LABELING REGIONS IN DEFORMATIONS OF GRAPHICAL ARRANGEMENTS

GÁBOR HETYEI

ABSTRACT. Combining a variant of the Farkas lemma with the Flow Decomposition Theorem we show that the regions of any deformation of a graphical arrangement may be bijectively labeled with a set of weighted digraphs containing directed cycles of negative weight only. Bounded regions correspond to strongly connected digraphs. The study of the resulting labelings allows us to add the omitted details in Stanley's proof on the injectivity of the Stanley-Pak labeling of the regions of the extended Shi arrangement and to introduce a new labeling of the regions in the a-Catalan arrangement. We also point out that Athanasiadis-Linusson labelings may be used to directly count regions in a class of arrangements properly containing the extended Shi arrangement and the Fuss-Catalan arrangement.

INTRODUCTION

Counting regions of a hyperplane arrangement is most often performed by computing its characteristic polynomial and by using Zaslavsky's formula [29]. This approach inspires combinatorial labelings of the regions in several important special cases: when all coefficients in all equations are integers, the finite field method [1, Theorem 2.2] (explained in detail by Stanley in [25, Lecture 5] and in [26, Section 3.11.4]) yields interesting models. For deformations of graphical (or affinographic) arrangements, in which all equations are of the form $x_i - x_j = c$, Whitney's formula [28] and the gain graph method [5, 10, 30] open a gateway to combinatorics. For certain deformations of the braid arrangement, Gessels formula on the generating function of labeled binary trees counted according to ascents and descents along left or right edges (shown in [7, 16]) has specializations that may be used to count the regions [4, 12].

A fundamentally different approach is to consider regions as sets defined by inequalities. Three important examples of this approach are the Stanley-Pak labeling [24] and the Athanasiadis-Linusson labeling [2] of the regions of the extended Shi arrangement, and the work of Hopkins and Perkinson [15] counting the regions in bigraphical arrangements.

The main purpose of this paper is to show that combining a variant of the Farkas lemma with the Flow Decomposition Theorem yields an automatic way to encode the regions with weighted digraphs in any deformation of a graphical arrangement in such a way that regions correspond exactly to the ones in which the weight of the directed cycles is negative. This approach generalizes the representation of the regions of the

Linial arrangement with semiacyclic tournaments and associates bounded regions to strongly connected digraphs. Using this approach one may easily fill in the omitted details in Stanley's proof of the injectivity of the Stanley-Pak labeling in the extended Shi arrangement, provide an alternative proof of Stanley's formula for exponential arrangements, and find a new labeling of the regions of the a-Catalan arrangement. This approach also generalizes the partial orientations introduced by Hopkins and Perkinson [15] to label the regions of a bigraphical arrangement (they used the same variant of the Farkas lemma) and naturally explains why sleek posets represent the regions of the Linial arrangement and why bounded regions of an interval order arrangement are in bijection with posets whose incomparability graph is connected. In an effort to keep the paper's size manageable, focus is on deformations of the braid arrangement, but the key ideas apply to the deformations of all graphical arrangements.

The paper is structured as follows. Section 2 contains the key bijection between regions of a graphical arrangement and weighted digraphs. Section 3 treats deformations of the braid arrangement which are *sparse* in the sense that at most two hyperplanes are associated to each edge. The Linial arrangement and interval order arrangement are important examples, but the bigraphical arrangements [15] are also related: they are also sparse in the above sense, however their underlying graph can be any simple graph, not only the complete graph. A key new idea idea is the notion of *gains*, using the fact that banning nonnegative directed cycles is equivalent to restricting our attention to weighted digraphs, in which negative weights represent costs, and the achievable maximum gain along any walk is finite. This idea is also used in Section 4 which treats deformations of the braid arrangement which contain the hyperplane $x_i - x_j = 0$ for each pair $\{i, j\}$: their regions refine the regions of the braid arrangement. Due to the presence of nonnegative weights, Dijkstra's algorithm can not be used, but for an important special case, when the weight function satisfies the *weak triangle inequality*, the gain function may be computed by building a tree recursively. This gain function is used in the final Section 6 to introduce a bijective labeling of the regions of an extended Shi arrangement with labeled a-Catalan paths. It is worth exploring in the future whether some variant of the gain function could play a role similar to parking functions in a broader setting. Section 5 focuses on integral deformations of the braid arrangement, in which the constants c appearing in the equations $x_i - x_j = c$ form a contiguous interval of integers. Among other results this section contains limits on the sizes of the directed cycles which we need to check to verify that a weighted digraph represents a nonempty region.

The last two sections contain applications of our approach to the extended Shi and a-Catalan arrangements, and also extend the use of Athanasiadis-Linusson diagrams to a broader class of arrangements. It turns out that this approach provides the fastest way to count the regions in the a-Catalan arrangement, and also in a class of hyperplane arrangements properly containing the extended Shi arrangements.

The highlighted variant of the Farkas lemma is likely suitable to count regions of a hyperplane arrangement in many other settings as well.

1. Preliminaries

1.1. **Two classical results.** The following variant of the Farkas lemma has also been used in [15]. It is originally due to Carver [6, Theorem 3], who stated it in a slightly different form. The formulation below may be found in [20].

