\
&= -Y^*AY + Y^*AY \qquad \text{by using } \frac{dY^*}{dt}, \frac{dY^*}{dt} \text{ and the fact that } J^2 = -I \\
&= 0
\end{aligned}$$

$\Rightarrow Y^*JY$ is constant in $t \geq 0$.

since, $Y(0) = I \Rightarrow Y^*JY = J$ that is $Y(t) \in SP(2d)$

Each symplectic matrix S is by the definition Skew-orthogonal $[Sy_1, Sy_2] \equiv [y_1, y_1]$. This is the fundamental identity, it implies that $detS = \pm 1$. We will study only connected component of unity, $detS = 1$.

If $S=\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ (all blocks are $d \times d$) then equation (1.8) implies the relations

$$a) A'B - B'A = 0; \tag{1.11}$$

$$b) C'D - D'C = 0; \tag{1.12}$$

$$c) A'D - B'C = I; \tag{1.13}$$

$$d) D'A - C'B = I \tag{1.14}$$

the transpose of (1.8) and vice-versa. For real valued matrices we will use A' instead of A^*.

For any two $(2d \times d)$ matrices E$=\begin{pmatrix} A \\ B \end{pmatrix}$, F$=\begin{pmatrix} C \\ D \end{pmatrix}$. One can define the corresponding skew-product(which will appear later as the Wronskian of two $(2d \times d)$ solutions of equation(2). W$=\begin{bmatrix} E & F \end{bmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}' J \begin{pmatrix} C \\ D \end{pmatrix} = A'D - B'C$.

If now E and F are two "halfs" of the symplectic matrix S, then (1.8) can be presented in the form

$$\begin{bmatrix} E & E \end{bmatrix} = \begin{bmatrix} F & F \end{bmatrix} = 0 \tag{1.15}$$
$$\begin{bmatrix} E & F \end{bmatrix} = A'D - B'C = I$$

Each $(2d \times d)$ matrix E$=\begin{bmatrix} A \\ B \end{bmatrix}$ of the maximal rank $r = d$ which is skew orthogonal to itself: $\begin{bmatrix} E & E \end{bmatrix} = 0$, we will call Lagrangian vector and the linear subspace, generated by its columns, will be the Lagrangian plane $\pi = \pi_E$. Dimension $\pi = d$. Any basis B in π has a form $B = EC$, where C is non-singular $d \times d$ matrix.

Remark. *If* $E = \begin{bmatrix} A \\ B \end{bmatrix}$ *is a Lagrangian vector then the vector* $E^{\perp} = \begin{bmatrix} -B \\ A \end{bmatrix}$ *is also Lagrangian and* $(E, E^{\perp}) = 0$*, that is,* E*,* $E^{\perp}$ *are orthogonal in the Euclidean sense.*

1.6 Symplectic representation on the spider quantum graph

We concentrate on the spectral theory related to the symplectic group $SP(2d)$, for the case $d = 3$ (see section 3). Let us repeat our definitions in a bit different form.

Consider $\mathbb{R}^6$, with vectors which we present in the form: $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1' \\ x_2' \\ x_3' \end{bmatrix}$ and $\vec{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_1' \\ y_2' \\ y_3' \end{bmatrix}$

where $\vec{x}$, $\vec{x'}$, $\vec{y}$, $\vec{y'} \in \mathbb{R}^3$ (here "primes" in the second half of each vector are only indices, not derivatives). In the space $\mathbb{R}^6$ we have the usual Euclidean dot-product

$$(\vec{X}, \vec{Y}) = (\vec{x}, \vec{y}) + (\vec{x'}, \vec{y'})$$

let us introduce the new skew-product

$$[\vec{X}, \vec{Y}] = (J\vec{X}, \vec{Y}) = -(\vec{x}, \vec{y'}) + (\vec{x'}, \vec{y})$$

where, $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and I is a $d \times d$ unit matrix

Let us consider matrix Schrödinger equation on $sp(3)$:

$$H\psi = -\psi'' + v\psi = \lambda\psi \tag{1.16}$$

Together with G.C. at $x = 0$ this operator can be represented by 6 equations of first order:

$$JY' = \lambda AY + VY \tag{1.17}$$

where in particular case of $sp(3)$

$$Y = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} = \begin{pmatrix} \vec{\psi} \\ \vec{\psi}' \end{pmatrix}$$

and J is a skew Hermitian matrix of order 6 given by

$$J = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

and,

$$A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

$$V = \left[\begin{array}{ccc|c} v_1 & 0 & 0 & \\ 0 & v_2 & 0 & 0 \\ 0 & 0 & v_3 & \\ \hline & 0 & & I \end{array}\right]$$

In fact,

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \times \begin{bmatrix} \vec{\psi} \\ \vec{\psi}' \end{bmatrix}' = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \times \begin{bmatrix} \vec{\psi}' \\ \vec{\psi}'' \end{bmatrix} = \begin{bmatrix} -\vec{\psi}'' \\ \vec{\psi}' \end{bmatrix}$$

$$= \begin{bmatrix} \lambda I + V & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} \vec{\psi} \\ \vec{\psi}' \end{bmatrix}$$

$$= \begin{bmatrix} \lambda\vec{\psi} + v\vec{\psi} \\ \vec{\psi}' \end{bmatrix}$$

That is

$$-\vec{\psi}'' = \lambda\vec{\psi} + v\vec{\psi}$$

It is our vector Strum-Liouville system.

We will assume that matrix potential V is bounded from below in the sense of eigenvalues ($\min \lambda_i(V) \geq c_0 > -\infty$)In our diagonal from, it means that $v_i(x) \geq c_0$ for $i = 1, 2, 3$

1.7 Boundary condition associated with the symplectic representation

In the case of scalar Strum-Liouville equation:

$$-y'' + v(x)y = \lambda y$$

where $x \in [0, \infty)$, $\lambda \in \mathbb{R}$

the boundary condition at the end point $x = 0$ define the solution up to a constant, say $y(0) = 0$ (Dirichlet condition) and also normalization constant $y'(0) = 1$ gives unique solution $y(\lambda, x)$. Using this solution one can introduce the generalized Fourier transform:

$$\hat{f}(\lambda) = \int_0^\infty y(\lambda, x) f(x) dx = \lim_{N \to 0} \int_0^N y(\lambda, x) f(x) dx$$

(For details see [11]).

For the system of Strum-Liouville equation (1.16) or equivalent symplectic system of order 1, the B.C. at $x = 0$ (say, Dirichlet or Neumann condition) define the family of the solutions, that is, the linear subspace in the functional space.Consider the system (1.17), equivalent to (1.16)

$$JY' = V(x)Y + \lambda AY, x \in \mathbb{R}^{2d}$$

or, more general system in $\mathbb{R}^{2d}$:

$$JY' = [B(x) + \lambda A(x)]Y \tag{1.18}$$

with appropriate assumptions on $2d \times 2d$ matrix functions B(x),A(x)(For details, see [3]).

Here, $Y = [\psi_1, ..., \psi_d, \psi_1', ...\psi_d']^*(x)$, $x \in [0, \infty)$ and $\lambda \in \mathbb{R}$

If for instance $\psi_1(0) = ... = \psi_d(0)$ (Dirichlet B.C.) then $\psi(\lambda, x)$ is the d-dimensional family of solutions with arbitrary values of $\psi_1'(0)$,...$\psi_d'(0)$.

It means that the B.C. at $x = 0$ is given if we fix the d-dimensional manifold π_0 in $\mathbb{R}^{2d}$ but we will assume more: this manifold is the Lagrangian plane, that is $J\pi_0 \perp \pi_0$.

We can introduce such π_0 as : $\{\pi_0 \in M(\vec{v}) : \vec{v} \in \mathbb{R}^{2d} \}$

Here, $2d \times 2d$ matrix M satisfies the equation $M^*JM = 0$, and rank $M = d$. Under such condition π_0 is a Lagrangian plane (i.e. $\pi_0 \perp J\pi_0$)

It means that for all $v_1, v_2 \in \mathbb{R}^{2d}$: $(JMv_1, Mv_2) = (M^*JMv_1, v_2) = 0$

Example 1:

$$M = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}$$

gives the Neumann condition at $x = 0$. and,

$$M = \begin{pmatrix} 0 & 0 \\ 0 & I_d \end{pmatrix}$$

gives Dirichlet B.C.

Example 2: for $d = 3$

$$M = \begin{pmatrix} E & 0 \\ \beta E & M_1 \end{pmatrix}$$

where,

$$M_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

and,

$$E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

given the G.C.: $\psi_1(0) = \psi_2(0) = \psi_3(0)$; $(\psi_1' + \psi_2' + \psi_3') = \beta\psi(0)$

Fundamental solution $Y(\lambda, x)$ of the systems (1.16) and (1.17) that is, solutions with condition $Y(\lambda, 0) = I_{2d}$ belongs to $SP(2d, \mathbb{R})$.

Solution with initial value $\psi(\lambda, 0) = Mv$ where $v \in \mathbb{R}^{2d}$ is given by

$$\psi(\lambda, x) = Y(\lambda, x)Mv$$

CHAPTER 2: BROWNIAN MOTION ON THE SPIDER GRAPH

2.1 Review on Brownian motion on $\mathbb{R}^1$

We will start by giving some reviews on the standard Brownian motion. Consider the space $\mathbb{C}$ of continuous functions $c : t \to x_t = x(t)$ from $[0, +\infty) \to \mathbb{R}^1$. Let us now consider the class of subsets S such that ,

$$S = x_t^{-1}(B) = x_{t_1,t_2,....t_n}^{-1}(B) \qquad\qquad \text{algebra of cylindric set}$$

where $t = (t_1, t_2, ...t_n)$ such that, $0 < t_1 < t_2 < ... < t_n$ and $B \in \mathbb{B}(\mathbb{R}^n)$, $n \geq 1$ of $\mathbb{C}$. $\mathbb{B}(\mathbb{R}^n)$ is the Borel algebra of subsets of R^n. x_t^{-1} is the map inverse to

$$x_t : c \to (x_{t_1}(c), x_{t_2}(c), ..., x_{t_n}(c)) \in \mathbb{R}^n$$

S is an algebra. Also,

$$\mathbb{C} = x_t^{-1}(\mathbb{R}^n)$$
$$\mathbb{C} - x_t^{-1}(B) = x_t^{-1}(\mathbb{R}^n - B)$$
$$x_t^{-1}(B_1) \cup x_t^{-1}(B_2) = x_t^{-1}(B_1 \cup B_2)$$

i.e. $x_t^{-1}\mathbb{B}(\mathbb{R}^1)$ is an algebra. Now consider, the Gauss kernel

$$g(t, a, b) = \frac{e^{-\frac{(b-a)^2}{2t}}}{\sqrt{2\pi t}} \qquad\qquad t > 0, a, b \in \mathbb{R}^1$$

where,

$$P_a[x(t) \in db] = g(t,a,b)db \qquad\qquad (t,a,b) \in (0,+\infty) \times \mathbb{R}^2$$

is the 1-dimensional Brownian motion starting from $a \in \mathbb{R}^1$ at time $t = 0$ and

$$P_t(C) = \int_B .... \int g(t_1, 0, b_1)db_1 g(t_2 - t_1, b_1, b_2)db_2 ....g(t_n - t_{n-1}, b_{n-1}, b_n)db_n$$

where

$$C = x_t^{-1}(B) \qquad\qquad \text{for} \qquad\qquad B \in \mathbb{B}(\mathbb{R}^1)$$

P_t is a probability measure on $x_t^{-1}\mathbb{B}(\mathbb{R}^n)$, which is the Markovian nature of Brownian motion.P_t is well defined.g is the source (Green) function of the problem

$$\frac{\delta u}{\delta t} = \frac{1}{2}\frac{\delta^2 u}{\delta a^2} \qquad\qquad t > 0$$

Since,

$$\begin{aligned} \int_{-\infty}^{\infty} g(t,a,b) &= 2\int_0^{+\infty} \frac{e^{-\frac{-b^2}{2}}}{\sqrt{2\pi}} \\ &= \left(\frac{2}{\pi}\int_0^{+\infty} da \int_0^{+\infty} dbe^{-\frac{a^2}{2}} e^{-\frac{b^2}{2}}\right)^{\frac{1}{2}} \\ &= \left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \int_0^{+\infty} e^{-\frac{r^2}{2}} rdr\right)^{\frac{1}{2}} \\ &= 1 \end{aligned}$$

This implies

$$P_t(\mathbb{R}^1) = 1$$

Also, the so-called Chapman- Kolmogorov equation,

$$\int_{-\infty}^{\infty} g(t-s,a,c)g(s,c,b)dc = \int_{-\infty}^{\infty} \frac{e^{-\frac{(a-c)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} \frac{e^{-\frac{(c-b)^2}{2s}}}{\sqrt{2\pi s}} dc$$
$$= \frac{e^{-\frac{(b-a)^2}{2t}}}{\sqrt{2\pi t}}$$
$$= g(t,a,b) \qquad t > s > 0, a, b \in \mathbb{R}^1$$

implies that, P is the probability measure of S, which can be extended to a Borel probability measure on the Borel extension of S. With that extension the triple $[\mathbb{C}, \mathbb{B}P]$ is called standard Brownian motion starting at 0. We used here the Kolmogorov's criterion of continuity of the random process $x(t)$, if there exist $(\alpha, \delta, c > 0)$ such that for any $t \in [0,T]$, $h > 0$ then,

$$E|x(t+h) - x(t)|^\alpha \leq ch^{1+\delta}$$

then there exists the continuous modification of $x(t)$. In our case we use this criterion in the form

$$E|(x(t+h) - x(t)|^2 = ch^2$$

(but $E|x(t+h) - x(t)|^2 = h$ is not enough for the continuity). P is called Wiener measure. Since, $g(t,a,b) = g(t,0,|b-a|)$, then for given $a \in \mathbb{R}^1$

$$P_a(B) = P_0(c + a \in B) \qquad , \qquad P_a(-c \in B) = P_{-a}(B) \qquad \text{for} \qquad B \in \mathbb{B}$$

where $c + a$ is the translated path $x(t, c+a) = x(t) + a$ and $-c$ is the reflected path $x(t, -c) = -x(t)$. (For more details on Brownian motion see [9]

2.2 Brownian motion on the spider quantum graph with N legs

We consider the following spider graph with N legs, denoted by Γ_N in figure2.1. Each leg of this graph is half axis $(0, \infty)$.

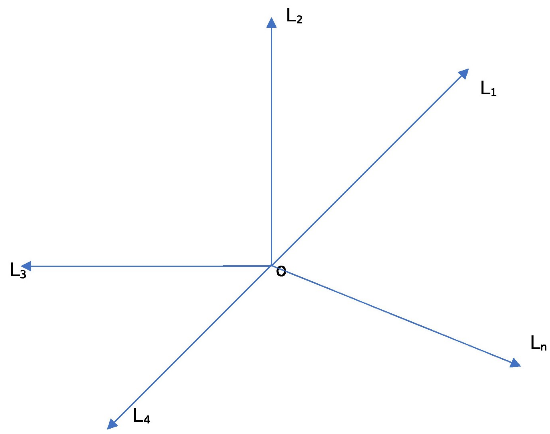

Figure 2.1: Γ_N, the N-legged finite spider graph

We define the Markov generator on each finite leg of length L_i by

$$Hf(x) = \frac{1}{2}\frac{d^2}{dx_i^2}f(x) \qquad x_i > 0 \qquad \text{for } i = 1, 2, ..., N$$

and $f(x) \in C^2(0, \infty)$ on any half axis for $x_i \in (0, \infty)$. $f(x)$ is continuous at $x = 0$, that is,

$$\lim_{x_i \to 0} f(x_i) = f(0)$$

The limits $\frac{dt}{dx_i}(0_+)$ exists for any i and the Kirchhoff boundary condition is satisfied:

$$\sum_{i=1}^{N} \frac{df}{dx_i}(0) = 0$$

Let us consider the Parabolic problem

$$\frac{\delta p}{\delta t} = \frac{1}{2}\frac{\delta^2 p}{\delta y^2} \qquad t > 0$$

$$p(0^+, .) = f$$

on each leg of Γ_N, plus Kirchhoff's gluing condition at the origin. Here $\overrightarrow{y}$ is the parameter on each leg given by $\overrightarrow{y} = (y_1, y_2, ...y_n)$. For the Brownian motion on $\mathbb{R}^1$ the most important Markov times (for more discussion on the Markov process and waiting time see [5]) are passage times, given by

$$m_y = min(t : x_t = y) \qquad\qquad y \in \mathbb{R}^1$$

P.Levy [12] has shown $[m_y, \geq 0, P_0]$ is the one sided stable process with exponent $\frac{1}{2}$ and rate $\sqrt{2}$ satisfying

$$P_0[m_x - m_y \leq t] = P_0[m_{x-y} \leq t] = \int_0^t \frac{x-y}{\sqrt{2\pi s^3}} e^{-\frac{(x-y)^2}{2s}} ds \qquad x \geq y \qquad t \geq 0$$

The reflection principle, proven by D.Andre,

$$P_0(m_y \leq t) = 2P_0(y_t \geq y) = 2\int_y^{+\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dy \qquad\qquad t, y \geq 0$$

helps prove that

$$P_0(m_y \leq t) = 2\int_y^{+\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx = \int_0^t \frac{y}{\sqrt{2\pi s^3}} e^{-\frac{y^2}{2s}} ds \tag{2.1}$$

that is, distribution density of m_y is equal to

$$P_y(t) = \frac{y}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}$$

Due to reflection principle the (1-D) Brownian motion on $\mathbb{R}^1_+ = [0, \infty]$ with reflection BC at $x = 0$ has the following transition probability

$$P_+(t, x, y) = \frac{1}{\sqrt{2\pi t}} \left[e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}} \right] \tag{2.2}$$

Here $x, y \in \mathbb{R}^1_+$ and $-y < 0$ is symmetric to y over the origin. In particular if $x = 0$ then

$$P_t(t, 0, y) = 2e^{-\frac{y^2}{2t}}$$

. Also (1-D) Brownian motion on $\mathbb{R}^1_+$ with Dirichlet boundary condition at $x = 0$ (the process disappears at the moment of the first passage time of $x = 0$) has the transition density

$$P_-(t, x, y) = \frac{1}{\sqrt{2\pi t}} \left[e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}} \right] \tag{2.3}$$