Lemma 1.1 (Carver)**.** *Let A be a real $m \times n$ matrix and let b be a real $n \times 1$ column vector. Then the system of inequalities $Ax < b$ has no solution if and only if there is a nonzero real $m \times 1$ row vector y satisfying $y \geq 0$, $yA = 0$ and $yb \leq 0$.*

The other classical result we need is the Flow Decomposition Theorem [17, Theorem 8.8], originally due to Gallai [11]. We apply it in the following setting. Consider a directed graph with edge set E. A *circulation* is a function $f : E \to \mathbb{R}_{\geq 0}$ satisfying $\sum_{v=t(e)} f(e) = \sum_{v=h(e)} f(e)$ at every vertex v. Here $t(e)$ and $h(e)$ denote the tail, respectively the head of e. We set no capacity constraint (upper bound) on the values $f(e)$. The simplest example of a nonzero circulation is supported by a *directed cycle* $(e_1, e_2, \ldots, e_k)$, where $h(e_i) = t(e_{i+1})$ holds for $i = 1, 2, \ldots, k-1$ and $h(e_k) = t(e_1)$. We identify the directed cycle $(e_1, e_2, \ldots, e_k)$ with the circulation f that assigns 1 to the edges $e_1, e_2, \ldots, e_k$ and zero to all other edges. The restriction of the Flow Decomposition Theorem to circulations is the following statement.

Theorem 1.2. *Every not identically zero circulation f can be written as a positive linear combination of directed cycles. Moreover, a directed edge e appears in at least one of these cycles if and only if $f(e) > 0$.*

We will apply Theorem 1.2 to digraphs with *a cost function* assigning a (possibly negative) cost $c(e)$ to each directed edge e, the cost of a circulation f is the sum $c(f) = \sum_{e \in E} c(e) \cdot f(e)$.

Corollary 1.3. *The everywhere zero flow is the unique minimum cost circulation if and only if the cost of every directed cycle is strictly positive.*

Indeed, if there is a directed cycle whose cost is zero or negative, then either the cost of the everywhere zero flow is not minimal or it is not the only circulation whose cost is zero. On the other hand, if all directed cycles have a positive cost then every nonzero circulation has a positive cost by Theorem 1.2.

1.2. **Deformations of a graphical arrangement.** A hyperplane arrangement $\mathcal{A}$ is a finite collection of hyperplanes in a d-dimensional real vector space, which partition the space into regions. We may use the poset $L_{\mathcal{A}}$ of nonempty intersections (ordered by reverse inclusion) of the hyperplanes to count the regions. The *characteristic polynomial* $\chi(\mathcal{A}, q)$ of the arrangement is defined as

$$\chi(\mathcal{A}, q) = \sum_{x \in L_{\mathcal{A}}} \mu(\widehat{0}, x) q^{\dim(x)}, \tag{1.1}$$

where $\mu(x, y)$ is the Möbius function of $L_{\mathcal{A}}$ and $\widehat{0}$ is the entire vector space. The numbers $r(\mathcal{A})$ and $b(\mathcal{A})$ of all, respectively bounded regions may be found using *Zaslavsky's formulas* [29], stating

$$r(\mathcal{A}) = (-1)^d \chi(\mathcal{A}, -1) \quad \text{and} \quad b(\mathcal{A}) = (-1)^{\mathrm{rk}(L_{\mathcal{A}})} \chi(\mathcal{A}, 1). \tag{1.2}$$

Various methods are known to compute the characteristic polynomial. In the case when all equations have only integer coefficients, the characteristic polynomial may be computed using the finite field method [1, Theorem 2.2]. When $\mathcal{A}$ is constructed from a graph, Whitney's formula [28] or the gain graph method [5, 10, 30] may be used. We focus on *deformations of the braid arrangement*, and our main reference is the work of Postnikov and Stanley [19]. The *braid arrangement* or *Coxeter arrangement of type* A_{n-1} is the collection of hyperplanes

$$x_i - x_j = 0, \quad 1 \le i < j \le n \tag{1.3}$$

in the subspace V_{n-1} of $\mathbb{R}^n$, given by $x_1 + x_2 + \cdots + x_n = 0$. The braid arrangement is also a special case of a *graphical arrangement* $\mathcal{A}_G$ induced by a simple connected undirected graph G with edge set $E(G)$ on the vertex set $\{1, 2, \ldots, n\}$. It consists of the hyperplanes

$$x_i - x_j = 0, \quad \{i, j\} \in E(G) \tag{1.4}$$

in V_{n-1}. Hence the braid arrangement is $\mathcal{A}_{K_n}$ where K_n is the complete graph on n vertices. A *deformation* of a graphical arrangement consists of replacing each hyperplane $x_i - x_j = 0$ with a set of hyperplanes

$$x_i - x_j = a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(n_{ij})}, \quad \{i, j\} \in E(G), \tag{1.5}$$

where the n_{ij} are nonnegative integers and the a_{ij}^k are real numbers.

Remark 1.4. Our choice to restrict our definition of a graphical arrangement to connected graphs only and restrict our hyperplanes to V_{n-1} is a consistent generalization of the notation and terminology in [19]. Another option is to consider graphical arrangements in the entire space $\mathbb{R}^n$, associate graphical arrangements to disconnected undirected graphs as well, but consider *relative bounded* regions instead. We refer the interested reader to [15, Definition 1.7] for the exact definitions of that approach. See also Remark 2.2.

Well-studied deformations of the braid arrangement are the *truncated affine arrangements* $\mathcal{A}_{n-1}^{a,b}$. The integer parameters a and b satisfy $a + b \ge 2$ and the hyperplanes are

$$x_i - x_j = 1 - a, 2 - a, \ldots, b - 1 \quad 1 \le i < j \le n.$$

In particular, $\mathcal{A}_{n-1}^{0,2}$ is the *Linial arrangement*, $\mathcal{A}_{n-1}^{1,2}$ is the *Shi arrangement* $\mathcal{A}_{n-1}^{a,a+1}$ with $a \ge 1$ is the *extended Shi arrangement*, $\mathcal{A}_{n-1}^{2,2}$ is the *Catalan arrangement*, and $\mathcal{A}_{n-1}^{a,a}$ with $a \ge 2$ is the *a-Catalan arrangement*.

Remark 1.5. In the study of truncated affine arrangements, without loss of generality we may assume that $a \le b$ holds: replacing each vector $(x_1, x_2, \ldots, x_n)$ with $(x_n, x_{n-1}, \ldots, x_1)$ sends the arrangement $\mathcal{A}_{n-1}^{ab}$ into the arrangement $\mathcal{A}_{n-1}^{ba}$, since the equation corresponding to $x_i - x_j = c$ is the equation $x_{n+1-j} - x_{n+1-i} = -c$ after the linear transformation.

2. Labeling regions in deformations of graphical arrangements

Describing a region in a deformation of a graphical arrangement amounts to determining whether a system of linear inequalities of the form

$$m_{ij} < x_i - x_j < M_{ij}, \quad 1 \le i < j \le n \tag{2.1}$$

has a solution in V_{n-1}. Here we assume that $m_{ij} < M_{ij}$ holds for all (i,j) and we allow $m_{ij} = -\infty$ and $M_{ij} = \infty$ respectively.

Definition 2.1. *We call the solution set of a system of linear inequalities of the form (2.1) in the set V_{n-1} a* weighted digraphical polytope.

Remark 2.2. When we count regions without regard to their boundedness, we may equivalently consider the solution set of (2.1) in $\mathbb{R}^n$. Here the solution set is either empty or unbounded: for any real number r, replacing each x_i with $x_i + r$ leaves all differences $x_i - x_j$ unchanged. On the other hand, if we subtract $\sum_{i=1}^n x_i$ from each x_i, we obtain a point in V_{n-1}.

Definition 2.3. *To each system of inequalities (2.1) we create its* associated weighted digraph *as follows. For each $i < j$, if $m_{ij} > -\infty$, we create directed edge $i \to j$ with weight m_{ij} and if $M_{ij} < \infty$ we also create a directed edge $i \leftarrow j$ with weight $-M_{ij}$. An* m-ascending cycle *in the associated weighted digraph is a directed cycle, along which the sum of the labels is nonnegative. We call the associated weighted digraph* m-acyclic, *if it contains no m-ascending cycle.*

The associated weighted digraph uniquely encodes the system (2.1) (but not its solution set). Our definition of an m-ascending cycle is designed to match the conventions of labeling the regions of the Linial arrangement using semiacyclic tournaments, see Example 2.9 below. To maintain this compatibility we make the following definition.

Definition 2.4. *The* cost *of an edge in an associated weighted digraph of a system of inequalities (2.1) is the negative of its weight. Equivalently, the weight is the amount we gain by using an edge.*

Hence an m-ascending cycle is a cycle with non-positive cost. Using Lemma 1.1 we obtain the following characterization of nonempty weighted digraphical polytopes.

Theorem 2.5. *A weighted digraphical polytope given by a system of inequalities of the form (2.1) is not empty if and only if the weighted digraph associated to (2.1) is m-acyclic.*

Proof. When we rewrite the inequalities (2.1) in the form $Ax < b$, we must rewrite all inequalities $-\infty \ne m_{ij} < x_i - x_j$ as $x_j - x_i < -m_{ij}$. Let us associate to each such inequality a distinct variable u_{ij}. We keep all remaining inequalities of the form $x_i - x_j < M_{ij} \ne \infty$ unchanged, and we associate to each such inequality a distinct variable v_{ij}. By Lemma 1.1 the system of inequalities (2.1) has no solution if an only if there is a pair of vectors (u,v) where $u = (u_{ij} \ : \ 1 \le i < j \le n, m_{ij} \ne -\infty)$ and $v = (v_{ij} \ : \ 1 \le i < j \le n, M_{ij} \ne \infty)$, such that the following are satisfied:

    (1) All coordinates u_{ij} and v_{ij} are nonnegative and at least one of the vectors u or v is not zero.