Note that,

$$P_-(t, 0, y) \equiv 0$$

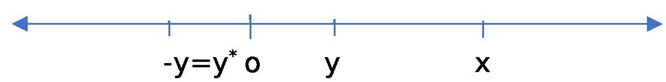

Figure 2.2: Reflection principle on the full real line

Lemma 2.2.1. *Transition density on the N legged spider is given by formulas* (2.4), (2.2.1) *below. It can be considered as reflection principle for the Brownian motion on* sp_N

Proof. We define now the Brownian motion $x(t)$ of the spider graph as follows, assume that we start from $x_i \in Leg_i$ and want to find transition density $P(t, x_i, y_j)$ where $x_i \in Leg_i$, $y_i \in Leg_i$, $i \neq j$ Note that due to the fact that from the starting point 0 process can reach any point $y_j = a \in Leg_j$ with the same probability as $y_{j_1} = a$. It gives,

$$P(t, 0, y_j) = \frac{1}{N} P^+(t, 0, y_j) = \frac{2}{N} \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}} \tag{2.4}$$

If $x_i \in Leg_i$, $y_j \in Leg_j$, $i \neq j$ then the process starting from x_i, must first reach point 0. at some Markove moment $\tau < t$ and in the remaining time $(t - \tau)$ from τ, it must enter y_j. Due to (2.1)

$$P_x\{\tau_0 \in (s+ds)\} = \int_0^t \frac{x}{\sqrt{2\pi s^3}} e^{-\frac{x^2}{2s}} ds$$

and due to (2.4)

$$P_0\{x_{t-s} \in (y, y+dy)\} = \frac{2}{N} \frac{e^{-\frac{y^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}$$

Using strong Markov property we can conclude that

$$P(t, x_i, y_j) = \int_0^t \frac{x_i}{\sqrt{2\pi s^3}} e^{-\frac{x_i^2}{2s}} \times \frac{2}{N} \frac{e^{-\frac{y_j^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} ds \tag{2.5}$$

The case when the final point y belongs to the same leg as x that is $x_i,\ y_i \in Leg_i$ is different. Here there are two options. Either process starting x_i enters to y_i before passing to 0. Corresponding density given by (2.3):

$$P_{-}(t, x_i, y_i) = \frac{1}{\sqrt{2\pi t}} \left[e^{-\frac{(x_i-y_i)^2}{2t}} - e^{-\frac{(x_i+y_i)^2}{2t}} \right]$$

or $\tau_0 < t$ then using (2.5) we will get additional probability.

$$\tilde{P_+} = \int_0^t \frac{x_i}{\sqrt{2\pi s^3}} e^{-\frac{x_i^2}{2s}} \times \frac{2}{N} \frac{e^{-\frac{y_i^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} ds \tag{2.6}$$

finally,

$$\begin{aligned} P(t, x_i, y_i) &= P_{-}(t, x_i, y_i) + \tilde{P_+}(t, x_i, y_i) \qquad (2.7) \\ &= P_{-}(t, x_i, y_i) + \int_0^t \frac{x_i}{\sqrt{2\pi s^3}} e^{-\frac{x_i^2}{2s}} \times \frac{2}{N} \frac{e^{-\frac{y_i^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} ds \\ &= \frac{1}{\sqrt{2\pi t}} \left[e^{-\frac{(x_i-\tilde{x_i})^2}{2t}} - e^{-\frac{(x_i+\tilde{x_i})^2}{2t}} \right] + \frac{2}{N} \int_0^t \frac{x_i}{\sqrt{2\pi s^3}} e^{-\frac{x_i^2}{2s}} \times \frac{e^{-\frac{\tilde{x_i}^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} ds \end{aligned}$$

Using result (2.5) we get,

$$P\left(t,x_i,y_i\right)=\frac{1}{\sqrt{2\pi t}}\left[e^{-\frac{(x_i-y_i)^2}{2t}}-\left(1-\frac{2}{N}\right)e^{-\frac{(x_i+y_i)^2}{2t}}\right]$$

This completes the lemma.

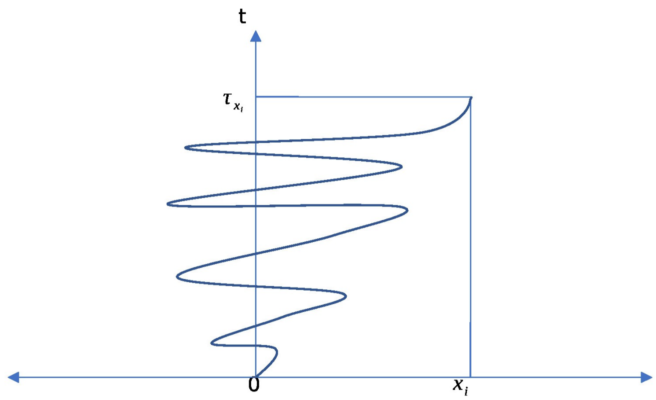

Figure 2.3: The Brownian motion reaching point x_i from 0

Let τ_x be the first entry moment from x to 0 for 1-D Brownian motion. It is clear, due to symmetry that τ_x has the same law as the moment it enters x from 0, that is,

$$P_0\{\tau_{x_i}\in(s,s+ds)\}=-\frac{x}{\sqrt{2\pi s^3}}e^{-\frac{x^2}{2s}}$$

See figure 2.4

Similar picture for $N=3$ (very rough similarity, see figure 2.5), since $x(t)$ visits all three planes infinitely many times.

Let τ_L be the first exit time from the L-neighbourhood of the origin of N-legged spider, that is,

$$\tau_L=min(t:x_i(t)=L)\qquad\text{for some}\qquad i=1,2,3,...,N$$

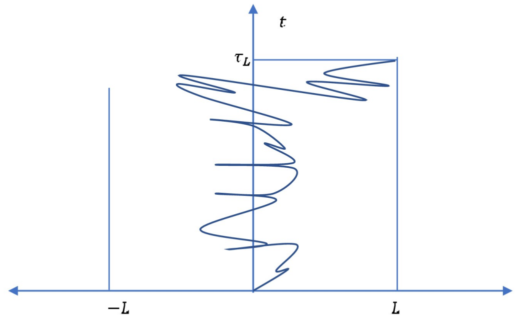

Figure 2.4: The first moment Brownian motion enters to one of the two end points $\pm L$ for $N = 2$

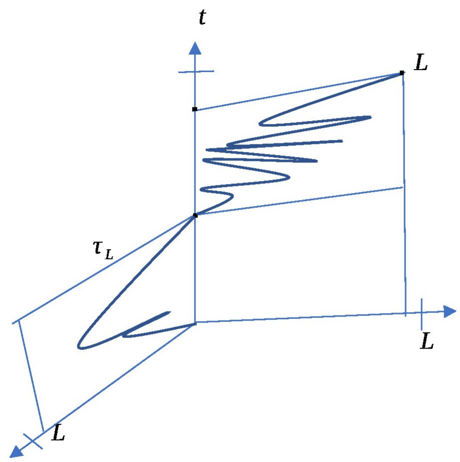

Figure 2.5: First moment of Brownian motion for $N = 3$

then,

$$E_{x_i} e^{-\lambda \tau_L} = \psi(x_i) = \psi(x_i, 1)$$

$$\psi_i(L) = 1$$

satisfies the parabolic problem

$$\frac{1}{2}\psi_i'' - \lambda \psi_i = 0 \qquad i = 1, 2, , ...N \qquad \text{with Kirchhoff's gluing condition} \tag{2.8}$$

then,

$$\psi_i(x_i) = \frac{\cosh\sqrt{2\lambda}x_i}{\cosh\sqrt{2\lambda}L}$$

Infact, since

$$\cosh'(0) = \sinh(0) = 0$$

we have the Kirchhoff's gluing condition condition at 0.

Let us note that

$$\max\psi(x_i) = \psi(0)$$

and we also have the $x(t)$'s self similarity property, which gives,

$$\psi_i(0) = E_0 e^{-\lambda\tau_1} = \frac{1}{\cosh\sqrt{2\lambda}L}$$

Infact,

$$E_0 e^{-\lambda\tau_L} = E_0 e^{-\lambda\frac{\tau_1}{L^2}} = E_0\frac{1}{\cosh\sqrt{2\frac{\lambda}{L^2}}L} = \frac{1}{\cosh\sqrt{2\lambda}}$$

where

$$\frac{\tau_1}{L^2} \sim \tau_1$$

One can calculate all moments of τ_L:

$$E_0\tau_L = -\frac{d\psi_L}{d\lambda}/_{\lambda=0} = L_0^2 E\tau_1 \tag{2.9}$$

$$E_0\tau_1 = -\left(\frac{1}{\cosh\sqrt{2\lambda}}\right)'/_{\lambda=0} = \frac{\sinh\sqrt{2\lambda}\frac{\sqrt{2}}{2\sqrt{\lambda}}}{\cosh^2\sqrt{2\lambda}} = 1$$

In general,

$$\begin{aligned}
E_0 e^{-\lambda\tau_1} &= (1 - \lambda E_0\tau_1 + \frac{\lambda^2}{2!}E\tau_1^2 + ...) \\
&= \frac{1}{\cosh\sqrt{2\lambda}} \\
&= \frac{1}{1 + \frac{(\sqrt{2\lambda})^2}{2!} + \frac{(\sqrt{2\lambda})^4}{4!} + ...} \\
&= \frac{1}{1 + \lambda + \frac{\lambda^2}{6} + ...} \\
&= 1 - (\lambda + \frac{\lambda^2}{6} + ...) + (\lambda + \frac{\lambda^2}{6} + ...)^2 + ... \\
&= 1 - \lambda + \frac{5}{6}\lambda^2 + ...
\end{aligned}$$

$E_0\tau = 1,\ E_0\tau^2 = \frac{5}{3},....$

Now we calculate the densities for τ_1.

Roots of $\cosh\sqrt{2\lambda}$ is given by the equation,

$$\begin{aligned}
\cosh\sqrt{2\lambda} &= 0 \\
\Rightarrow \sqrt{2\lambda} &= i(\frac{\pi}{2} + \pi n) \\
\Rightarrow \lambda_n &= -\frac{\pi^2(2n+1)^2}{8} \qquad\qquad n \geq 0
\end{aligned}$$

It gives the infinite product,

$$\cosh\sqrt{2\lambda} = \left(1 + \frac{8\lambda}{\pi^2}\right) \cdot \left(1 + \frac{8\lambda}{(3\pi)^2}\right) \cdot \left(1 + \frac{8\lambda}{(5\pi)^2}\right) .... \left(1 + \frac{8\lambda}{((2n+1)\pi)^2}\right) .... \tag{2.10}$$

Hence for the Brownian motion to visit the end point and come back to 0 on one of the legs of spider, we get

$$E_0 e^{-\lambda(\tau_1 + \tilde{\tau_1})} \sim E_0 e^{-\lambda(\frac{\tau_1}{L^2} + \frac{\tilde{\tau_1}}{L^2})} = \frac{1}{\cosh^2\sqrt{2\lambda}} \tag{2.11}$$

where $\tilde{\tau}_1$ is the time Brownian motion takes to come back to point 0 after visiting end point L on one of the legs.

Then,

$$E_{x_i}\tilde{\tau}_1 = \psi(x)$$

where $\psi(x_i)$ satisfies:

$$\frac{1}{2}\frac{d\psi}{dx_i} = -1$$

with $\psi(x_i)/_{x_i=L} = 0$

that is,

$$\psi(x_i) = L^2 - x_i^2 \qquad i = 1, 2, ..N$$

and

$$P_0\{x_{\tau_{\tilde{\tau}_L}} = L_i\} = \frac{1}{N}$$

Let us find the expansion of Laplace transform of $\frac{1}{\cosh\sqrt{2\lambda}}$ into simple form. It is known that,([7])

$$\frac{1}{\cos\frac{\pi x}{2}} = \frac{4}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k-1}{(2k-1)^2 - x^2}$$

Using the substitution

$$\frac{\pi x}{2} = z \Rightarrow x = \frac{2z}{\pi}$$

and formula

$$\frac{1}{\cosh\sqrt{2\lambda}} = \frac{1}{\cos i\sqrt{2\lambda}}$$

we will get,

$$\frac{1}{\cosh\sqrt{2\lambda}} = \sum_{k=1}^{\infty}(-1)^k \frac{4(2k-1)\pi}{\pi^2(2k-1)^2+8\lambda}$$
$$= \sum_{k=1}^{\infty}(-1)^{k+1} \frac{(2k-1)\pi 2}{\lambda + \frac{\pi^2(2k-1)^2}{8}}$$

Now applying inverse Laplace transform, we have

$$P_\tau(s) = \sum_{k=1}^{\infty}(-1)^k \frac{(2k-1)\pi}{2} e^{-s} \frac{\pi^2(2k-1)^2}{8}$$

This series converges extremely fast.

Let us now consider

$$T_N = \xi_1 + \xi_2 + ... + \xi_N \qquad \text{where} \qquad \xi_1 = (\tau_1 + \tilde{\tau_1}), ..., \xi_N = (\tau_n + \tilde{\tau_N})$$

ξ_1, ξ_2, ...,ξ_N generate complete Brownian motion cycles on the corresponding spider legs.

Then,

$$E_0 e^{-\lambda T_N} \sim E_0 e^{-\lambda \frac{T_N}{L^2}} = (\frac{1}{\cosh^2\sqrt{2\lambda}})^N \tag{2.12}$$

CHAPTER 3: A BRIEF REVIEW ON THE CLASSICAL SPECTRAL THEORY

In this chapter I will give some review on the classical spectral theory in the spirit of Strum-Liouville theory.

3.1 Spectral theory on the finite interval

Let's consider the spectral problem (1.4).on $\mathbb{L}^2(0, L)$ with the boundary conditions $Y(0) \in \pi_0$, $Y(L) \in \pi_L$ where π_0, π_L are fixed Lagrangian planes. If π_0, π_L are given by the basis $E_0 = \begin{bmatrix} A \\ B \end{bmatrix}$ (for π_0) and $E_L = \begin{bmatrix} C \\ D \end{bmatrix}$ (for π_L) then we can specify two particular $(2d \times d)$ matrix solutions $Y^{\pm}(x)$,$x \in [0, L]$ for (2) by conditions $Y^+(0) = E_0$, $Y^-(L) = E_L$. It is equivalent to the system (1) with conditions,

$$y^+(0) = A, \dot{y}^+(0) = B, \qquad Y^+(x) = \begin{bmatrix} y^+(x) \\ \dot{y}^+(x) \end{bmatrix} \tag{3.1}$$

$$y^-(L) = C, \dot{y}^-(L) = D, \qquad Y^-(x) = \begin{bmatrix} y^-(x) \\ \dot{y}^-(x) \end{bmatrix}$$

let $M_\lambda(x)$ be the propagator for the canonical system (2), that is,

$$-J\dot{M}_\lambda = (v + \lambda Q)M_\lambda; x \geq 0, M_\lambda(0) = I_{2n} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and the fundamental fact is $M_\lambda(0, x) \in SP(2d, \mathbb{R})$:to prove it just differentiate

$$M_\lambda^{'}(x)JM_\lambda(x) = s(x)$$

and check that $\dot{s}(x) = 0,\ s(0) = J$

It gives for any two matrix $2d \times d$ solutions $v_1,\ v_2$ of(2), the important relation

$$[v_1(x), v_2(x)] = [M_\lambda(0,x)v_1(0), [M_\lambda(0,x)v_2(0)] = [v_1(0), v_2(0)]$$

In particular it means

$$[Y_\lambda^+(x), Y_\lambda^-(x)] = y_\lambda^+(x)\dot{y}_\lambda^-(x) - \dot{y}_\lambda^+(x)y_\lambda^-(x) = W(\lambda)$$

where $W(\lambda)$ is $d \times d$ matrix (Wronskian). According to classical result from the linear system of ODE, the propagator $M_\lambda(L)$ is analytical function of λ.
It gives the discreteness of the spectrum of problem (1.5) with boundary condition this spectrum is real due to standard symmetry of the Hamiltonian H and corresponding eigenfunctions are orthogonal. The orthogonality of the eigenfunctions is corresponding to the different real eigenvalues (but for multiple eigenvalues it can be selected).

The spectral problem (1) with boundary Lagrangian planes π_0, π_L is equivalent to the integral equation

$$(\lambda - \lambda_0) \int_0^L G_{\lambda_0}(x_1, x_2) y(x_2) dx_2 = y(x_1) \tag{3.2}$$

with symmetric Green's kernel

$$G_{\lambda_0}(x_1, x_2) = \begin{cases} y_{\lambda_0}^+(x_1) W^{-1}(\lambda_0) \dot{y}_{\lambda_0}^-(x_2) 0, & x_1 < x_2 \\ y_{\lambda_0}^-(x_1) W^{-1}(\lambda_0) \dot{y}_{\lambda_0}^+(x_2) 0, & x_1 \geq x_2 \end{cases} \tag{3.3}$$

To prove (3.3) we can consider the matrix

$$s(x) = \begin{bmatrix} y^+_{\lambda_0}(x) & y^-_{\lambda_0}(x) & W^{-1}(\lambda_0) \\ \dot{y}^+_{\lambda_0}(x) & \dot{y}^-_{\lambda_0}(x) & W^{-1}(\lambda_0) \end{bmatrix}$$

it is easy to see (by using 1.11-1.14) that $s(x) \in SP(2d, \mathbb{R})$ that is $s'(x) \in SP(2d, \mathbb{R})$. It implies that $\begin{bmatrix} \dot{y}^+_{\lambda_0}(x) \\ W^{-1}\dot{y}^-_{\lambda_0}(x) \end{bmatrix}$ is a Lagrangian vector. This implies also the continuity relation for Green's Kernel $G_{\lambda_0}(x_1, x_2)$ on the diagonal $x_1 = x_2$.

also,

$$I = \begin{bmatrix} y^+_\lambda & \dot{y}^+_\lambda \\ W^{-1}y^-_\lambda & \dot{y}^-_\lambda W^{-1} \end{bmatrix}$$

that is,

$$y^+_\lambda W^{-1}\dot{y}^-_\lambda - \dot{y}^+_\lambda W^{-1} y^-_\lambda = I$$

and it is the condition for the jump of derivative of the kernel on the diagonal.

Classical result on the compact symmetric operators gives us now the completeness of the eigenbasis for (3.2).