(2) For each $i \in \{1, 2, \ldots, n\}$ we have

$$-\sum_{j>i} u_{ij} + \sum_{k<i} u_{k,i} - \sum_{k<i} v_{k,i} + \sum_{j>i} v_{ij} = 0.$$

(3) The inequality

$$-\sum_{i<j} u_{ij} \cdot m_{ij} + \sum_{i<j} v_{ij} \cdot M_{ij} \leq 0 \quad \text{holds.} \tag{2.2}$$

Condition (2) amounts to stating the following: if for each $i < j$ satisfying $m_{ij} \neq -\infty$ we let $u_{ij} \geq 0$ units flow from i to j and for each $i < j$ satisfying $M_{ij} \neq \infty$ we let $v_{ij} \geq 0$ units flow from j to i, we obtain a circulation. Let us call a circulation m-ascending if it satisfies (2.2). If we think of the numbers $-m_{ij}$ and M_{ij} as costs, then an m-ascending circulation is simply a circulation whose cost is not positive. By Corollary 1.3 there is a nonzero m-ascending circulation if and only if there is also and m-ascending cycle. $\square$

Remark 2.6. The net flow between i and j does not change if we decrease both u_{ij} and v_{ij} by the same positive real number r, whereas the sum on the left hand side of (2.2) decreases by $M_{ij} - m_{ij} \geq 0$. Hence if there is an m-ascending circulation, then there is also such a circulation in which for any $i < j$ at most one of u_{ij} and v_{ij} is positive. A related observation is that there is an m-ascending cycle of length 2 if and only if $M_{ij} - m_{ij} < 0$ holds for some $i < j$: in this case $m_{ij} < x_i - x_j < M_{ij}$ has no solution.

Remark 2.7. We may also introduce an edge $i \to j$ (respectively $i \leftarrow j$) of weight $-\infty$ whenever $i < j$ and $m_{i,j} = -\infty$ (respectively $M_{i,j} = \infty$) holds, and we may extend the addition operation to $\mathbb{R} \cup \{-\infty\}$ by setting $r + (-\infty) = (-\infty)$ for any $r \in \mathbb{R} \cup \{-\infty\}$. This addition makes no substantial difference as no m-ascending cycle could contain a directed edge of weight $-\infty$.

Theorem 2.5 may be rephrased as follows.

Corollary 2.8. *Consider the associated weighted digraph D encoding a system of inequalities of the form (2.1) and think of the weight $w(e)$ as money we gain when we walk from the tail of the edge e to its head. Then the system of inequalities (2.1) has a nonempty solution set if and only if we lose money along any closed walk.*

In other words, m-acyclic weighted digraphs are exactly the ones to which the Floyd-Warshall algorithm may be applied to find a unique minimum cost directed path between any pair of vertices.

Example 2.9. Each region of the Linial arrangement $\mathcal{A}_{n-1}^{0,2}$ is described by a set of inequalities

$$m_{ij} < x_i - x_j < M_{ij}, \quad 1 \leq i < j \leq n$$

where each of these inequalities is either $-\infty < x_i - x_j < 1$ or $1 < x_i - x_j < \infty$. The associated weighted digraph is a *tournament*: for each pair $\{i, j\}$, exactly one of the directed edges $i \to j$ and $i \leftarrow j$ belongs to the digraph. This tournament contains no m-ascending cycle if and only if it is *semiacyclic*, as defined in [19]. The observation that semiacyclic tournaments are in bijection with the regions of the Linial arrangement was independently made by Postnikov and Stanley and by Shmulik Ravid.

Our next result helps identify the bounded regions in a deformation of the braid arrangement.

Theorem 2.10. *A weighted digraphical polytope, given by a system of inequalities of the form* (2.1), *is not empty and bounded if and only if the associated weighted digraph is m-acyclic and it is strongly connected.*

Proof. By Theorem 2.5 we may assume that the associated weighted digraph is m-acyclic: this is equivalent to assuming that our polytope is not the empty set.

Assume first that the associated weighted digraph is not strongly connected. Then the set $\{1, 2, \ldots, n\}$ may be partitioned into two disjoint subsets V_1 and V_2 such that for each $v_1 \in V_1$ and $v_2 \in V_2$ the directed edge $v_1 \leftarrow v_2$ does not belong to the associated weighted digraph. Let $(x_1, \ldots, x_n)$ be any point satisfying (2.1). We claim that the point $(x_1', \ldots, x_n')$ given by

$$x_v' = \begin{cases} x_v + \frac{t}{|V_1|} & \text{if } v \in V_1 \\ x_v - \frac{t}{|V_2|} & \text{if } v \in V_2 \end{cases}$$

also satisfies (2.1) for all $t > 0$. Indeed $x_i' - x_j' = x_i - x_j$ holds if both i and j belong to V_1 or both of them belong to V_2. We are left to consider the inequalities where one of the indices belongs to V_1 and the other to V_2. Without loss of generality we may assume $i \in V_1$ and $j \in V_2$. If there is any bound on $x_i - x_j$, it is of the form $m_{ij} < x_i - x_j$ if $i < j$ and it is of the form $x_j - x_i < M_{j,i}$ if $i > j$. In either case, the inequality is even more valid if we increase x_i to x_i' and we decrease x_j to x_j'. Observe finally that we increased $|V_1|$ coordinates by $t/|V_1|$ and we decreased $|V_2|$ coordinates by $t/|V_2|$, hence $\sum_{i=1}^{n} x_i' = \sum_{i=1}^{n} x_i = 0$. The value of t is not bounded from above, hence our digraphical polytope must be unbounded.

Assume next that the associated weighted digraph D defining a weighted digraphical polytope $P \subset V_{n-1}$ is strongly connected. We show by induction on n that P is bounded. There is nothing to prove for $n = 1$: V_0 consists of a single point and every directed graph with a single vertex is strongly connected. For $n = 2$, the region defined by $m_{1,2} < x_1 - x_2 < M_{1,2}$ is an open interval, and it is bounded exactly when both $m_{1,2} \neq -\infty$ and $M_{1,2} \neq \infty$ hold, which is precisely the case when the vertices 1 and 2 are linked by directed edges both ways. Assume from now on that $n > 2$ holds. Since the digraph is strongly connected, neither the set $\text{In}(n) = \{i : i \to n\}$ nor the set $\text{Out}(n) = \{j : n \to j\}$ is empty. Each weighted arrow $i \to n$ represents an inequality $m_{i,n} < x_i - x_n$ which we rearrange as $x_n < x_i - m_{i,n}$. Each weighted directed edge $n \to j$ represents an inequality $x_j - x_n < M_{j,n}$ which we rearrange as $-x_n < M_{j,n} - x_j$. For each pair (i, j) with $i \in \text{In}(n)$ and $j \in \text{Out}(n)$, we take the sum of the inequalities represented by $i \to n$ and $j \leftarrow n$ and obtain $0 < x_i - m_{i,n} + M_{j,n} - x_j$. We rearrange this inequality as $m_{i,n} - M_{j,n} < x_i - x_j$ if $i < j$ and as $x_j - x_i < -m_{i,n} + M_{j,n}$ if $j < i$. We add these inequalities, remove the inequalities involving x_n, and we obtain the definition of a weighted digraphical polytope $P' \subset V_{n-2}$. (Note that we may have created several upper and lower bounds for the same $x_i - x_j$ but we only need to consider the greatest lower bound and the least upper bound.) Its associated weighted digraph D' is still strongly connected: if a directed path, connecting two elements of the set $\{1, 2, \ldots, n-1\}$ does not pass through n then the same walk is also present in D', if it

passes through n, we may replace the subwalk $i \to n \to j$ in D with a single edge $i \to j$ in D' as we created the edge $i \to j$ when we took the sum of the inequalities associated to $i \to n$ and $n \to j$. By the induction hypothesis P' is a bounded (or empty) polytope. Since P' is bounded, there is a cube $[-R, R]^{n-1}$ containing it for some $R > 0$. Consider now any point $(x_1, x_2, \ldots, x_n) \in P$ and let $s = (\sum_{i=1}^{n-1} x_i)/(n-1)$. We claim that the point $(x_1 - s, x_2 - s, \ldots, x_{n-1} - s)$ belongs to P'. Indeed, for any $i, j \in \{1, 2, \ldots, n-1\}$ the difference $(x_i - s) - (x_j - s)$ is the same as $x_i - x_j$, and every bound imposed on $x_i - x_j$ in P' is a consequence of the linear inequalities defining P. Furthermore, we have $\sum_{i=1}^{n-1}(x_i - s) = 0$. (In particular, P' is not empty.) Using any $i \in \text{In}(n)$ and any $j \in \text{Out}(n)$, we may write

$$-R - M_{j,n} \leq x_j - s - M_{j,n} < x_n - s < x_i - s - m_{i,n} \leq R - s - m_{i,n}.$$

As a consequence, $x_n - s$ is also bounded, there is an $R^* > 0$ such that $(x_1 - s, x_2 - s, \ldots, x_n - s) \in [-R^*, R^*]^n$. Observe finally that $\sum_{i=1}^{n} x_i = 0$ implies $(n-1)s + x_n = 0$, that is, $x_n - s = -ns$. Since $x_n - s$ is bounded, so is $s = (x_n - s)/(-n)$ and the vector $(x_1, x_2, \ldots, x_n)$, obtained by adding $(s, s, \ldots, s)$ to $(x_1 - s, x_2 - s, \ldots, x_n - s)$, is also bounded. $\qquad\square$

Remark 2.11. A variant of Theorem 2.10 is stated in [15, Theorem 1.8]. As noted in Remark 1.4, the paper of Hopkins and Perkinson considers relative bounded regions in $\mathbb{R}^n$ instead of bounded regions in V_{n-1}. Their result may be generalized to deformations of graphical arrangements as follows: if we define the associated weighted digraphs the same way as before, then relative bounded regions correspond to digraphs whose strongly connected components are the same as the weakly connected components. We leave the proof of this variant to the reader.