3.2 Spectral theory on the finite interval for the spider graph

Let $x \in [0, L]$ that is, on the graph $\gamma_3(L)$, we must introduce two B.C. at the end points $x = 0$ and $x = L$.

They will have the form :

$$y(0) = Mv; y(L) = Nv \tag{3.4}$$

for $v \in \mathbb{R}^{2d}$

matrix N satisfies the same condition as M : $N^*JN = 0$ rank $N = 3$

If $Y(\lambda, x)$ is the fundamental solution of our system for fixed (real) spectral parameter λ then,

$$det[N - Y(\lambda, L)M] = 0 \tag{3.5}$$

This is the characteristic equation for λ. It follows from our boundary condition (3.4). Since, for fixed L the fundamental solution $Y(\lambda, L)$ is the matrix valued analytic function of λ, the spectrum of our system on the finite interval is discrete and corresponding system of eigenfunctions is complete in $\Gamma_3(L)$. We can then construct the spectral measure in $\mathbb{L}^2(\Gamma_3, dx)$ using passing to the limit approach.

3.3 General spectral theory

For simplicity let us consider Neumann's boundary condition $\dot{y}(0) = 0$ and consider for any λ the $(d \times d)$ matrix solution y_λ^+ of the problem $Hy = \lambda y$ with initial data $y_\lambda^+(0) = I$, $y_\lambda^+(0) = 0$

For any compactly supported function $\phi(x) = [\phi_1,\phi_2, ..., \phi_n]' \subset \mathbb{C}^2(\mathbb{R}^+)$ we can define its generalized Fourier transform

$$\hat{\phi}(\lambda) = \int_0^L y_\lambda^+(x)\phi(x)dx$$

which is independent of L iff $support(\phi) \subset [0, L]$.

If $y_1(x), y_2(x), ...$ are the eigenfunctions of the problem $Hy = \lambda y$ (with $\dot{y}(0) = 0$ and some boundary condition $[y(L), \dot{y}(L)]' \in \pi_L$) and $\lambda_1, \lambda_2, ...$ are corresponding eigenvalues then we can define the matrix-valued spectral measure $\mu_{\lambda(d\lambda)}$.

by the formula,

$$\mu_L(d\lambda) = \sum_{i=1}^{\infty} d\lambda\delta(\lambda - \lambda_i(L))\frac{y_\lambda(0)y_\lambda'(0)}{\int_0^L y_\lambda^2(x)dx} \tag{3.6}$$

$$Tr\left(\mu_L(d\lambda)\right) = \sum_{i=1}^{\infty} \frac{\left(y_\lambda(0)\right)^2 \delta(\lambda - \lambda_i(L))d\lambda}{\int_0^L y_\lambda^2(x)dx} \tag{3.7}$$

Now we can present $\phi(x)$ using the expansion over eigenfunctions $y_i(x, L)$.

$$\begin{aligned}\phi(x) &= \sum_{i=1}^{\infty} \frac{y_{\lambda_i}(x)}{\int_0^L y^+_{\lambda_i}(x)\phi(x)dx} \\ &= \int_0^L y^+_\lambda(x)\overline{\mu}_L(d\lambda)\hat{\phi}(\lambda)\end{aligned} \tag{3.8}$$

Due to completeness we have the Parseval identity

$$\int_0^L \phi^2(x)dx = \int_{\mathbb{R}} \hat{\phi}'_\lambda(\lambda)\mu_L(d\lambda)\hat{\phi}(\lambda) \tag{3.9}$$

Lemma 3.3.1. *If* $\int_x^{x+1} \|v(z)\|dz = L_0 \leq \sup_{x+1}(\int_0^x \|v(z)\|^2 dz)^{\frac{1}{2}}$ *then*

$$E_0 = min \sum(H) \geq -L_0(L_0+1)$$

This is Birman's type estimation (see [6])

Proof. To prove this, we can say, due to Neumann-Dirichlet condition, it is sufficient to show that for the unit interval we have estimation $\lambda_0 \geq -L_0(L_0+1)$ for principle eigenvalue of the Hamiltonian $Hy = \lambda y$ with $\dot{y}(0) = \dot{y}(1) = 0$.

but,

$$\lambda_0 = \min_{y:\|y\|_2=1} \int_0^1 [\dot{y}^2 + vy.y]dx$$

Now one can find point $x_0 \in [0, 1]$ such that $|y(x_0)| = 1$

then

$$y^2(x) - y^2(x_0) = 2\int_{x_0}^x (y, \dot{y})dz$$

That is for any $x \in [0,1]$

$$|y^2(x)| \leq 1 + \frac{1}{\epsilon}\int_0^1 y^2 dz + \epsilon \int_0^1 \dot{y}^2 dz$$

now,

$$\lambda_0 \geq \min_{y:\|y\|_2=1}\left[\int_0^1 \dot{y}^2 dz - L_0(1+\frac{1}{\epsilon}) - \epsilon L_0 \int_0^1 \dot{y}^2 dz\right]$$

If, $\epsilon = \frac{1}{L_0}$ then $\lambda_0 \geq -L_0(1+L_0)$

Lemma 3.3.2. *(Uniform Bound of the Spectral Measure)*

For the Hamiltonian in (1.4) *with Neumann's boundary condition* $[0, L]$ *and for any* $\wedge \geq 0$ *and appropriate constant* $c_0 > 0$

$$Tr\mu_L(-L_0, \wedge) \leq c_0(1+\sqrt{\wedge})$$

Proof. Solution $y_\lambda(x)$ for $\lambda \in [-L_0, \wedge]$ satisfies the integral equation

$$y_\lambda(x) = \cos\sqrt{\lambda I x} + \int_0^x \frac{\sin\sqrt{\lambda I}(x-z)}{\sqrt{\lambda I}} v(z) y_\lambda(z) dz$$

If $x\sqrt{\wedge} \leq 1$ then Bellman-Gronwall estimation gives,

$$y_\lambda(x) = \cos\sqrt{\lambda I}x(1+R_\lambda) \qquad \|R_\lambda\| \leq \frac{1}{2} \tag{3.10}$$

Now let us select test function $\psi_n(x)$ such that $\|\psi_n^2\|_2 = 1$, $Support(\psi_n) \in [0, h]$, $h\sqrt{\wedge} \leq 1$.

Standard application of the Parseval identity to the functions $\phi_n(x), \hat{\psi_n}(\lambda)$ provides the desirable estimation (we can compare this result with [11]).

Now we can pass to the limit $L \to \infty$ using Hallie's lemma and prove that $\mu_L(d\lambda) \to \mu(d\lambda)$ in weak limit (on $\mathbb{C}_0(\mathbb{R}^1_+)$). The limiting matrix spectral measure $\mu(d\lambda)$ is unique which does not depend on $L_n \to \infty$ and boundary conditions. It satisfies the estimations of the previous lemmas.. For any $\phi(x) \in L^2(\mathbb{R}_+)$ we can define the generalized Fourier transform in the Parseval sense, that is, if

$$\hat{\phi}_L(\lambda) = \int_0^L y_\lambda^+(x)\phi(x)dx \Rightarrow \lim \hat{\phi}_L(\lambda) = \hat{\phi}(\lambda)$$

with respect to spectral measure $\mu(d\lambda)$

We can reconstruct $\phi(x)$ using the inverse Fourier transform:

$$\phi(x) = \int_{-E_0}^{\infty} y_\lambda^+(x)\mu(d\lambda)\hat{\phi}(\lambda)$$

(again, in the Parseval sense) together with Parseval identity

$$\int_0^\infty \phi^2(x)dx = \int_{-E_0}^{\infty} \hat{\phi}'(\lambda)\mu(d\lambda)\hat{\phi}(\lambda)$$

3.4 Construction of the spectral measure on the spider graph

Construction of the spectral measure is based on the transition from the spectral measure on $\Gamma_3(L)$ to its weak limit if L $\to \infty$. Consider the spectral problem:

$$H\psi = J\psi' + (\lambda A + V)\psi = 0$$

with the B.C. $\psi(0) = Mv$, $\psi(L) = Nv$

Let, λ_n be eigenvalues and $\psi_n(x)$ are eigenfunctions with normalization condition

$$(\psi_n, A\psi_m) = \delta_{mn}$$

They form the complete system in $\mathbb{L}^2\left(\Gamma_3(L), dx\right)$

Let, $u_n = (A\psi_n)(0)$ and $\mu_L(d\lambda) = \sum_n \delta(\lambda - \lambda_n) u_n u_n^*$

Note that, $u_n \times u_n^*$ is a 3×3 positive definite matrix: the tensor squares of the vectors u_n, $n \geq 1$

The general theory contains the theorem on the existence of the weak limit of the measures $\mu_L(d\lambda)$, L$\rightarrow \infty$ (for details see [3] chapter 9) This approach is different from scalar Strum-Liouville theory, based on the generalized direct and inverse Fourier transform [11].

For some classes of the matrix self-adjoint operators, one can also develop the spectral theory based on the Fourier type integral transformation.

Consider the matrix Strum-Liouville spectral problem

$$-\vec{\psi}''(x) + Q(x)\vec{\psi} = \lambda\vec{\psi}(x), x \geq 0 \tag{3.11}$$

$$\vec{\psi}(x) = (\psi_1(x), ..., \psi_d(x))^* \qquad \text{and} \qquad Q(x) = Q^*(x)$$

Let also take $d \times d$ matrix potential $Q(x) > 0$, in the sense of quadratic form:

$$(Q\vec{a}, \vec{a}) > 0 \qquad \text{for all } x \in [0, \infty); \text{ a} \in \mathbb{R}^d$$

This system (like the previous case) can be represented as the canonical from :

$$J\frac{d\vec{\psi}}{dx} = (\lambda A + \tilde{Q})\vec{\psi} \tag{3.12}$$

where,

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} ; \vec{\psi}(\lambda, x) = \begin{pmatrix} \vec{\psi} \\ \vec{\psi}' \end{pmatrix} ; A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} ; \tilde{Q} = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}$$

Corresponding fundamental solution belongs to $S_p(2d)$ and we can use the approach used in [3] to explain the problem presented above.

However, there is another approach : Consider spectral theory for the equation (3.12), with, say Neumann B.C. $\vec{\psi}'(0) = 0$.

It defines the d-dimensional Lagrangian plane π_0 of the functions $(\psi_1, \ldots, \psi_d) = \vec{\psi}$: $\vec{\psi}(0) = 0$ but $\vec{\psi}'(0) \in \mathbb{R}^d$ is an arbitrary vector.

Let us select basis in $\pi(0)$:

$$\vec{\psi}_{i,0}(\lambda, x) : \vec{\psi}_{i,0}(\lambda, 0) = 0, \psi'_{i,0}(\lambda, 0) = (0, \ldots, 0, 1, 0, \ldots, 0)^* \qquad \text{for, } i = 1, \ldots, d$$

For arbitrary vector function $\phi(x) \in L^2\left([0, \infty), dx\right)$ we can define the Fourier transform

$$\hat{\phi}_i(\lambda) = \int_0^\infty \left(\vec{\phi}(x), \vec{\psi}_{i,0}\right) dx \qquad\qquad \text{for } i = 1, \ldots, d$$

(in the beginning, for functions with bounded support and after, using $\mathbb{L}^2$-approximation of the general function)

now we can introduce,

$$\mu_L(d\lambda) = \sum_{i=1}^{\infty} d\lambda \delta(\lambda - \lambda_i(L)) \frac{\vec{\psi}_i(0)\vec{\psi}_i^*(0)}{\int_0^L \vec{\psi}_i^2 dx}$$

and

$$tr\mu_L(d\lambda) = \sum \frac{\vec{\psi}_i^2(0) d\lambda \delta(\lambda - \lambda_i)}{\int_0^L \vec{\psi}_i(\lambda, x)}$$

note that: $\psi_i(0) \times (\psi_i(0))^*$ is the $d \times d$ matrix (tensor product of vectors) and $\psi_i^* \psi_i$ is the dot product, such that :

$$\phi(x) = \int_0^L \psi_i(\lambda, x) \bar{\mu_L}(d\lambda) \hat{\phi}_i(\lambda)$$

then, due to completeness, we have the Parseval's identity

$$\int_0^L \phi^2(x) dx = \int_{\mathbb{R}} \hat{\phi}_i^* \mu_L(d\lambda) \hat{\phi}_i \tag{3.13}$$

If we take $\phi_0(x)$ supported on $[0, h]$, $h << 1$ and solve over system (3.13) on $[0, h]$ using the equivalent integral equation and iterations as in [11] then we will get the weak compactness of μ_L on each interval of λ-axis. Now if one takes $L \to \infty$, then by Hellie's lemma it can be proved $\mu_L(d\lambda) \to \mu(d\lambda)$ in the weak sense on $C_0(\mathbb{R}^+)$. The limiting spectral measure $\mu(d\lambda)$ is unique. It does not depend on L or boundary conditions. The generalized Fourier transformation is given by,

$$\hat{\phi_L}(\lambda) = \int_0^L \psi_i(\lambda, x) \phi(x)$$

This implies

$$\lim \hat{\phi_L} = \hat{\phi}(\lambda)$$

with respect to the spectral measure $\mu(d\lambda)$

CHAPTER 4: THE SPECTRAL THEORY OF THE SCHRÖDINGER OPERATOR ON THE SPIDER-LIKE QUANTUM GRAPHS

4.1 Introduction to the spectral theory of Laplacian

Here we will consider the Schrödinger operator on the special case of quantum graphs. There are two versions of this theory : continuous and lattice cases. We will study here only the the continuous case. Consider the graph sp_N for $N \geq 2$, which consists of half-line $[0, \infty)$ connected at the fixed point 0 (origin). To simplify notations we will take,in some cases $N = 3$).

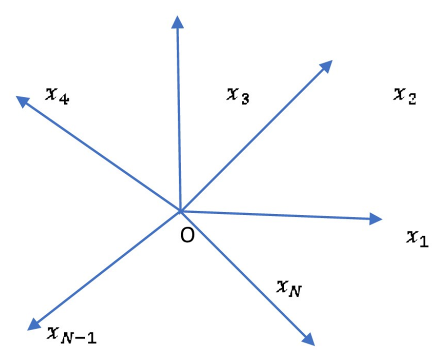

Figure 4.1: Spider graph with N legs

On each leg of this spider like graph we introduce the coordinates $x_1, x_2, ..., x_N$ and they are increasing in the corresponding directions from 0 to ∞. The Lebesgue measure on each leg of s_{p_N} is defined as $dm = (dm_1, dm_2, ...dm_N)$ on each leg with differentials dx_1,dx_2,..., dx_N.Consider on s_{p_N} the space of compactly supported smooth functions like,

$$f(x_1, x_2, ..., x_N) = [f_1(x_1) : x_1 \in (0, \infty), f_2(x_2) : x_2 \in (0, \infty), ..., f_N(x_N) : x_N \in (0, \infty)]$$

with $\sum_{i=1}^{N}\int_0^\infty |f_i|^2(x_i)dx_i = \|f\|_2^2$. f is not a vector function. $f \in \mathbb{L}^2(sp)_N)$ are simply restrictions of f along the legs (without the origin).Space $\mathbb{L}^2(sp_{N)}$ contains, in general, only measurable functions but it is the closure of the class of compactly supported $\mathbb{C}^\infty$ functions $f = (f_1, ...f_N)$ on each leg with the appropriate gluing conditions at 0. Let us describe this conditions.

a) First we assume that the following limits exist and equal. It confirms the continuity of f on sp_N.

$$f(0) = \lim_{x_1 \to 0} f_1(x_1) = \lim_{x_2 \to 0} f_2(x_2) = ... = \lim_{x_N \to 0} f_N(x_N) \tag{4.1}$$

b) also, we will assume that f has right derivatives at point 0,that is, $\frac{df}{dx_1}(0)$, $\frac{df}{dx_2}(0)$.....$\frac{df}{dx_N}(0)$ exist on each half-axis correspondingly.

and ,

$$\sum_i \frac{df}{dx_i}(0) = 0 \qquad \text{Kirchhoff's condition} \tag{4.2}$$

Note: We will have $N - 1$ continuity conditions for $f(.)$:

$$\lim_{x_1 \to \infty} f(x_1) = \lim_{x_2 \to \infty} f(x_2), \lim_{x_1 \to \infty} f(x_1) = \lim_{x_3 \to \infty} f(x_3)......., \lim_{x_1 \to \infty} f(x_1) = \lim_{x_N \to \infty} f(x_N)$$

It means that vector $(\vec{f}(0), \vec{f}'(0))$ with $2N$ components, satisfying the gluing condition (4.1) and 4.2 at the origin 0.

Function f on the spider is defined as follows: $f(x_1, x_2, ...x_n) = \{f(x_1), f(x_2), ...f(x_N)\}$; $x_1 \in$Leg$_1$,$x_2 \in$ Leg$_2$,...,$x_N \in$ Leg$_N$ and $f(x_1)$,$f(x_2)$,...$f(x_N)$ are functions on Leg_i for $i = 1, 2, 3..., N$.

The last condition (4.2) means that

$$\sum_{i=1}^{N} \frac{df_i}{dx_i}(0) = 0$$

The operator $-\Delta$ (the Laplacian), on Γ with the gluing conditions (4.1) and (4.2) is given by

$$-\Delta f = \begin{cases} -\frac{d^2 f}{dx_1^2}, & \text{if } x \in (0,\infty)\text{along leg 1} \\ -\frac{d^2 f}{dx_2^2}, & \text{if } x_2 \in (0,\infty)\text{along leg 2} \\ \dots\dots\dots\dots & \\ -\frac{d^2 f}{dx_N^2} & \text{if } x_N \in (0,\infty)\text{along leg N} \end{cases} \tag{4.3}$$

Let us look at the Laplacian $-\Delta$ from the functional analysis perspective.

Let $\mathbb{L}^2(sp_N, dm)$ is the Hilbert space of square integrable functions on sp_N (in our particular case we consider N=3) with the dot product defined as :

$$< f, g >= \int_{sp_N} f.\bar{g}dm = \sum_{i=1}^{N} \left(\int_0^\infty f_i \bar{g}_i dm_i \right) \tag{4.4}$$

For $N = 3$ that is, in our case,

$$< f, g >= \int_0^\infty (f_1.\bar{g_1})(x)dx + \int_0^\infty (f_2.\bar{g_2})(y)dy + \int_0^\infty (f_3.\bar{g_3})(z)dz \tag{4.5}$$

Consider on $\mathbb{L}^2(sp_N, dm)$, the dense set of compactly supported $\mathbb{C}^\infty$-functions on each leg with gluing conditions (4.1), (4.2). On such functions we already defined the Laplacian $-\Delta = -\frac{d^2}{di^2}$ on each Leg_i. We will now give the sketch of the spectral theory of the Laplacian $-\Delta$ on $L^2(sp_N, dm)$. For each $\lambda \in \mathbb{R}$ we define the fundamental system of solutions of the equation $-\Delta f = \lambda f$ with gluing conditions (4.1),(4.2).