Corollary 2.12. *The bounded regions of the Linial arrangement are in bijection with the strongly connected semiacyclic tournaments.*

Remark 2.13. Athanasiadis computed the number $b(\mathcal{L}_n)$ of bounded regions of the Linial arrangement [1, Theorem 4.2] and raised the question whether there is a combinatorial interpretation of the numbers $b(\mathcal{L}_n)$ similar to the combinatorial interpretation of the number of all regions given by Postnikov and Stanley [19]. Corollary 2.12 provides a new response to this question. Previous models were provided by Tewari [27, Theorem 1.1] and by Flórez and Forge [9, Theorem 2.2].

We may generalize our observations regarding the Linial arrangement to an arbitrary deformed graphical arrangement $\mathcal{A}$ as follows. Assume $\mathcal{A}$ is given by (1.5). Without loss of generality we may assume that for each ordered pair (i, j) satisfying $i < j$ the numbers $a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(n_{ij})}$ are listed in increasing order. Each region of $\mathcal{A}$ is a weighted digraphical polytope and it is described by a system of inequalities (2.1) where each m_{ij} is either $-\infty$ or some element of the set $\{a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(n_{ij})}\}$ and M_{ij} is given by the following formula:

$$M_{ij} = \begin{cases} a_{ij}^{(1)} & \text{if } m_{ij} = -\infty; \\ a_{ij}^{(k+1)} & \text{if } m_{ij} = a_{ij}^{(k)} \text{ for some } k < n_{ij}; \\ \infty & \text{if } m_{ij} = a_{ij}^{(n_{ij})}. \end{cases} \qquad (2.3)$$

Keeping in mind these possibilities, we define a *valid weighted digraph* associated to a deformation of a graphical arrangement in V_{n-1} as follows.

Definition 2.14. *Let $\mathcal{A}$ be a deformation of a graphical arrangement, given by (1.5). We call a weighted digraph with vertex set $\{1, 2, \ldots, n\}$ valid if for each (i, j) satisfying $1 \leq i < j \leq n$ exactly one of the following holds:*
  1. *$n_{i,j} = 0$ and there is no directed edge between i and j.*
  2. *$n_{i,j} > 0$, there is no directed edge $i \to j$, and there is exactly one directed edge $i \leftarrow j$ which has weight $-a_{ij}^{(1)}$.*
  3. *$n_{i,j} > 0$, there is a directed edge $i \to j$ of weight $a_{ij}^{(k)}$, and there is exactly one directed edge $i \leftarrow j$ which has weight $-a_{ij}^{(k+1)}$, for some for some $k < n_{ij}$.*
  4. *$n_{i,j} > 0$, there is exactly one directed edge $i \to j$ which has weight $a_{ij}^{(n_{i,j})}$, and there is no directed edge $i \leftarrow j$.*

Since the associated weighted digraphs contain at most one directed edge $i \to j$ for any ordered pair (i, j), we may uniquely encode each such weighted digraph with a *weight function*
$$w : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to \mathbb{R} \cup \{-\infty\}$$
by setting $w(i, j)$ to be the weight of the directed edge $i \to j$, if $i \to j$ is present in the weighted digraph and $w(i, j) = -\infty$ otherwise. The m-acyclic condition may be then rephrased as follows:
$$w(i_1, i_2) + w(i_2, i_3) + \cdots + w(i_{m-1}, i_m) + w(i_m, i_1) < 0 \qquad (2.4)$$
must hold for any cyclic list $(i_1, i_2, \ldots, i_m)$ of elements of $\{1, 2, \ldots, n\}$. Here we extend the rules of addition to $\mathbb{R} \cup \{-\infty\}$ as in Remark 2.7. To simplify the notation in all subsequent proofs, we introduce the shorthand notation
$$w(i_1, i_2, \ldots, i_m) = w(i_1, i_2) + w(i_2, i_3) + \cdots + w(i_{m-1}, i_m) + w(i_m, i_1) \qquad (2.5)$$
for the total weight of all directed edges along the closed walk $i_1 \to i_2 \to \cdots \to i_m \to i_1$. As a consequence of Theorems 2.5 and 2.10 we have the following result.

Corollary 2.15. *Let $\mathcal{A}$ be a deformation of a graphical arrangement, given by (1.5). Then the regions of $\mathcal{A}$ are in bijection with the valid m-acyclic weighted digraphs on $\{1, 2, \ldots, n\}$ in such a way that bounded regions correspond to strongly connected valid m-acyclic weighted digraphs.*

Indeed, each region created by the hyperplanes of $\mathcal{A}$ may be uniquely encoded by exactly one valid m-acyclic weighted digraph of the given form: for each $i, j \in \{1, 2, \ldots, n\}$ satisfying $i < j$, the value of $x_i - x_j$ belongs to exactly one interval created by the set $\{a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(n_{ij})}\}$ and this interval is the same for all points of the same region. The assignment of valid weighted digraphs to regions is thus injective. By Theorem 2.5, exactly the m-acyclic valid weighted digraphs encode sets of inequalities with a nonempty solution set, and by Theorem 2.10 exactly the strongly connected m-acyclic valid weighted digraphs represent bounded regions.