Let, $\lambda = k^2 > 0$ then on each leg, the general solution of $-\frac{d^2 f}{dx^2} = k^2 f$ has the form:

$$f_i(x_i) = c_i \cos kx_i + d_i \sin kx_i \qquad\qquad i = 1, ..., N$$

where, $\cos kx_i$ and $\sin kx_i$ are two linearly independent solutions on Leg_i Note: For the N-legged spider we will have $2N$ solutions, two linearly independent solutions on each leg.

Due the gluing condition (4.1), $c_i = c_0 = f(0)$

$$f = c_0 \cos kx_i + d_i \sin kx_i \tag{4.6}$$

Now, the gluing condition (4.2), (4.6) implies $\sum d_i = 0$

4.2 Spectral theory on the finite spider graph

Let us now describe $(N-1)$ solutions with Dirichlet boundary condition at 0. First fix the central leg 1. then,

$$\psi_i = \begin{cases} \sin kx_1, & x_1 > 0 \\ -\sin kx_i, & x_i > 0 \ i \neq 1 \\ 0, & x_j > 0, i = 2, .., N \end{cases} \tag{4.7}$$

we have $(N-1)$ such solutions.

The last $i = N$'s solution $\psi_1 = \cos kx_i$ for $i = 1, 2, ...N$. Here $\psi_1(0) = 1$ and $\frac{d\psi_i}{dx_i} = 0$. We will develop the spectral theory of the the of the Laplacian on sp_N passing to the limit from the finite spider. Let us consider first the truncated graph (sp_N, L) where all legs have length L and $\psi_i(L) = 0$.

Let us show that for $\lambda < 0$ there are no eigenvalues.

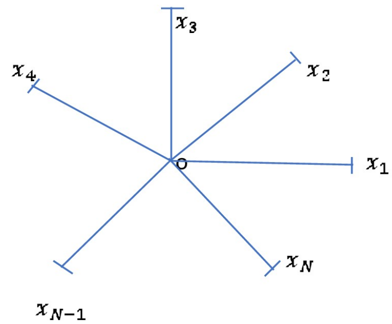

Figure 4.2: sp_N with N finite length legs

If $\lambda = -k^2$ then on each Leg_i, solution has the form

$$\psi_i(x_i) = c_i \sinh k(x_i - L)$$

that is,

$$\psi_i(0) = -c_i \sinh L, \qquad \sinh L > 0$$

due to condition (4.1)

$$c_1 = c_2 = ...c_N$$

but, due to condition (4.2),

$$c_1 \sum_{i=1}^{N} \frac{d}{dx_i} \sinh k(x - L)/_{x=0} = c_1 N k \cosh kL = 0$$

that is, $c_1 = 0$ and hence $\psi \equiv 0$.

Assume now that $\lambda = k^2 > 0$ where k is strictly positive and solve,

$$-\frac{d^2\psi}{dx^2} = k^2\psi \qquad \text{with boundary condition} \qquad \psi(L) = 0 \tag{4.8}$$

Then on each leg we have $\psi_i(x) = c_i \sin k(x-L)$ for $i = 1, 2, 3, ...N$. We will consider later, for simplicity, $N = 3$.

Asuume first that $\sin k(x-L)/_{x=0} = -\sin kL \neq 0$,

Then from condition (4.1) we have $c_1 = c_2 = c_3$ and from condition (4.2)

$$
\begin{aligned}
3c_1 k \cos kL &= 0 \\
k_n L &= \frac{\pi}{2} + n\pi \qquad\qquad n \geq 0 \\
k_n &= \frac{\pi(2n+1)}{2L}
\end{aligned}
$$

Now, ***A***) if, $\sin kL \neq 0$ that is $c_i \neq 0$ then $\lambda_n = k_n^2 = \frac{\pi^2(2n+1)^2}{4L^2}$, without any loss of generality, $c_i = 1$. This gives the first series of eigenfunctions. For each k_n, $n = 0, 1, 2, ...$, there is only one eigenfunction.

$$
\psi_n = \pm \sin k_n(x_i - L) = \cos k_n x_i, \qquad x_i \geq 0 \tag{4.9}
$$

with $k_n = \frac{\pi(2n+1)}{2L}$ where $n \geq 0$ which implies $\lambda_n = k_n^2 = \frac{\pi^2(2n+1)^2}{4L^2}$

B) if, $\sin kL = 0$ that is, c_i, can be different, then condition (4.2) gives

$$
\begin{aligned}
\sum_{i=1}^{3} c_i k \cos kL &= 0 \\
\Rightarrow \sum_{i=1}^{3} c_i &= 0
\end{aligned}
$$

This implies, that there are two linearly independent solutions corresponding to,

$$k_m L = \pi m$$
$$\Rightarrow k_m = \frac{\pi m}{L}$$

We have the following eigenvalues and eigenfunctions. For eigenvalues $\lambda_m = \frac{m^2\pi^2}{L^2}$ corresponding series of eigenfunctions are given by

$$\psi_{L,m,i}(x) = \begin{cases} \frac{\sin k_m(x_1-L)}{\sqrt{L}}, & x_1 \in [0, L] \\ \frac{-\sin k_m(x_i-L)}{\sqrt{L}}, & x_i \in [0, L] \\ 0, & \text{for remaining legs} \end{cases} \tag{4.10}$$

also,

$$\|\psi_{L,m,i}\| = 1 \qquad i = 2, ...N$$

but these functions are not orthogonal:

$$(\psi_{L,m,i}, \psi_{L,m,j}) = \frac{L}{2}, \qquad i \neq j, \qquad i, j \in (2, ..., N)$$

for different m we will have orthogonality associated with gluing condition

and, for $\lambda_n = \frac{\pi^2(2n+1)^2}{4L^2}$ with $n \geq 0$, corresponding eigenfunctions are given by

$$\psi_n = \frac{\cos k_n x_i}{\sqrt{\frac{LN}{2}}}, i = 1, 2, ..N \tag{4.11}$$

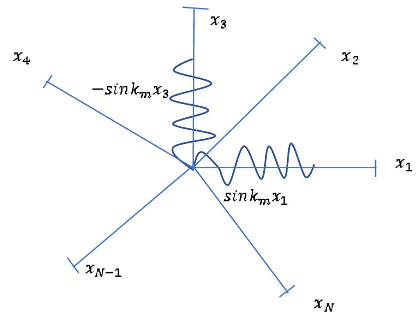

Figure 4.3: Eigenfunctions $\psi_{L,m,i}(x)$

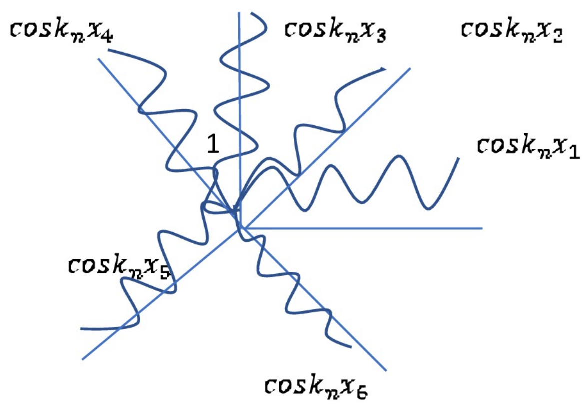

Figure 4.4: Eigenfunctions $\psi_n(x)$

Let us take function $f(x)$ on $sp_{N,L}$ and expand it over the eigenbasis.

$$f(x) = \sum_{n=0}^{\infty} \psi_n b_n + \sum_{m=1}^{\infty} \sum_{i=1}^{N} \psi_{m,i} a_{m,i} \tag{4.12}$$

To find coeficients b_n multiply (4.12) by ψ_n, we get,

$$b_n = (\psi_n, f)$$

To get $a_{m,i}$, multiply (4.12) by $\psi_{m,i}$ and we get,

$$a_{m,i}.(\psi_{m,i}, \psi_{m,j}) + \sum_{i \neq j} a_{m,j}(\psi_{m,i}, \psi_{m,j}) = (f, \psi_{m,i}) \qquad i, j \in [2, ..N]$$

$$a_{m,i}.1 + \sum_{i \neq j} a_{m,j}.\frac{1}{2} = (f, \psi_{m,i})$$

This gives,

$$\frac{N}{2}(a_{m,1} + a_{m,2} + ...a_{m,n-1}) = (f, \psi_{m,1}) + ... + (f, \psi_{m,n-1})$$

hence,

$$(a_{m,1} + a_{m,2} + ...a_{m,n-1}) = \frac{2}{N}\left[(f, \psi_{m,1}) + ... + (f, \psi_{m,n-1})\right]$$

This implies $a_{m,1} = \frac{2}{N}(f, \psi_{m,1})$ and $a_{m,i} = \frac{2}{N}(f, \psi_{m,i})$ for $i = 2, ..., N$ and 0 otherwise. But the eigen functions are not orthogonal and as a result spectral measure will not be diagonal.

So, for $f(x) \in sp_N$, consider,

$$\hat{F}_{m,i}(\lambda) = \begin{cases} \int_{s_{p_N}} f(x)\psi_{m,1}dx & x_1 \in [0, L] \qquad i = 1 \\ \int_{s_{p_N}} f(x)\psi_{m,i}dx & x_i \in [0, L] \qquad i \in [2, N] \\ 0, & \text{for remaining legs} \end{cases}$$

and

$$\hat{F}_n(\lambda) = \int_{sp_N} f(x)\psi_n dx \tag{4.13}$$

which gives the generalized Fourier transforms of the function f(x) on sp_N in the case of zero potential and from the weak compactness of the measure on each finite interval, we can conclude that as $L \to \infty$ the spectral measure tend weakly to the limiting measure

Our next goal is to give the qualitative spectral analysis of the general spider type

Hamiltonian. Using information about potential on each leg of the spider graph we would be able to describe the structure of the spectral measure, that is, its representation as the sum of absolutely continuous, singular continuous and point (discrete) components.

4.3 Spectral theory of sp_3 with fast decreasing potential

In this section as well as in the section about the periodic potentials, $v_j(x_j)$ for $j = 1, 2, 3$, we will use the fundamental fact : the change of the gluing condition at the point 0 (which is the rank 1 perturbation of the operator) cannot change the fact of existence of the absolute continuous component of the spectral measure as well as its support (that is, minimal closed set such that absolute continuous measure is 0. If on the spider sp_3, the potentials $v_j(x_j)$ are decreasing fast enough, then the standard assumptions are

$$\int_0^\infty x_j |v_j(x_j)| dx_j < \infty \qquad j = 1, 2, 3 \qquad \text{(Bargmann's condition)}$$

then under Dirichlet condition at point 0:

$$\psi_1(\lambda, 0) = \psi_2(\lambda, 0) = \psi_3(\lambda, 0) = 0$$

We can split the spectral problem on sp_3 into three spectral problems on legs Leg_j for $j = 1, 2, 3$ which have pure absolute continuous spectra for $\lambda > 0$, supported on $[0, \infty)$ and at most finite discrete spectra for $\lambda < 0$.

Hence, the initial problem on sp_3 with our gluing conditions has the absolute continuous spectrum of multiplicity 3, supported on $[0, \infty)$ and finite spectrum for $\lambda < 0$. Our goal in this section is to give the construction of the absolute continuous part.

sp_3 contains three legs, which starts from O(origin) and have coordinates x_j for $j = 1, 2, 3$ and $x_j \geq 0$. Let us denote O_j for $j = 1, 2, 3$, the part of the origin attributed to Leg_j. The Schrödinger operator on sp_3 has the form

$$H = -\Delta + v(x) \tag{4.14}$$

where,

$$-\Delta = -\frac{\delta^2}{\delta x_j^2} \qquad j = 1, 2, 3$$

and

$$v(x) = v_j(x_j) \qquad j = 1, 2, 3$$

Now, let us consider the following problem on the spider graph with three legs :

$$Hy = -\Delta y + vy = \lambda y \tag{4.15}$$

$$f(0) = \lim_{x_1 \to 0} f_1(x_1) = \lim_{x_2 \to 0} f_1(x_2) = \lim_{x_3 \to 0} f_1(x_3) \tag{4.16}$$

$$\frac{df}{dx_1}(0) + \frac{df}{dx_2}(0) + \frac{df}{dx_3}(0) = 0 \tag{4.17}$$

Let L be the truncation parameter. For simplicity, we consider compactly supported potentials v_1, v_2,v_3 on open semi axes x_1, x_2, x_3. For each $v_j(x_j)$, $j = 1, 2, 3$ we will introduce the scattering solution for $\lambda = k^2 > 0$, $k > 0$.

For waves, moving from right side to left:

$$\psi_1(x) = e^{-ikx_j} \qquad \text{for} x < x_j^- \tag{4.18}$$

$$= A_j(k)e^{-ikx_j} + B_j(k)e^{ikx_j} \qquad x > x_j^+ \tag{4.19}$$

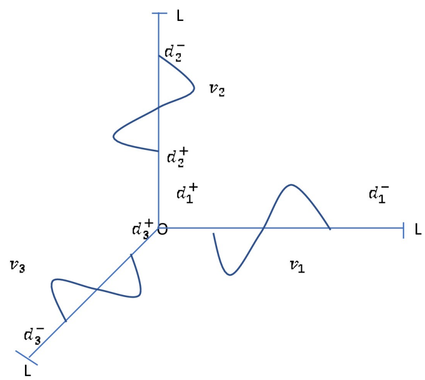

Figure 4.5: Three legged spider with fast decreasing potential

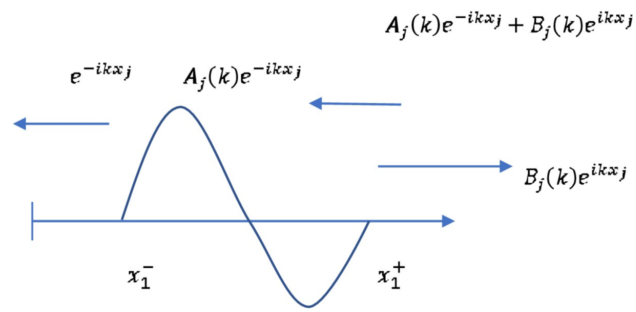

Figure 4.6: Wave propagation along the legs from right to left

Here $A_j(k)e^{-ikx_j}$ is the incidented wave component with magnituted $A_j(k)$ and frequency k. $B_j(k)e^{ikx_j}$ is the reflected wave component with magnitude $B_j(k)$ and frequency k, e^{-ikx_j} is the transmitted wave component. $A_j(k)$, $B_j(k)$ are the transamission and reflection coefficients.

It is well known, that,

$$|A_j(k)|^2 = 1 + |B_j(k)|^2 \qquad \text{(the conservation of energy law)}$$

Let,

$$A_j(k) = (a_{j1}(k) + ia_{j2}(k))$$

$$\text{and} \qquad B_j(k) = (b_{j1}(k) + ib_{j2}(k)) \qquad \text{complex form for } j = 1, 2, 3$$

After separation of the real and imaginary part, we find two solutions:

$$\cos kx_j(\text{near O}) \to (a_{j1} + b_{j1})\cos kx_j + (a_{j2} - b_{j2})\sin kx_j(\text{near } \infty)$$

and

$$\sin kx_j(\text{near O}) \to (a_{j2} + b_{j2})\cos kx_j + (-a_{j1} + b_{j1})\sin kx_j(\text{near } \infty)$$

At the origin $O = O_j$ for $(j = 1, 2, 3)$ we have, two gluing conditions:

a) if $\psi(x) \in D(H) \Rightarrow \psi(O_j) = \psi(0)$, $j = 1, 2, 3$ that is, $(\psi(x_j) \to \psi(0)$, $j = 1, 2, 3$, continuity of $\psi(x)$ at the origin)

b) $\sum_{j=1}^{3} \frac{\delta\psi}{\delta x_j}(O_j) = 0$, Kirchhoff's condition.

There are three solutions (for fixed $\lambda = k^2 > 0$) which satisfy the gluing condition at the origin and scattering information near infinity. The first solution $\psi_1(x)$ (given by figure 4.7):

This solution is supported on three legs satisfying $\psi_1(0) = 1$, $\sum_j \frac{\delta\psi_1}{\delta x_j}(O_j) = 0{+}0{+}0 =$

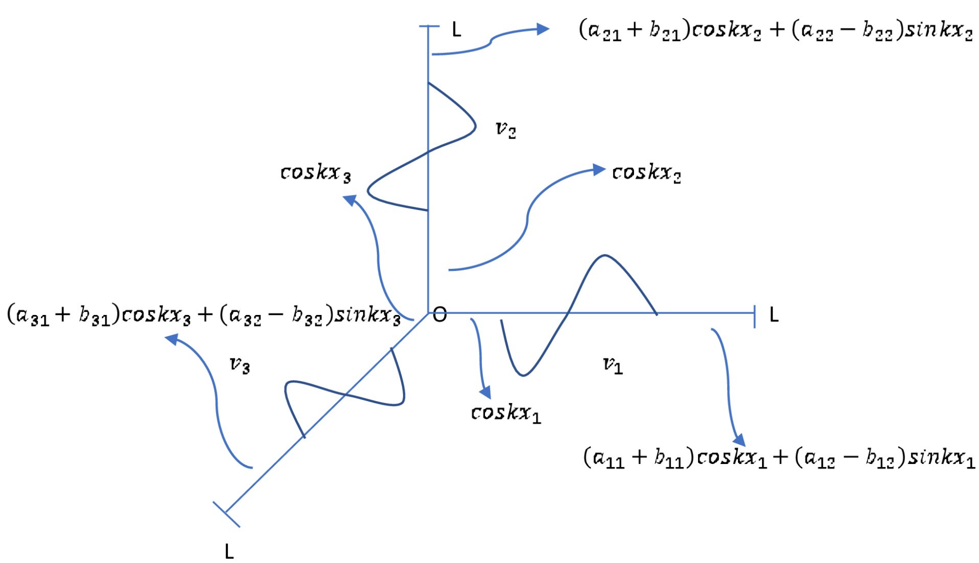

Figure 4.7: Solution ψ_1

0.