The m-acyclic property can be independently verified within each strong component of the weighted digraph. Hence we have the following structure theorem.

Theorem 2.16. *Assume a deformation of a graphical arrangement given by (1.5) has the property that $n_{ij} > 0$ holds for all (i,j) satisfying $1 \leq i < j \leq n$. Then associated weighted digraph of any region may be uniquely constructed as follows.*

   (1) *We fix an ordered set partition $(N_1, N_2, \ldots, N_k)$ of the set $\{1, 2, \ldots n\}$. The parts of the ordered set partition will be the vertex sets of the strong components. For any (i,j) having the property that the part containing i precedes the part containing j there is a of weight directed edge $i \to j$ but no directed edge $i \leftarrow j$. The label on $i \to j$ must be $a_{ij}^{(n_{ij})}$ if $i < j$ and it must be $-a_{ij}^{(1)}$ if $i > j$.*
   (2) *On each strong component N_i we independently select a strongly connected valid m-acyclic weighted digraph.*

Proof. By our assumption each valid weighted digraph has at least one directed edge between any two vertices, hence the strong components may be linearly ordered in such a way that for any i in a smaller strong component and for any j in a larger strong component there is a directed edge $i \to j$ but no directed edge $i \leftarrow j$. The weighting of these edges can only be as stated. As noted above, the m-acyclic property only needs to be verified on the strongly connected components. $\square$

In [23] Stanley considers a sequence $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots)$ of deformations of the braid arrangement, such that each $\mathcal{A}_n$ is a hyperplane arrangement in $\mathbb{R}^n$ and for each $S \subseteq \{1, 2, \ldots\}$ he defines $\mathcal{A}_n^S$ as the subcollection of hyperplanes $x_i - x_j = c$ of $\mathcal{A}_n$ satisfying $\{i, j\} \subseteq S$. He calls such a sequence *exponential* if the number $r(\mathcal{A}_n^S)$ of regions of $\mathcal{A}_n^S$ depends only on $k = |S|$ and it is the number $r(\mathcal{A}_k)$ of regions of $\mathcal{A}_k$. Introducing the exponential generating functions

$$R_{\mathcal{A}}(t) = \sum_{n \geq 0} r(\mathcal{A}_n) \cdot \frac{t^n}{n!} \quad \text{and} \quad B_{\mathcal{A}}(t) = \sum_{n \geq 1} b(\mathcal{A}_n) \cdot \frac{t^n}{n!}$$

for all, respectively the bounded regions, Stanley [23, Theorem 1.2]

$$B_{\mathcal{A}}(t) = 1 - \frac{1}{R_{\mathcal{A}}(t)}. \tag{2.6}$$

The outline of the proof cites Zaslavsky's formula (1.2), Whitney's formula [28] and the exponential formula in enumerative combinatorics. Formula (2.6) may also be derived from Theorem 2.16 as follows. Consider an exponential sequence $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots)$ of deformations of the braid arrangements. As a consequence of Zaslavsky's formula (1.2), for each $n \geq 1$ and for each $S \subseteq \{1, 2, \ldots, n\}$ the number $b(\mathcal{A}_n^S)$ of bounded regions of $\mathcal{A}_n^S$ also depends only on $k = |S|$ and it is the number $b(\mathcal{A}_k)$ of bounded regions of $\mathcal{A}_k$. To construct an associated weighted digraph of a region of $\mathcal{A}_n$ we first fix an ordered set partition $(N_1, N_2, \ldots, N_k)$ of the set $\{1, 2, \ldots n\}$. After fixing k and the size n_i of each N_i, the number of ways we can select an ordered set partition is $\binom{n}{n_1, n_2, \ldots, n_k}$. In the next step we must select a strongly connected valid m-acyclic weighted digraph on each part of our set partition. There are $b(\mathcal{A}_{n_i})$ ways to perform this step on the part N_i. Hence we obtain

$$r(\mathcal{A}_n) = \sum_{k=1}^{n} \sum_{\substack{n_1 + \cdots + n_k = n \\ n_1, \ldots, n_k > 0}} \binom{n}{n_1, n_2, \ldots, n_k} \prod_{i=1}^{k} b(\mathcal{A}_{n_i}) \quad \text{for all } n \geq 1. \tag{2.7}$$

This formula implies $R_A(t) = \sum_{k \geq 0} B_A(t)^k$.

3. The poset of gains and sparse deformations

Consider a deformation of a graphical arrangement given by (1.5) and one of its valid m-acyclic associated weighted digraphs. We may use the weights to define a partial order on the vertex set $\{1, 2, \ldots, n\}$ as follows.

Definition 3.1. *Given a valid m-acyclic weighted digraph D on $\{1, 2, \ldots, n\}$, we define $i <_D j$ if there is a directed path $i = i_0 \to i_1 \to \cdots \to i_k = j$ such that the weight of each directed edge $i_s \to i_{s+1}$ is nonnegative. We call the set $\{1, 2, \ldots, n\}$, ordered by $<_D$ the* poset of gains induced by D.