Other two solutions vanish at 0. The solution $\psi_2(x)$ (given by figure 4.8) and

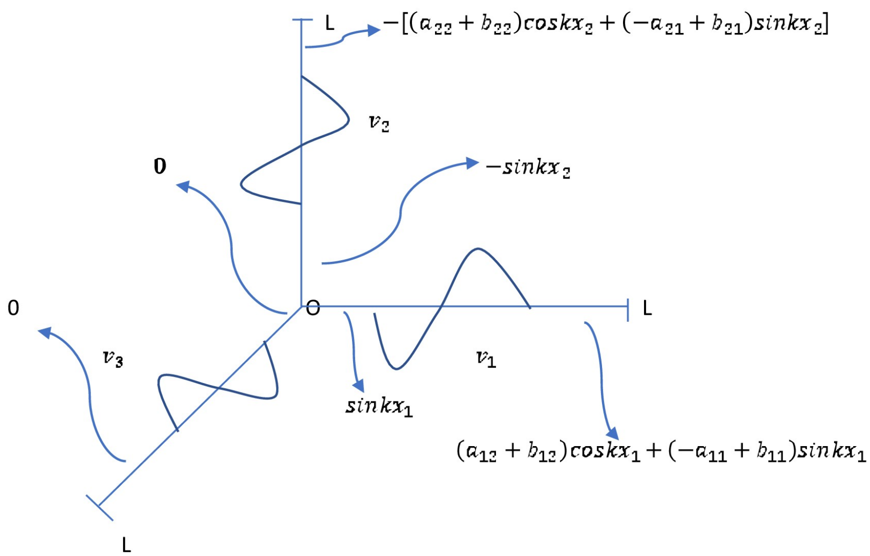

Figure 4.8: Solution ψ_2

the solution $\psi_3(x)$ (given by figure 4.9) are equal on Leg_1. The solution ψ_2 vanishes at x_3 axis and solution ψ_3 vanishes at x_2 axis.

Linear combination of $\psi_j(x)$, $j = 1, 2, 3$ satisfies the gluing condition at $x = 0$

Consider,

$$\psi(x) = \xi_1\psi_1(x) + \xi_2\psi_2(x) + \xi_3\psi_3(x)$$

with some normalization condition at 0 (say) :

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$$

Let us now impose, at the end points $x_j = L$, $j = 1, 2, 3$, the Dirichlet boundary conditions $\psi(x_j)/_{x_j=L} = 0$.

Later we will pass to the limit $L \to \infty$. It is well known that the limiting spectral measure (which we will derive later) is independent of the boundary conditions at

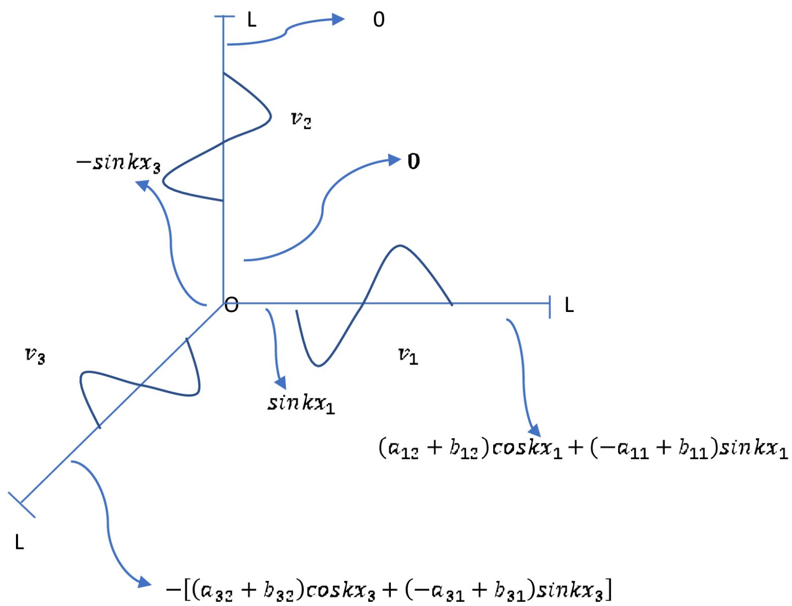

Figure 4.9: Solution ψ_3

the end points if potentials $v(x)=\{v_j(x_j), j=1,2,3, x_j\in[0,L]\}$ are bounded from below, in the sense, $v_i(x_i)\geq c_0$ (for $i=1,2,3$). This is sufficient condition for the uniqueness of the spectral measure but not necessary condition.

For fixed L we have three free parameters, $\lambda(=k^2>0)$ and ξ_2, ξ_3. Which implies the relation,

$$1-\xi_1^2=\xi_2^2+\xi_3^3$$

and three Dirichlet conditions at the end points $x_j=L$, for $j=1,2,3$.

As a result we will find the discrete spectrum for the restriction of H on $sp_3(L)$. To calculate the eigenvalues $\lambda_n(L)$, we have to use three Dirichlet equations at the end points.

$$0=\xi_1\psi_1(x)+\xi_2\psi_2(x)+\xi_3\psi_3(x)/_{x_j=L} \qquad\qquad j=1,2,3$$

We will start from the equations at the point $L_2(x_2 = L)$:

$$\xi_1[(a_{21} + b_{21})\cos kL + (a_{22} - b_{22})\sin kL] - \xi_2[(a_{22} + b_{22})\cos kL + (-a_{21} + b_{21})\sin kL] = 0 \tag{4.20}$$

Put

$$t = \frac{\cos kL}{\sin kL} = \cot kL$$

then,

$$\frac{\xi_2}{\xi_1} = \frac{(a_{21} + b_{21})t + (a_{22} - b_{22})}{(a_{22} + b_{22})t + (-a_{21} + b_{21})} \tag{4.21}$$

at point $L_3(x_3 = L)$, we will find

$$\frac{\xi_3}{\xi_1} = \frac{(a_{31} + b_{31})t + (a_{32} - b_{32})}{(a_{32} + b_{32})t + (-a_{31} + b_{31})} \tag{4.22}$$

and at point $L_1(x_1 = L)$,

$$-\frac{\xi_2 + \xi_3}{\xi_1} = \frac{(a_{11} + b_{11})t + (a_{12} - b_{12})}{(a_{12} + b_{12})t + (-a_{11} + b_{11})} \tag{4.23}$$

Now, adding (4.21)-(4.23)

$$\frac{(a_{21} + b_{21})t + (a_{22} - b_{22})}{(a_{22} + b_{22})t + (-a_{21} + b_{21})} + \frac{(a_{31} + b_{31})t + (a_{32} - b_{32})}{(a_{32} + b_{32})t + (-a_{31} + b_{31})} + \frac{(a_{11} + b_{11})t + (a_{12} - b_{12})}{(a_{12} + b_{12})t + (-a_{11} + b_{11})} = 0 \tag{4.24}$$

(4.24) produces the equation for $t = \cot kL$ and finally for k_n, such that $\lambda = k_n^2$

Note that,

$$\frac{at + b}{ct + d} = \frac{a}{c} - \frac{1}{c^2}\frac{(ad - bc)}{t + \frac{d}{c}} \tag{4.25}$$

and apply (4.25) in (4.24). In all three cases (4.21),(4.22),(4.23) determinants are equal -1, due to well known identity (law of the conservation of energy):

$$|A_j(k)|^2 = 1 + |B_j(k)|^2$$
$$\text{i.e.}(a_{j1}^2 - b_{j1}^2) + (a_{j2}^2 - b_{j2}^2) = 1$$

This means that the equation for unknown parameter $t = \cot kL$ has generically three simple real roots $t_1(k)$,$t_2(k)$,$t_3(k)$.In some limiting cases we can get one root of multiplicity 2 and one simple root (for instance $v(x) \equiv 0$)

4.4 Spectral analysis on the $sp_3(L)$ with Dirichlet boundary condition

Lemma 4.4.1. *For each $k > 0$ one can find three real roots of the cubic equation:(by using* (4.24), (4.25)*)*

$$\frac{\alpha_1}{t - a_1} + \frac{\alpha_2}{t - a_2} + \frac{\alpha_3}{t - a_3} = h$$

Under the generic condition $\alpha_1, \alpha_2, \alpha_3 > 0$, and $a_1 < a_2 < a_3$ and any h.

Proof. The proof follows from the graph of the equation given by figure 4.10

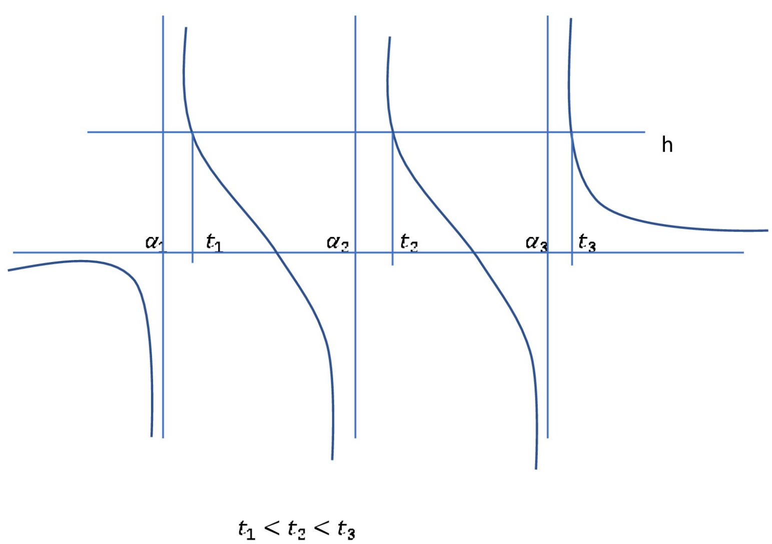

Figure 4.10: Graph of the cubic equation

Remark. *The parameters* α_1*,* α_2*,* α_3*;* a_1*,* a_2*,* a_3*,* h *can be expressed in the terms of reflection-transmission coefficients* A_j*,* B_j *for* $j = 1, 2, 3$ *(see* (4.24)*) and they are continuous functions of* k*.*

Remark. *Note that if* α_i *have the different signs then the situation is different.*

We can then find coefficients $c_2(t)$*,* $c_3(t)$ *such that,*

$$\xi_2 = c_2(t)\xi_1 \qquad\qquad \xi_3 = c_3(t)\xi_1$$

ξ_1 *is arbitrary and* $t = t_j$ *for* $j = 1, 2, 3$ *where* $t = \cot kL$*.*
then, for arbitrary $i = 1, 2, 3$ *we can construct eigenfunctios*

$$\psi(x) = \xi_1\psi_1 + \xi_2\psi_2 + \xi_3\psi_3 = \xi_1\left(\psi_1 + c_2(t)\psi_2 + c_3(t)\psi_3\right)$$

with boundary condition $\psi(L_i) = 0$ *+ gluing condition at the point* 0*.*

One can put,

$$\alpha(t) = \frac{\xi_1}{\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} \quad \beta(t) = \frac{\xi_1}{\sqrt{\xi_2^2 + \xi_2^2 + \xi_3^2}} \quad \text{and} \quad \gamma(t) = \frac{\xi_2}{\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} \ \text{(for normalization)}$$

Lemma 4.1 gives $t_j = \cos kL$ *and* $k_jL = \pi n + \cot^{-1} t_j$ *which give,*

$$\lambda_{n,j} = k_{n,j}^2 \qquad \text{and} \qquad k_{n,j}L = \pi n + \cot^{-1} t_j$$

For any n *there are three eigenvalues corresponding to three roots* t_1*,*t_2*,*t_3

According to the Strum-Liouville theory by [11] real eigenvalues corresponding to

Dirichlet gluing condition are discrete with finite multiplicity. If L $\to \infty$ then eigenvalues become more and more dense, and the then discrete measure concentrated at the eigenvalues will tend to limit which is called the spectral measure.

Let, for $f(x) \in sp_3$, *and* $\lambda > 0$ *then by* (4.13) *we can introduce the generalized Fourier transform as follows:*

$$\begin{aligned}
\hat{F}_1(\lambda) &= \int_{sp_3} f(x)\psi_1(\lambda, x)dx \\
\hat{F}_2(\lambda) &= \int_{sp_3} f(x)\psi_2(\lambda, x)dx \\
\hat{F}_3(\lambda) &= \int_{sp_3} f(x)\psi_3(\lambda, x)dx
\end{aligned} \tag{4.26}$$

Let, $\phi_n(\lambda, L, x)$ *be the orthonormalized eigenfunctions on the finite interval of* $sp_3(L)$, *then by the Strum-Liouville theory,*

$$\phi_n(\lambda, L, x) = \alpha_n\psi_1(\lambda_n, x) + \beta_n\psi_2(\lambda_n, x) + \gamma_n\psi_3(\lambda_n, x)$$

The normalization condition gives: $\alpha_n^2 + \beta_n^2 + \gamma_n^2 = 1$

Here, behind the potentials v_1, v_2, v_3, *that is near end points* L,

$$\begin{aligned}
\psi_1(\lambda_n, x) &= \frac{c_1 \cos k_n x_1 + c_2 \sin k_n x_1}{\sqrt{\frac{L}{2}}} \\
\psi_2(\lambda_n, x) &= \frac{c_3 \cos k_n x_2 + c_4 \sin k_n x_2}{\sqrt{L}} \\
\psi_3(\lambda_n, x) &= \frac{c_5 \cos k_n x_3 + c_6 \sin k_n x_3}{\sqrt{L}}
\end{aligned}$$

the c_i *s,* $i = 1, ..., 6$ *are given by the real and imaginary components of* A_j *and* B_j, *that is* $a_{j,1}$, $a_{j,2}$, $b_{j,1}$ $b_{j,2}$ *where* $j = 1, 2, 3$

For the finite spider $sp_{L,3}$, due to completeness of the set of eigenfunctions $\phi_n(\lambda_n, L, x)$ for all sufficiently large L and compactly supported f,

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \phi_n(f, \phi_n) \\
&= \sum_{n=1}^{\infty} [\alpha_n\psi_1(\lambda_n, x) + \beta_n\psi_2(\lambda_n, x) + \gamma_n\psi_3(\lambda_n, x)] \times \int_0^L f(y)(\alpha_n\psi_1 + \beta_n\psi_2 + \gamma_n\psi_3)dy \\
&= \int_0^L f(y) \sum_{n=1}^{\infty} ([\alpha_n\psi_1(\lambda_n, x) + \beta_n\psi_2(\lambda_n, x) + \gamma_n\psi_3(\lambda_n, x)](\alpha_n\psi_1 + \beta_n\psi_2 + \gamma_n\psi_3)dy)
\end{aligned}$$

Applying the Parseval equality to $f(x)$ we get,

$$\begin{aligned}
||f||_2^2 &= \int_0^L f^2(x)dx \qquad (4.27) \\
&= \sum_{n=1}^{\infty} \left\{ \int_0^L f(x)[\alpha_n\psi_1(\lambda_n, x) + \beta_n\psi_2(\lambda_n, x) + \gamma_n\psi_3(\lambda_n, x)] \right\}^2 \\
&= \sum_{n=1}^{\infty} \alpha_n^2 \left\{ \int_0^L f(x).\psi_1(\lambda_n, x) \right\}^2 + 2\sum_{n=1}^{\infty} \alpha_n\beta_n \int_0^L f(x).\psi_1(\lambda_n, x)dx \int_0^L f(x).\psi_2(\lambda_n, x)dx \\
&+ 2\sum_{n=1}^{\infty} \alpha_n\gamma_n \int_0^L f(x).\psi_1(\lambda_n, x)dx \int_0^L f(x).\psi_3(\lambda_n, x)dx + \sum_{n=1}^{\infty} \beta_n^2 \left\{ \int_0^L f(x).\psi_2(\lambda_n, x) \right\}^2 \\
&+ 2\sum_{n=1}^{\infty} \beta_n\gamma_n \int_0^L f(x).\psi_2(\lambda_n, x)dx \int_0^L f(x).\psi_3(\lambda_n, x)dx + \sum_{n=1}^{\infty} \gamma_n^2 \left\{ \int_0^L f(x).\psi_3(\lambda_n, x) \right\}^2
\end{aligned}$$

Let us now introduce the matrix valued measure following Parseval's equality to $f(x)$ (Strum-Liouville Theory)

$$\rho_L(\lambda) = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix}(\lambda) = \begin{bmatrix} \sum_{\lambda_n<\lambda} \alpha_n^2 & \sum_{\lambda_n<\lambda} \alpha_n\beta_n & \sum_{\lambda_n<\lambda} \alpha_n\gamma_n \\ \sum_{\lambda_n<\lambda} \alpha_n\beta_n & \sum_{\lambda_n<\lambda} \beta_n^2 & \sum_{\lambda_n<\lambda} \beta_n\gamma_n \\ \sum_{\lambda_n<\lambda} \alpha_n\gamma_n & \sum_{\lambda_n<\lambda} \beta_n\gamma_n & \sum_{\lambda_n<\lambda} \gamma_n^2 \end{bmatrix} \qquad (4.28)$$

Note: This is a 3×3 symmetric matrix. From the weak compactness of the measure on each finite interval, we can conclude that as $L \to \infty$ the spectral measure $\rho_n(d\lambda)$

tend weakly to the limiting measure $\rho(d\lambda)$ on any spectral interval. The off diagonals charges can be negative but the measure matrix is positive definite.

Let us note that, say,

$$\sum_{\lambda_n<\lambda} \alpha_n^2 = \sum_{k_n<\sqrt{\lambda}} \frac{c_1^2(k_n)}{L} \to \int_{-\infty}^{\lambda} c_1^2(k)dk \qquad \text{similarly } \beta_n \text{ and } \gamma_n$$

It means that the limiting spectral measure $\rho(d\lambda)$ is absolute continuous with multiplicity 3. Unfortunately the coefficients c_1, c_2, c_3 as the roots of the cubic equation from lemma 4.1 cannot be calculated explicitly. So, we do not have any clear formula for $\rho(d\lambda) = \rho(\lambda)d\lambda$.

The inverse Fourier transform is given by:

$$\begin{aligned} f(x) &= \int_0^\infty < \hat{F}_1(\lambda), \hat{F}_2(\lambda), \hat{F}_3(\lambda) > \rho_L(d\lambda) \\ &= \int_0^\infty \hat{F}_1\psi_1(\lambda,x)\rho_{11}(d\lambda) + \int_0^\infty \hat{F}_1\psi_2(\lambda,x)\rho_{12}(d\lambda) + \int_0^\infty \hat{F}_1\psi_3(\lambda,x)\rho_{13}(d\lambda) \\ &+ \int_0^\infty \hat{F}_2\psi_1(\lambda,x)\rho_{21}(d\lambda) + \int_0^\infty \hat{F}_2\psi_2(\lambda,x)\rho_{22}(d\lambda) \int_0^\infty \hat{F}_2\psi_3(\lambda,x)\rho_{23}(d\lambda) \\ &\int_0^\infty \hat{F}_3\psi_1(\lambda,x)\rho_{31}(d\lambda) + \int_0^\infty \hat{F}_3\psi_2(\lambda_n,x)\rho_{32}(d\lambda) + \int_0^\infty \hat{F}_3\psi_3(\lambda,x)\rho_{33}(d\lambda) \end{aligned} \tag{4.29}$$

4.5 Negative eigenvalues

Consider on sp_3 the problem

$$-\frac{d^2\psi}{dx_j^2} + v_j(x_j)\psi = \lambda\psi \qquad \lambda = -k^2$$

If $\psi(0) = 0$ on each leg and

$$\int_0^\infty x_j|v_j(x_j)|dx_j < \infty \qquad j = 1, 2, 3$$

(Bargmann's condition)Then number of negative eigenvalues will be finite.

Total number of negative eigenvalues on sp_3 *is less or equal to*

$$N_0(H) \leq 1 + \sum_{j=1}^{3} \int_0^{\infty} x_j |v_j(x_j)| dx_j$$

This is the Bargmann's estimate plus rank one perturbation at $x = 0$*. The change in gluing condition at* $x = 0$ *can provide only one additional negative eigenvalue. Here, the number of negative eigenvalues equal to the number of negative* $\sigma_i s$ *for* $i = 1, 2, 3$.

4.6 Solvable model

Here we consider

$$v(x) = \sigma\delta(x - a)$$

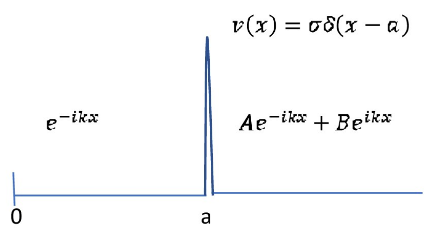

Figure 4.11: Wave component on half axis with a positive delta potential

For the continuity condition at a:

$$\begin{aligned} e^{-ika} &= Ae^{-ika} + Be^{ika} \\ 1 &= A + Be^{2ika} \end{aligned} \tag{4.30}$$

and for jump of the derivative at a:

$$\psi'(a - 0) - \psi'(a + 0) = \sigma\psi(a)$$

then we have,

$$-ike^{-ika} - (A(-ike^{-ika}) + B(ike^{ika})) = \sigma e^{-ika}$$
$$-ik + Aik - Bike^{2ika} = \sigma$$
$$A - Be^{2ika} = 1 + \frac{\sigma}{ik} = 1 - \frac{\sigma i}{k} \tag{4.31}$$

(4.30) *and* (4.31) *gives*

$$A(k) = 1 - \frac{\sigma i}{2k}$$

and

$$B(k) = \frac{\sigma i}{2k} e^{-2ika}$$

Now

$$|A(k)|^2 = 1 + \frac{\sigma^2}{4k^2} = 1 + |B(k)|^2 \tag{4.32}$$

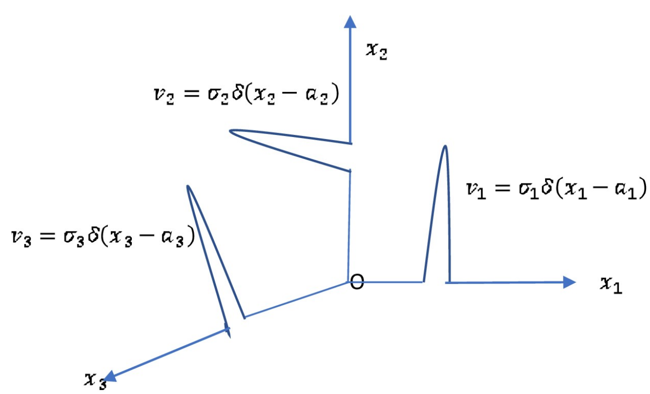

Figure 4.12: Positive delta potential on the legs of the three legged quantum spider graph

We already pointed out that (more or less) explicit formulas for the spectral measure $\rho(d\lambda) = \rho(\lambda)d\lambda$ *where* $\rho(\lambda)$ *is* (3×3) *positive definite function of* λ*, do not exist (like the similar formulas in the case of* $\mathbb{R}^1$*, that is,* sp_2*).*

There are two reasons: there is no simple formulas for the roots of the roots of *the cubic equation from lemma 4.1 and in general there is no simple formulas for the reflection-transmission coefficient $A(k)$ and $B(k)$ except for some simple situations.*

In this section we will give example of the solvable model. Consider sp_3 and potentials $v(x_1) = \sigma\delta(x_1 - a)$, $v(x_2) = \sigma\delta(x_2 - a)$ and $v(x_3) = \sigma\delta(x_3 - a)$. Let us stress on the fact that, all potentials are equal and model is invariant with respect to interchange of the legs. Let us start from $sp_{3,L}$.

In this case there are two invariant subspaces in $\mathbb{L}^2(sp_3)$: set of the functions

1)

$$\psi(0) = 0 \quad \sum \frac{d\psi}{dx_i}(0) = 0 \quad \textit{(that is, } \mathbb{L}^2_{\mathcal{D}}(sp_3), \textit{ corresponding to Dirichlet condition)}$$

and

$$2) \quad \psi(0) > 0 \quad \sum \frac{d\psi}{dx_i} = 0 \quad \textit{(that is, } \mathbb{L}^2_{\mathcal{N}}(sp_3), \textit{ corresponding to Neumann's condition)}$$

hence,

$$\mathbb{L}^2(sp_3) = \mathbb{L}^2_{\mathcal{D}} \oplus \mathbb{L}^2_{\mathcal{N}}$$

Then, as result, the spectral problem can be reduced to two independent spectral problems.

$$-\frac{d^2\psi}{dx_i} + \sigma\delta(x_i - a)\psi = \lambda\psi$$

$$\psi(0) = 0 \qquad \textit{and} \qquad \psi(L) \qquad = 0$$

Now the solution on each leg with delta potential has the following form.

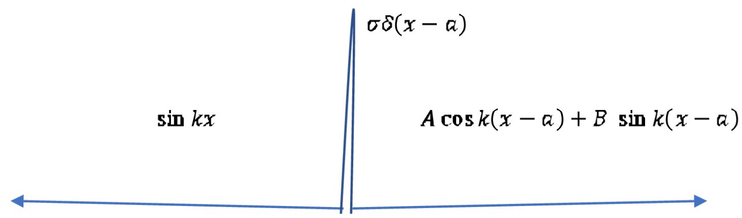

Figure 4.13: Solution on both side of a delta potential for a solvable model

This implies

$$A = \sin ka \qquad \textit{(by the continuity condition of } \psi \textit{ at point a)}$$

and

$$k\cos ka - kB = \sigma \sin ka \qquad \textit{(continuity of the derivative at point a)}$$
$$\Rightarrow B = \cos ka - \frac{\sigma}{k}\sin ka$$

Eigenvalues for this Dirichlet component of our spectral problem are given by the following equation $(if x = L)$

$$\psi(x-a)/_{x=L} = 0$$
$$\left[\sin ka\cos k(x-a) + (\cos ka - \frac{\sigma}{k}\sin ka)\sin k(x-a)\right]/_{x=L} = 0$$
$$\sin kL - \frac{\sigma}{k}\sin ka\sin k(L-a) = 0$$
$$\sin kL - \frac{\sigma}{k}\sin ka(\sin kL\cos ka - \cos kL\sin ka) = 0$$
$$(1 - \frac{\sigma}{k}\sin ka\cos ka)\sin kL + \frac{\sigma}{k}\sin^2 ka\cos kL = 0$$
$$b\sin kL + c\cos kL = 0$$
$$\Rightarrow \sqrt{b^2+c^2}\sin(kL+\phi) = 0$$

where ϕ is given by

$$\tan\phi = \frac{\frac{\sigma}{k}\sin^2 ka}{1-\frac{\sigma}{k}\cos ka \sin ka} = F(k)$$
$$\Rightarrow \phi = \tan^{-1}\left(\frac{\frac{\sigma}{k}\sin^2 ka}{1-\frac{\sigma}{k}\cos ka \sin ka}\right)$$

Note that,

$$\cos\phi = \frac{b}{\sqrt{b^2+c^2}} \qquad \textit{and} \qquad \sin\phi = \frac{c}{\sqrt{b^2+c^2}}$$

where

$$b = (1-\frac{\sigma}{k}\sin ka \cos ka) \qquad \textit{and} \qquad c = \frac{\sigma}{k}\sin^2 ka$$

For the eigenvalues,

$$\sin(kL+\phi(k)) = 0 \qquad \textit{where} \qquad \phi = \phi(k) = \sin^{-1}\frac{c}{\sqrt{b^2+c^2}} \qquad \textit{etc.}$$
$$\Rightarrow kL + \phi(k) = \pi n$$
$$\Rightarrow k_n = \frac{n\pi}{L} - \frac{\phi(k)}{L}$$

The eigenvalues are given by

$$k_n^2(L) = \lambda_n(L)$$

There are two normalized eigenfunctions associated with $k_n(L)$, given by figure 4.14 and 4.15.

There are also eigenfunctions associated with Neumann's condition such that

$$A_1 = \cos ka \qquad \textit{(due to continuity condition)}$$

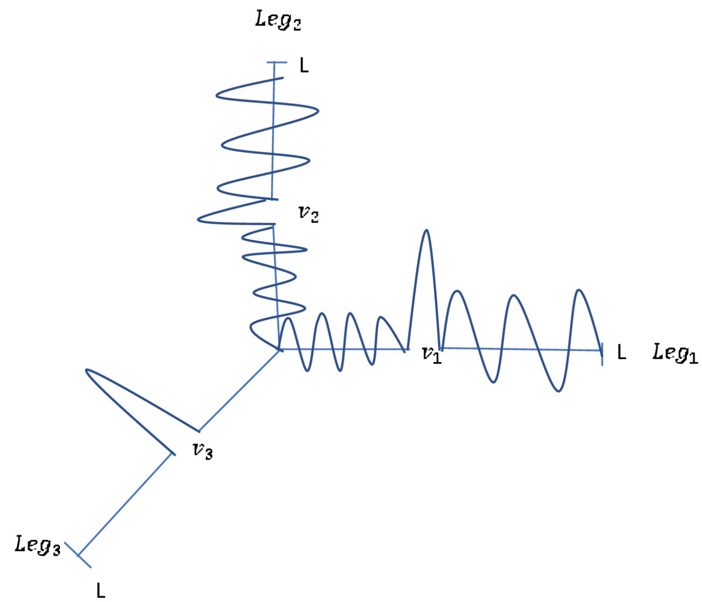

Figure 4.14: Normalized eigenfunction ψ_1

and,

$$B_1 = -\sin ka - \frac{\sigma}{k}\cos ka$$

The eigenfunctions are given by figure 4.17). For related work see also [14]

4.7 Spectral theory of sp_3 with increasing potential

Let us consider potential v_1 on leg 1, v_2 on leg 2 and v_3 on leg 3 and $V(x) = v_i(x_i)$ for $i = 1, 2, 3$. $v_j(x) \in \mathbb{C}_{loc}$ and

$$v_i(x_i) \to +\infty \qquad as\ x_i \to \infty\ for\ i = 1, 2, 3 \tag{4.33}$$

Let us consider in the beginning instead of Kirchhoff gluing condition, the Dirichlet boundary condition at point 0, $\psi(0) = 0$ for $\psi \in \mathbb{C}^2(sp_3)$. This Splits the spider graph into 3 one-dimensional spectral problem on $[0, \infty)$

According to the classical Strum-Liouville theory by [11], (4.14) *with the Dirichlet*

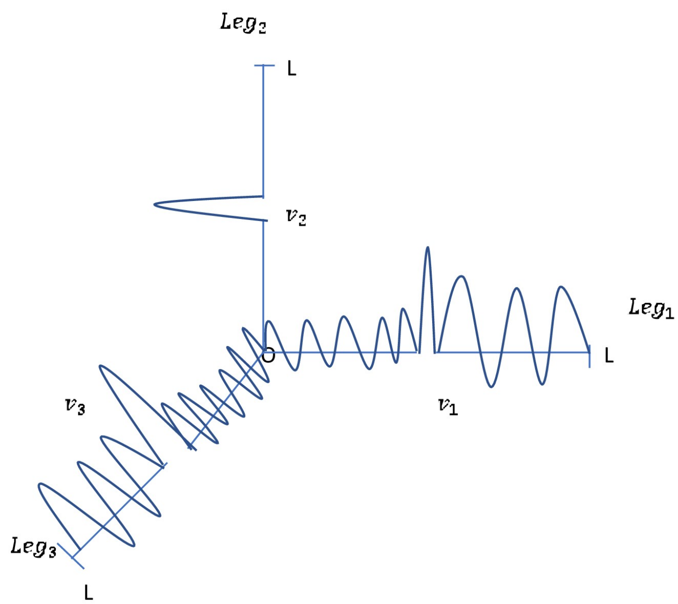

Figure 4.15: Normalized eigenfunction ψ_2

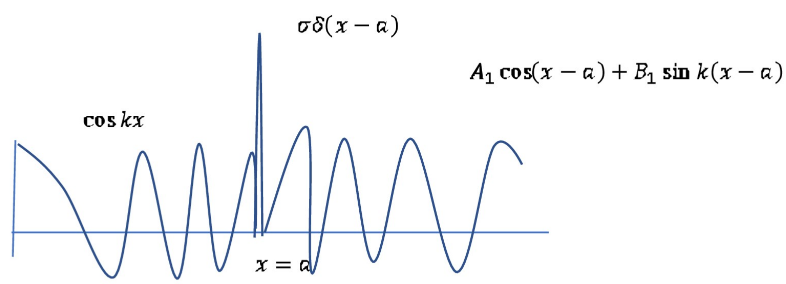

Figure 4.16: Solution on both side of a delta potential associated to Neumann's condition

boundary condition at point 0*, any solution for every fixed* λ *has finite number of zeros. It implies the discreteness of the spectrum on each leg of the spider that is there exist sequence* $\lambda_1 < \lambda_2 < ... < \lambda_n$ $(\lambda \to \infty)$ *of eigenvalues of* H *and corresponding eigenfunctions* $\psi_n(x)$*,* $x > 0$ *form an orthogonal basis in* $\mathbb{L}^2[0,\infty)$ *on each leg and decays super-exponentially. The spectral measure on each leg* Leg_i *is given by,*

$$\rho(d\lambda) = \sum_{0}^{\infty} \alpha_n \delta(\lambda - \lambda_n) d\lambda$$

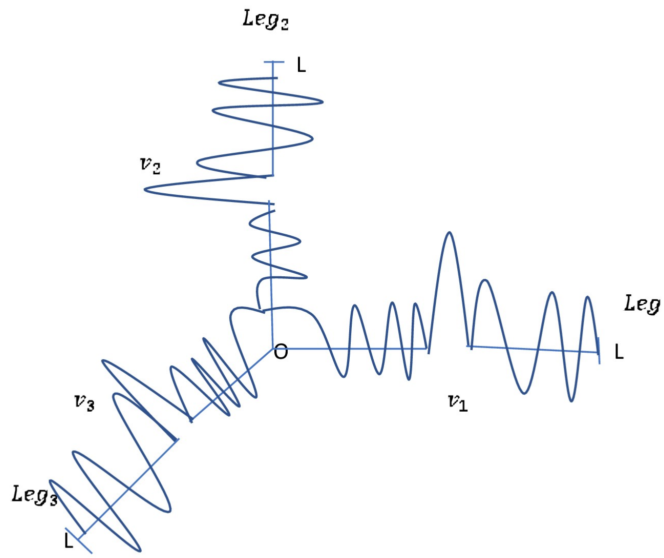

Figure 4.17: Eigenfunction ψ_3 associated to Neumann's condition

Corresponding atoms α_n have the following meaning, let us consider on $[0, \infty)$ the spectral problem (in fact, three problems on Leg_i for $i = 1, 2, 3$)

$$y_\lambda'' = (v_i(x) - \lambda) y_\lambda \qquad \textit{for } i = 1, 2, 3$$

with conditions

$$y_\lambda(0) = 0 \qquad \textit{and} \qquad y_\lambda'(0) = 1$$

There exists only finitely many eigenvalues $\lambda_{n,i}$ for $i = 1, 2, 3$ and $n \geq 0$ in any spectral interval $[0, \wedge]$ on each leg Leg_i. Corresponding solutions $y_{\lambda_{i,n}}(x)$ are decreasing on Leg_i super-exponentially, for all other λ solutions (that is, there magnitudes)

$$r_{\lambda_{i,n}} = \sqrt{\left(y_{\lambda_{i,n}}'^2 + y_{\lambda_{i,n}}^2\right)(x)}$$

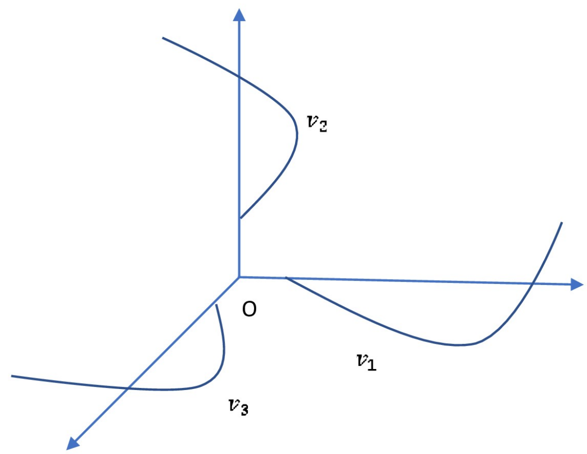

Figure 4.18: A three legged spider quantum graph with increasing potentials on each leg

are growing super exponentially. Then,

$$\alpha_{n,i} = \left(\int_0^\infty y^2_{\lambda_{n,i}} dx \right)^{-\frac{1}{2}}$$

Since the sets of eigenvalues $\{\lambda_{i,n}, n \geq 1\}$ *are different for different* $i = 1, 2, 3$ *and the discrete spectrum is unstable with respect to rank one perturbation (change of the Kirchhoff's gluing condition on Dirichlet gluing condition) the result, presented above, cannot prove the discreteness of the spectrum on* sp_3 *for initial conditions of the continuity and Kirchhoff gluing condition.*

However, the general compactness arguments give the desirable discreteness theorem. Let us assume, with out loss of generality, that $v_i(x_i) \geq 0$ *for* $i = 1, 2, 3$ *and fix the spectral interval* $[0, \wedge]$. *For given* $\wedge$ *one can find such* L, *that for any* $i = 1, 2, 3$ $v_i(x_i) > \wedge + 1$ *if* $x_i > L = L(\wedge)$. *Then, any solution* $y_\lambda(x)$ *our initial equation*

$$y_\lambda^{''}(x) = (V - \lambda) y_\lambda \qquad \lambda \leq \wedge$$

with continuity and Kirchhoff's conditions on each leg L_i, $i = 1, 2, 3$ *has at most one zero (non-oscillating) if* $x_i \geq L$.

The member of eigenvalues on $[0, \wedge]$ *for the truncated spider with the legs of the length* L *and any boundary condition at the endpoints* $x_i = L$, $i = 1, 2, 3$ *is uniformly bounded by constant, depending only on* $\wedge$ *and potentials* $v_i(x_i)$, $i = 1, 2, 3$, $x_i \in [0, L]$. *As result, the spectral problem*

$$y_\lambda^{''}(x) = (V - \lambda) y_\lambda \qquad x_i \in [0, L] \qquad for \qquad i = 1, 2, 3 \qquad with \qquad y_\lambda = L$$

on each leg plus the continuity and Kirchhoff's conditions at the origin has spectral measure $\rho_L(d\lambda)$, *containing on* $[0, \wedge]$ *is uniformly bounded (that is, independent of* L*). Number of atoms (say,* $N(\wedge)$ *(constant)), total mass of the spectral measure is also uniformly bounded (this is true for any locally continuous and bounded from below potential [see [3]])*

The proof of these statements is based (like in [11]) on two Strum lemmas.

Lemma 4.7.1. *Any solution of the equation*

$$-y^{''} + g(x) y = 0 \qquad\qquad x \in [a, b] \subset \mathbb{R}^1_+$$

with condition $g(x) \geq m^2 > 0$ *has at most one zero on* $[a, b]$.

Lemma 4.7.2. ***Comparison theorem***

Consider the sp_3 *and two equations*

$$-y_1^{''} + g_1(x) y_1 = 0$$
$$-y_2^{''} + g_2(x) y_2 = 0 \qquad\qquad x \geq 0$$

with the same initial conditions at the origin, that is, continuity and Kirchhoff's condition plus 3 initial data, say, value of $y(0)$ *and* $\frac{dy}{dx_1}(0)$, $\frac{dy}{dx_2}(0)$ *etc., such 6 equations uniquely determine the solutions* y_1, y_2. *Assume that* $g_1(x) < g_2(x)$ *and solution* $y_1(x)$ *has zeros* $x_1^{(1)}$, $x_2^{(1)}$, $x_3^{(1)}$ *on each leg,* Leg_i *for* $i = 1, 2, 3$. *Then* y_2 *has also zero on one* *of the intervals* $[0, x_i^{(1)}]$ *for* $i = 1, 2, 3$

Proof of this lemma is like the proof of the Strum theorem (see [11] (theorem 3.1)) is based on the integration by the parts of the expression $y_1^{''} y_2 - y_1 y_2^{''}$ *over the finite spider with the legs* $[0, x_1^{(1)}]$, $[0, x_2^{(1)}]$, $[0, x_3^{(1)}]$ *using the gluing conditions at 0.*

The following lemma is obvious

Lemma 4.7.3. *If on the fixed interval* $[0, \wedge]$ *there is the family of the discrete measures* $\mu_L(d\lambda)$ *depending on the parameter* $L \geq 0$ *and*

$$i) \int_0^{\wedge} d\mu_L \leq M$$

$$ii) \text{number of atoms of } \mu_L \text{ or } [0, \wedge] \leq N$$

(M,N are constant and independent of L) then,

a) Family $\mu_L(\cdot)$ *is weakly compact*

b) If $\mu_L(d\lambda) \Rightarrow \mu(d\lambda)$ *(weakly) then limiting measure* $\mu(d\lambda)$ *is discrete and satisfies the same inequalities i), ii).*

It implies the following result

Theorem 4.7.4. *If* $v_i(x_i) \rightarrow +\infty$ *for* $i = 1, 2, 3$ *and at the origin we have the usual gluing conditions (continuity + Kirchhoff gluing condition) then the spectrum is discrete, corresponding eigenfunction are decreasing super-exponentially and have multiplicity at most 3.*

Theorem 4.7.5. *The condition* $v_i(x_i) \rightarrow \infty$, *for* $i = 1, 2, 3$ *can be replaced by the* *conditions* $v_i(x) \geq 0$ *for* $i = 1, 2, 3$ *and for arbitrary small* l *and any* $i = 1, 2, 3$

$$\int_{x_i}^{x_i+l} v_i(x_i)dx_i \to +\infty \qquad\qquad as\ x_i \to \infty$$

(condition that A.M Molčanov [13] proved, which is necessary and sufficient for the discreteness of the spectrum for 1-D Schrödinger operator with bounded from below potential).

Proof. The proof for the spider case is the same as on $\mathbb{R}^1$. The central idea here is to check that for $\lambda < \wedge$ on each leg any solution $y_\lambda(x)$ has finitely many zero.

4.7.1 Phase and amplitude

Let us consider the problem

$$H\psi(x) = -\psi^{''} + v(x)\psi = \lambda\psi \tag{4.34}$$

$$\psi(0) = \sin\theta_0, \psi^{'}(0) = \cos\theta_0$$

The solution of (4.34) in the form of phase-amplitude form can be given by the standard formulas [3]

$$\psi(x) = \rho_\lambda(x)\sin\theta_\lambda(x) \qquad \text{and} \qquad \psi^{'}(x) = \rho_\lambda(x)\cos\theta_\lambda(x)$$

Then,

$$\theta_\lambda^{'} = \cos^2\theta_\lambda + (\lambda - v(x))\sin^2\theta_\lambda, \quad \theta_\lambda(0) = \theta_0 (=0 \text{ for Dirichlet gluing condition})$$

$$\rho_\lambda^{'} = \frac{1}{2}\rho_\lambda(x)(1 + v(x) - \lambda)\sin 2\theta_\lambda, \qquad \rho_\lambda(0) = 1$$

$$\rho_\lambda = e^{\left(\frac{1}{2}\int_0^x (1+v(z)-\lambda)\sin 2\theta_\lambda(z)dz\right)}$$

The spectral properties for H^{θ_0} depend on the behavior of $\rho_\lambda(L)$ where $L \to \infty$. The results on the negative part of the spectrum ($\lambda < 0$) of H^{θ_0} are simpler. For positive energies λ ($\lambda > 0$), it is useful to work with frequency $k = \sqrt{\lambda} > 0$. The WKB approach suggests the following definition of phase amplitude, which is called Prüfer transformation.

$$\psi_k(x) = r_k(x) \sin t_k(x)$$
$$\psi_k'(x) = k r_k(x) \cos t_k(x)$$

Then,

$$t_k'(x) = k - \frac{v(x) \sin^2 t_k(x)}{k}$$
$$r_k'(x) = \frac{v(x) \sin 2t_k(x)}{2k} r_k$$

with initial conditions

$$\cot t_k(0) = \frac{1}{k} \cot \theta_0$$
$$r_k(0) = \sqrt{sin^2\theta_0 + \frac{1}{k^2} \cos^2 \theta_0}$$

In particular, if $\theta_0 = \frac{\pi}{2}$, then,

$$t_k(0) = 0, \qquad r_k(0) = 1$$

Then the prespectral measure $\bar{\mu_L}(d\lambda)$ can be represented as (see [3])

$$\bar{\mu_L}(d\lambda) = \frac{2k dk}{\rho_{k^2}^2(L)}$$

If $\Delta = [a, b] \subset (0, \infty)$ is a fixed interval on the positive energy axis then on the frequency axis it transforms to $\tilde{\Delta} = [\sqrt{a}, \sqrt{b}]$. Then for appropriate constants $c^{\pm}(\Delta)$ and $L > 0$ (the following result follow from [3])

$$c^{-}(\Delta)\frac{1}{r_k^2(L)} \leq \frac{1}{\rho_{k^2}^2(L)} \leq c^{+}(\Delta)\frac{1}{r_k^2(L)}$$

The spectral measure on $\tilde{\Delta}$ can be given by:

$$\tilde{\mu_L}(dk) = \frac{dk}{r_k^2(L)}$$

For $L \to \infty$, $\tilde{\mu_L}(dk)$ has the same property as the properties of $\mu(d\lambda)$ on the corresponding interval Δ of the energy axis.

The following results follow from [15]

If $v(x) \geq v_0(x) > -\infty$ that is, the potential is bounded from below, then for any bounded interval Δ on the energy axis, for $x_0 = x_0(v_0, \Delta)$, $c_0 = c_0(v_0, \Delta)$ and $\delta_0 = \delta_0(v_0, \Delta)$ one can give the estimation for $\psi(x, \lambda)$ as

$$\int_{\Delta} \psi_\lambda^2(x)\mu(d\lambda) \leq c_0$$

and $$|\psi_\lambda(z) \geq \frac{1}{2}|\psi_\lambda(x)| \qquad \text{for } z \in [x_0, x_0 - \delta_0] \text{ or for } z \in [x_0, x_0 + \delta_0]$$

for $x \geq x_0$ and $\lambda \in \Delta$ If the potential is uniformly bounded, that is, $||v(\boldsymbol{.})||_\infty \leq v_0 < \infty$ then the estimation for $\psi^{'}(\lambda, x)$ can be written as

$$|\psi_\lambda^{'}(x)|^2 \leq \frac{c_0}{2\delta}\int_{x-\delta}^{x+\delta} \psi_\lambda^2(z)dz$$

It gives (extension of Schnoll's lemma)

$$\int_{\Delta} \rho_\lambda^2(x)\mu(d\lambda) \leq c_0$$

for $x_0(v_0, \Delta)$, $c_0(v_0, \Delta)$ and $\forall x \geq x_0$. Then for fixed sequence $\{x_n\}$, where $x_n \to \infty$

and μ-a.e, $\lambda \in \Delta$, $\epsilon > 0$

$$\rho_\lambda(x_n) \leq c(\lambda, \epsilon) n^{\frac{1}{2}+\epsilon}$$

and $$\rho_\lambda(x) \leq c(\lambda, \epsilon) x^{\frac{1}{2}+\epsilon} \qquad \forall x \geq x_0$$

We will now introduce the counting function $N(\lambda)$ for $\lambda_i < \lambda$. The following formula goes to Neils Bohr. It states that under some condition for $\lambda \to \infty$

$$N(\lambda) \sim B(\lambda) \equiv \frac{1}{\pi} \int_0^\infty \sum_0^d \sqrt{\lambda - v_j(x)}_+ dx \tag{4.35}$$

where $d = 3$ for three legged spider graph.

Let us recall the standard approach (by Kac [10]). Consider $p(t, x, y)$ be the fundamental solution of the parabolic problem on sp_3:

$$\frac{\delta \psi}{\delta t} = \mathcal{L}\psi + V(x)\psi \qquad t, x > 0$$

$$p(0, x, y) = \delta(x - y)$$

Here $\mathcal{L} = \frac{\delta^2 \psi_i}{\delta x_i}$ and $V(x) \equiv v_i(x_i)$ for $i = 1, 2, 3$ with gluing condition at $x_i = 0$ Fourier transform gives: $p(t, x, y) = \sum_{i \geq 1} e^{-\lambda_i t} \psi_i(x) \psi_j(x)$

which implies,

$$Tre^{-itH} = \int_0^\infty p(t, s, s) ds = \int_0^\infty e^{-\lambda t} dN(\lambda)$$

also the Kac-Feynman formula gives:

$$p(t, x, x) = \frac{1}{\sqrt{\pi t}} E_x \left(e^{-\int_0^t V(B(s))} \right) ds$$

where $B(s)$ is the Wiener process at time $t=0$ at point x and $B(t)=x$ (Wiener bridge). Under minimum regularity condition $p(t,x,x) \sim \frac{1}{\sqrt{\pi t}} e^{-tV(x)}$ for $t \geq 0$ (see [10])

Now for $t \to 0$

$$\int_0^\infty e^{-\lambda t} dN(\lambda) \sim \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-tV(s)} ds = \frac{1}{\pi} \int_{-\infty}^\infty \left(\int_0^\infty e^{-t(p^2+V(s))} \right) ds = \int_0^\infty e^{-\lambda t} d\mu(\lambda)$$

where

$$\mu(\lambda) = \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{\lambda - V(s)}_+ ds$$

applying so-called Tauberian theorem to the Laplace transform for $t \to 0$ we will get

$$N(\lambda) \sim \mu(\lambda) = \frac{1}{\pi} \int_0^\infty \sqrt{(\lambda - V(s))_+} ds$$

For details see Holt and Molchanov's work in [8] Kac [10] and [16] Hence on the spider graph with three legs

$$N(\lambda) \sim \mu(\lambda) = \frac{1}{\pi} \int_0^\infty \sum_0^3 \sqrt{(\lambda - v_j(s))_+} ds$$

Studying asymptotic is convenient using phase amplitude formalism. So, For $N(\lambda)$ where $\lambda > 0$

set

$$\psi_\lambda(x) = \rho_\lambda(x) \sin \theta_\lambda(x) \qquad \text{and} \qquad \psi'_\lambda(x) = \rho_\lambda(x) \cos \theta_\lambda(x)$$

then solve the Cauchy problem

$$\theta_\lambda' = \cos^2\theta_\lambda(x) + (\lambda - V(x))\sin^2\theta_\lambda(x), \qquad \theta_\lambda(0) = \theta_0$$
$$\rho_\lambda'(x) = \frac{1}{2}\rho_\lambda(x)(\lambda + 1 - V(x))\sin 2\theta_\lambda(x), \qquad \rho_\lambda(0) = 1$$

Let $a(\lambda) = \max\{x : v_j(x) \leq \lambda\}$

by Strum theory,

$$N(\lambda) = \lfloor \frac{1}{\pi}\theta_\lambda(a(\lambda))\rfloor + R(\lambda) \qquad |R(\lambda)| \leq 1$$

[8] proved that for strictly increasing sequence of non-negative real numbers,

$$N(\lambda) = B(\lambda) + \hat{R}(\lambda)$$

implies

$$N(\lambda) \sim B(\lambda)$$

Where $B(\lambda) = \int_0^{a(\lambda)} \frac{(\lambda - v(s))^{\frac{1}{2}}}{\pi} ds$ and $|\hat{R}(\lambda)| \leq a(\lambda) + 1$ and we will use the approximation for our increasing potential on the spider legs.

Let us define, $v_j^+(x) = \max_{y \leq x} v_i(y)$ and $v_j^-(x) = \min_{y \geq x} v_j(y)$. Let $a^\pm(\lambda) = \max x : v_j^\pm(x) \leq \lambda$ and denote $N^\pm(\lambda)$ for the eigenvalues $\{\lambda_i^\pm \leq \lambda\}$ for $v_j^\pm$

The following theorem is applicable for general non-monotonic increasing potential.

Theorem 4.7.6. *suppose $v_j(x) \to \infty$ as $x \to \infty$ for $j = 1, 2, 3$ and Bohr asymptotic holds for $v_j^\pm(x)$. Let, for $\lambda > 0$ there exists $L(\lambda)$ and $\epsilon(\lambda) \to 0$ as $\lambda \to \infty$ such that,*

$$\mathbf{1)} \qquad \sqrt{\lambda}L(\lambda) = \mathcal{O}(N^-(\lambda))$$

$$\mathbf{2)} \qquad 1 \leq \frac{v_j^+(x)}{v_j^-(x)} \leq 1 + \epsilon(\lambda) \qquad \forall x \in [L(\lambda), a^-(\lambda)]$$

$$\textit{and} \qquad \mathbf{3)} \frac{N^-\left(\frac{\lambda}{1+1+\epsilon(\lambda)}\right)}{N^-(\lambda)} \to 1 \qquad \textit{as } \lambda \to \infty$$

The Bohr asymptotic holds for v_j*.*

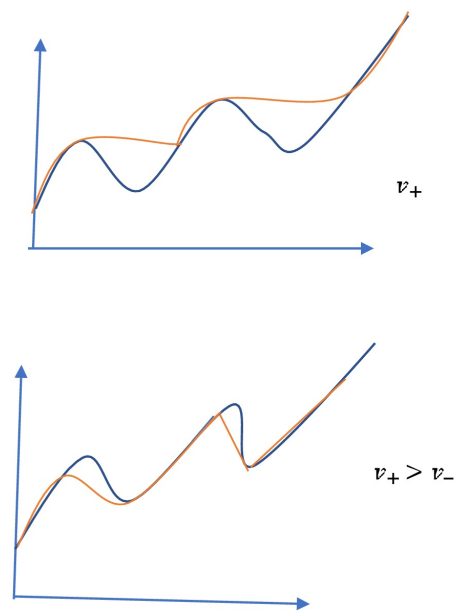

Figure 4.19: Positive and negative part of the potential which tends to ∞

Proof.

$$N^+(\lambda) \leq N(\lambda) \leq N^-(\lambda) \qquad \forall \lambda > 0 \tag{4.36}$$

and

$$B^{+}(\lambda) \leq B(\lambda) \leq B^{-}(\lambda) \forall \lambda > 0 \tag{4.37}$$

Let $\epsilon > 0$ be arbitrary and $\lambda > 0$ such that

$$1 \leq \frac{v_j^{+}(x)}{v_j^{-}(x)} \leq 1 + \epsilon(\lambda)$$

for all $x \in [L(\lambda), a^{-}(\lambda)]$.

Then

$$\begin{aligned} B^{+}(\lambda) &\geq \frac{1}{\pi} \int_{L(\lambda)}^{\infty} \left(\lambda - v_j^{+}(s)\right)_{+}^{\frac{1}{2}} ds \\ &\geq \left(\frac{1+\epsilon}{\pi^2}\right)^{\frac{1}{2}} \left(\int_0^{\infty} (\frac{\lambda}{1+\epsilon} - v_j^{-})_{+}^{\frac{1}{2}} ds - \int_0^{L(\lambda)(\frac{\lambda}{1+\epsilon} - v_j^{-})_{+}^{\frac{1}{2}}} ds \right) \\ &\geq \left(\frac{1+\epsilon}{\pi^2}\right)^{\frac{1}{2}} \left(\int_0^{\infty} (\frac{\lambda}{1+\epsilon} - v_j^{-})_{+}^{\frac{1}{2}} ds - L(\lambda)\sqrt{\lambda} \right) \end{aligned}$$

Assumption on v_j^{-} gives

$$N^{-}(\frac{\lambda}{1+\epsilon}) \sim B(\frac{\lambda}{1+\epsilon}) \qquad \text{as } \lambda \to \infty$$

This together with condition **2** and condition **3** give

$$\frac{B^{+}(\lambda)}{N^{-}(\lambda)} \geq (1+\epsilon)^{\frac{1}{2}} \tag{4.38}$$

Relation (4.38) and the assumption that Bohr asymptotic holds for v_j^{+} implies

$$\frac{N^{+}(\lambda)}{N^{-}(\lambda)} \geq (1+\epsilon)^{\frac{1}{2}}$$

since $N^{+} \leq N^{-}$ and ϵ is arbitrary we have $N^{-}(\lambda) \sim N^{+}(\lambda)$ for $\lambda \to \infty$ Bohr asymptotic for v_j follow from (4.36) and (4.37).

The following theorem works for the general monotonically increasing potential.

Theorem 4.7.7. *Let $v_j(x_j) \to \infty$ for $x_j \to \infty$ for $j = 1, 2, 3$ be increasing potential on the spider leg $[0, \infty)$. Let us consider $\{x_n\}$, a monotonically increasing sequence of non-negative real numbers on each leg of the spider and construct $v_j^+(x) = v_j(x_n - 0) \equiv v_{jn}^+$ and $v_j^-(x) = v_j(x_{n-1}) \equiv v_{jn}^-$ for $x \in [x_{n-1}, x_n)$ such that* **a**) $\left(v_{jn}^+ - v_{jn}^-\right)^{\frac{1}{2}}(x_n - x_{n-1}) \leq c$ *where c is a constant and* **b**) $v_j(x) - v_j(dn(x)) \to \infty$ *as $x \to \infty$ where d is a constant and $n(x)$ is an unique integer such that $x_{n(x)} \leq x \leq x_{n(x)+1}$. Then Bohr asymptotic formula holds for v_j.*

Proof. For fixed $\lambda > 0$ there exists a real number $a = a(\lambda)$ such that

$$v_j(x) > \lambda \quad \text{for} \quad x > a$$
$$v_j(x) < \lambda \quad \text{for} \quad x < a$$

Let $b = b(\lambda)$ be the unique integer such that $x_b \leq a \leq x_{b+1}$.

Let $a^\pm = a^\pm(\lambda)$ be the unique real number such that

$$v_j^\pm(x) \leq \lambda \quad \text{for} \quad x \leq a^\pm$$
$$\text{and} \quad v_j^\pm(x) > \lambda \quad \text{for} \quad x > a^\pm$$

This implies $a^+ = x_b$ and $a^- = x_{b+1}$ except for $v_{jn}^+ < v_{jn+1}^-$ for some n and $v_{jn}^+ \leq \lambda < v_{jn+1}^-$. Then we have $x_b = a = a^+ = a^-$ then by Strum theory

$$\frac{1}{\pi}\int_0^{x_b}(\lambda - v_j^+)^{\frac{1}{2}}ds \leq \frac{1}{\pi}\int_0^{a}(\lambda - v_j)^{\frac{1}{2}}ds \leq \frac{1}{\pi}\int_0^{x_{b+1}}(\lambda - v_j^-)_+^{\frac{1}{2}}ds \tag{4.39}$$

This implies

$$N^{+}(\lambda) \leq N(\lambda) \leq N^{-1}(\lambda)$$

Now for the phase rotation of $N(\lambda)$ over $[0, a^{\pm}]$ we can write

$$N^{+}(\lambda) = \frac{1}{\pi}\int_{0}^{x_b} \left(\lambda - v_j^{+}\right)^{\frac{1}{2}} ds + \mathcal{O}(b(\lambda)) \tag{4.40}$$

$$N^{-}(\lambda) = \frac{1}{\pi}\int_{0}^{x_{b+1}} \left(\lambda - v_j^{-}\right)_{+}^{\frac{1}{2}} ds + \mathcal{O}(b(\lambda)) \tag{4.41}$$

(4.39),(4.40),(4.41) together implies, Now condition **a**) gives $N(\lambda) = \frac{1}{\pi}\int_0^{a(\lambda)} (\lambda - v_j(s))^{\frac{1}{2}} ds + \mathcal{O}(b(\lambda))$ as $\lambda \to \infty$.

For large λ condition (**b**) gives,

$$\frac{1}{b\pi}\int_0^a (\lambda - v_j)^{\frac{1}{2}} ds \geq \frac{c}{\pi}\{v_j(a) - v_j(cb)\}^{\frac{1}{2}} \geq \frac{c}{\pi}\{v_j(x_b - 0) - v_j(cb)\}^{\frac{1}{2}}$$

Such that $\frac{1}{b(\lambda)\pi}\int_0^{a(\lambda)} (\lambda - v_j(s))^{\frac{1}{2}} ds \to \infty$ for $\lambda \to \infty$. Hence the Bohr asymptotic formula holds that is $N(\lambda)$ holds on the three leg of the spider as $\lambda \to \infty$

Example 5. ***Airy function [1]***

The linearly independent solutions of the equation

$$-y''(x) + xy(x) = 0 \qquad \textit{on } (-\infty, \infty) \tag{4.42}$$

is given by

$$y_1(x) = \frac{1}{\pi}\int_0^{\infty} \cos(tx + \frac{t^3}{3})dt \qquad \textit{for } x \to \infty$$

which is called Airy function of first kind and

$$y_2(x) = \frac{1}{\pi}\int_0^{\infty} [e^{(tx - \frac{t^3}{3})} + \sin(tx + \frac{t^3}{3})]dt \qquad \textit{for } x \to -\infty$$

which is called the Airy function of second kind which differs by phase $\frac{\pi}{2}$

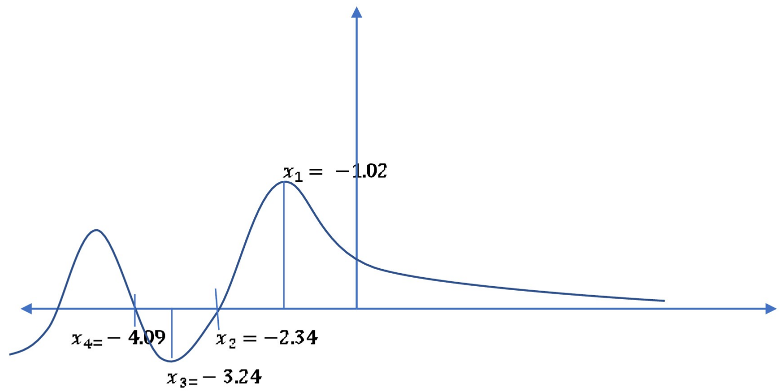

Figure 4.20: Graph of the zeros of Airy function of first kind and its derivative

The asymptotic for the Airy function is given by

$$y(x) \sim \frac{1}{2\sqrt{\pi}} \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{x^{\frac{1}{4}}}(1 + O(\frac{1}{x^{\frac{3}{2}}})) \qquad as\ x \to +\infty$$

and

$$y(x) \sim \frac{x^{-\frac{1}{4}}}{\sqrt{\pi}} \sin(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}) \qquad as\ x \to -\infty$$

Let us now consider the spectral problem on the full axis $[0, \infty)$*:*

$$-\psi^{''} + x\psi = \lambda\psi \qquad with\ \psi(0) = 0 \tag{4.43}$$

assume, $\lambda_n = x_n$ *where* $-x_n$ *is the* n^{th} *negative root of* $y(x)$ *with* $y_n(0) = 0$*, that is, Dirichlet condition at point 0, on* $[0, \infty)$

The Bohr formula can be given by,

$$N(\lambda) \sim \frac{1}{\pi}\int_0^{\lambda} \sqrt{\lambda - x}dx$$
$$(\lambda - x)^{\frac{3}{2}}\frac{2}{3\pi}/_{\lambda}^{0}$$
$$= \lambda^{\frac{3}{2}}\frac{2}{3\pi}$$
$$\Rightarrow x_n \sim (\frac{3}{2}\pi n)^{\frac{2}{3}}$$

finally we can write,

$$x_n = \left(\frac{3}{2}\pi(n - \frac{1}{4}) + O(\frac{1}{n})\right)^{\frac{2}{3}}$$

and

$$x_n^{'} = \left(\frac{3}{2}\pi(n - \frac{3}{4}) + O(\frac{1}{n})\right)^{\frac{2}{3}}$$

The solution of (4.42) *is given by:*

$$y(x) = \frac{1}{\pi}\int_0^{\infty} \cos(tx + \frac{t^3}{3})dt$$

and its derivative is

$$y^{'}(x) = -\frac{1}{\pi}\int_0^{\infty} t\sin(tx + \frac{t^3}{3})dt$$

then,

$$y(0) = \frac{1}{\pi}\int_0^{\infty} \cos(\frac{t^3}{3})dt$$
$$= \frac{3^{-\frac{2}{3}}}{\pi}\int_0^{\infty} z^{-\frac{2}{3}}\cos zdz \qquad = \frac{3^{-\frac{2}{3}}}{\pi}\Gamma(\frac{1}{3})\cos(\frac{\pi}{6}) = \frac{3^{-\frac{1}{6}}}{2\pi}\Gamma(\frac{1}{3})$$

and

$$y^{'}(0) = -\frac{1}{\pi}\int_0^{\infty} t\sin(\frac{t^3}{3})dt$$
$$= -\frac{3^{-\frac{1}{3}}}{\pi}\int_0^{\infty} z^{-\frac{1}{3}}\sin zdz \qquad = -\frac{3^{-\frac{1}{3}}}{\pi}\Gamma(\frac{2}{3})\sin(\frac{\pi}{3}) = -\frac{3^{\frac{1}{6}}}{2\pi}\Gamma(\frac{2}{3})$$

now, the solution of (4.43) *can be given by* $\psi_1 = y(x - x_1)$, $\psi_2 = y(x - x_2)$, $\psi_3 = y(x - x_3)$,*.....on* $[0, \infty)$. *4.21 gives the zeros of* (4.43) *on* $[0, \infty)$.

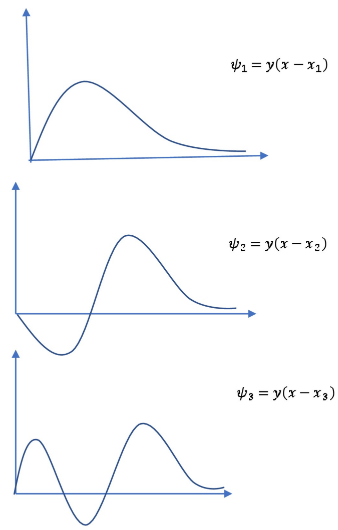

Figure 4.21: Graph of the zeros of (4.43) on $[0, \infty)$

4.8 Spectral theory of spider graph with mixed potential

In this section we will consider mixed type potentials on our spider graph with three legs. Consider first the case of increasing potentials on one leg and summable potential on other two legs with Bargmann's condition $\int_0^\infty x_i|v_i(x_i)|dx < \infty$ *for* $i = 2, 3$

If we split the spider sp_3 *onto three half axis by the Dirichlet boundary condition at 0 and potentials* $v_i(x_i)$ *for* $i = 1, 2, 3$ *such that* $v_i(x_i) \to \infty$, $v_1 \geq 0$ *and* $\int_0^\infty x_i|v_i(x_i)|dx$ *for* $i = 2, 3$ *then due to classical results of 1-D Strum-Liouville spectral theory we will get the mixed spectrum. The Dirichlet spectrum of our operator on* Leg_1 *will be discrete with super-exponentially decreasing eigenfunctions. On the legs* Leg_2, Leg_3 *the spectrum will be absolutely continuous and supported on* $[0, \infty)$ *plus (maybe) the finite discrete spectrum for* $\lambda < 0$. *Due to Bargmann's condition, if we have only the*

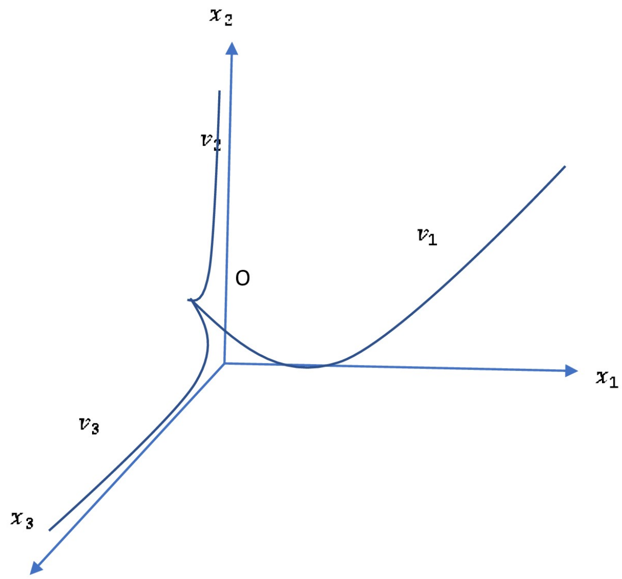

Figure 4.22: A three legged Spider quantum graph with fast increasing potential along leg 1 and fast decreasing potential along leg 2 and leg 3

summability of v_i, $i = 2, 3$, then the discrete spectrum for $\lambda < 0$ can be infinite.

When we will return to our initial conditions (Kirchhoff's gluing condition + continuity at point 0) that is, rank one perturbation, then due to general theory the absolute continuous part of the spectral measure will be preserved with some perturbations, that is, the operator will have the absolute continuous spectrum of multiplicity 2, but what will happen with the discrete part of the spectrum?

The following theorem gives the answer

Theorem 4.8.1. *Consider the Hamiltonian $Hy = -y^{''} + v(x)y$ on the quantum graph sp_3 with standard conditions at $x = 0$ (continuity of $\overrightarrow{y}$ and Kirchhoff's gluing condition) has the potentials $v_i(x_i)$, $i = 1, 2, 3$ such that, $v_1 \geq 0$ where $v_1(x_1) \to +\infty$ as $x_1 \to +\infty$. $v_{2,3}$ satisfy Bargmann's conditions $\int_0^\infty x_i|v_i| < \infty$ for $i = 1, 2, 3$. Then the spectral measure of H for positive energies, $\lambda \in [0, \infty)$ is purely absolute continuous with multiplicity 2., for $\lambda \in (-\infty, 0]$ can appear in the finite discrete spectrum.*

Proof. The essential spectrum of H equals $[0,\infty)$ (since, $v_1 \geq 0, v_{2,3} \in \mathbb{L}^1$). For any fixed $\lambda > 0$ on Leg_1, there is only one solution $y_{\lambda,1}(x)$ which tends to 0 very fast and all other solutions have the magnitude $r_{1,\lambda}(x_1) = \sqrt{y_{1,\lambda}^2 + (y_{1,\lambda}^{'})^2}(x_1)$ tending to $+\infty$ super-exponentially. Due to Schnoll's theorem [see [6]] which tells that absolute continuous spectrum with respect to spectral measure, the generalized eigenfunctions of H have estimations $|y_{1,\lambda}| \leq c|x|^{\frac{1}{2}+\epsilon}$ for any $\epsilon > 0$. It means that the generalized eigenfunction on Leg_1 must decay, that is equal to $y_{1,\lambda}(x_1)$. Let us assume that $r_{1,\lambda}(0) = 1$, $y_{1,\lambda}(0) = \cos\alpha$, $y_{1,\lambda}^{'}(0) = \sin\alpha$, where the phase $\alpha = \alpha(\lambda)$ is at least measurable function of the spectral parameter $\lambda > 0$. On Leg_2, Leg_3 we can consider solutions $y_{2,\lambda}(x_2)$, $y_{3,\lambda}(x_3)$ such that

$$r_{2,\lambda}(0) = r_{3,\lambda}(0) = r_{1,\lambda}(0) = 1$$

and dues to Kirchhoff's gluing condition

$$y_{1,\lambda}^{'}(0) + y_{2,\lambda}^{'} + y_{3,\lambda}^{'}(0) = \sin(\alpha\lambda) + y_{2,\lambda}^{'}(0) + y_{3,\lambda}^{'}(0) =$$

Of course, in the case of the multiple spectrum (in our case of the multiplicity 2) the selection of $y_{2,\lambda}^{'}(0)$, $y_{2,\lambda}^{'}(0)$ is not unique, one can put, say,

$$y_{2,\lambda}^{'}(0) = -\sin\alpha(\lambda), \qquad\qquad y_{3,\lambda}^{'}(0) = 0$$

The conditions

$$r_{2,\lambda}(0) = 1 \qquad\qquad y_{2,\lambda}^{'}(0) = -\sin\alpha$$

$$r_{3,\lambda}(0) = 1 \qquad\qquad y_{3,\lambda}^{'}(0) = 0$$

uniquely define on Leg_2, leg_3, the pair of the bounded solutions $y_{2,\lambda}(x_2)$, $y_{3,\lambda}(x_3)$.

The asymptotics for the solutions, for $x_i \to +\infty$, $i = 1, 2, 3$ can be expressed in terms of the transmission-reflection coefficients and functions $\alpha(\lambda)$, $A_i(k)$, $B_i(k)$, $|A_i(k)|^2 = 1 + |B_i(k)|^2$ where $i = 1, 2, 3$. Like in the scalar case ($\mathbb{R}^1_+$, see [11]) from the last fact, it follows that for $\lambda > 0$ the spectral measure is absolute continuous and has multiplicity 2.

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