The fact that $i <_D j$ is a partial order is a direct consequence of the m-acyclic property stated in Theorem 2.5. In terms of Corollary 2.8, $i <_D j$ holds if we can find a directed path from i to j such that we do not lose money by walking from i to j using that path. In general, the poset of gains carries less information than recording the actual weights, but in some special cases all information may be reconstructed from it. One example is the Linial arrangement: its posets of gains are the *sleek posets*, see [19, Section 8.2]. Semiacyclic tournaments and sleek posets are in bijection: for $i < j$, the relation $i <_D j$ holds exactly when the $w(i,j) = 1$, if $w(i,j) = -1$ then i and j are incomparable. It has been shown in of [19, Theorem 8.6] that the length of a minimal ascending cycle is at most 4, equivalently, sleek posets may be characterized by a finite set of excluded subposets on at most 4 elements.

A similar approach may be taken to the *semiorder arrangement*, defined as the set of hyperplanes

$$x_i - x_j = -1, 1 \quad \text{for } 1 \leq i < j \leq n \text{ in } V_{n-1}.$$

For these, in any associated valid weighted digraph, regardless of the order of the numbers i and j, there is either a single directed edge of weight 1 between i and j, or there are two edges of weight -1, one in each direction. The elements i and j are comparable in the poset of gains exactly when there is a single directed edge between them, pointing toward the larger element. These are exactly the *semiorders* as defined in [19, Section 7]. In analogy to [19, Theorem 8.6] one may directly show the following:

Proposition 3.2. *In a valid weighted digraph associated to the semiorder arrangement, the length of a shortest m-ascending cycle is at most 4.*

The equivalent statement for semiorders was first shown by Scott and Suppes [22]. The Linial arrangement and the semiorder arrangement are both examples of the following class of hyperplane arrangements.

Definition 3.3. *We call $\mathcal{A}$ a deformation of the braid arrangement, given by (1.5) sparse if $1 \leq n_{i,j} \leq 2$ holds for all $i < j$, and the signs of the numbers $a_{i,j}^{(k)}$ satisfy the following for all $i < j$:*

   (1) $a_{i,j}^{(1)} > 0$ *holds, whenever $n_{i,j} = 1$,*
   (2) $a_{i,j}^{(1)} < 0 < a_{i,j}^{(2)}$ *holds, whenever $n_{i,j} = 2$.*

We call $\mathcal{A}$ an interval order arrangement *if $n_{i,j} = 2$ holds for all $i < j$.*

Note that interval order arrangements are precisely the hyperplane arrangements associated to interval orders in [23]. Generalizing our observations made so far, we have the following.

Proposition 3.4. *Consider a sparse deformation of the braid arrangement and any valid m-acyclic weighted digraph D associated to it. In the induced poset of gains, $i <_D j$ holds exactly when there is a single directed edge $i \to j$ of positive weight. For any pair $\{i, j\}$ of incomparable vertices satisfying $i < j$, the edge $j \to i$ is always present, and any edge between i and j has negative weight.*

The straightforward verification is left to the reader. An interesting property of sparse deformations of the braid arrangement is that a bounded region corresponds to a poset of gains with a connected *incomparability graph*.

Definition 3.5. *The* incomparability graph *of a partially ordered set is the undirected graph whose vertices are the elements of the poset, and the edges are the incomparable pairs of elements.*

Theorem 3.6. *Let D be a valid m-acyclic weighted digraph associated to a sparse deformation of the braid arrangement in V_{n-1}. If D is strongly connected then the incomparability graph of the induced poset of gains is connected. The converse is also true when $n_{i,j} = 2$ holds for all $1 \le i < j \le n$.*

Proof. Assume first D is strongly connected. We show for any pair of vertices $\{i, j\}$ that there is a path between them in the incomparability graph of $<_D$. There is nothing to prove if i and j are incomparable. Without loss of generality we may assume that $i <_D j$ holds, by Proposition 3.4 this implies the presence of a single directed edge $i \to j$ in D. Since D is strongly connected, there is a path $j = i_0 \to i_1 \to \cdots \to i_k = i$ from j to i in D in which each edge $i_s \to i_{s+1}$ is either between a pair of incomparable vertices, or corresponds to the relation $i_s <_D i_{s+1}$. It suffices to show that in a shortest such path all edges correspond to incomparable pairs. Assume, by contradiction, that $i_s <_D i_{s+1}$ holds for some $i_s \to i_{s+1}$, and consider first the case when this edge is succeeded by an edge $i_{s+1} \to i_{s+2}$. The edge $i_{s+1} \to i_{s+2}$ must correspond to an incomparable pair of vertices, otherwise $i_s <_D i_{s+1} <_D i_{s+2}$ holds, and the segment $i_s \to i_{s+1} \to i_{s+2}$ in our path may be shortened to $i_s \to i_{s+2}$. A similar shortening may also occur if the vertex i_{s+2} forms an incomparable pair with i_s, or if $i_s <_D i_{s+2}$ holds. Hence we must have $i_{s+2} <_D i_s$ and $i_{s+2} <_D i_s <_D i_{s+1}$ in contradiction with the presence of the edge $i_{s+1} \to i_{s+2}$. A similar contradiction may be reached if $i_s \to i_{s+1}$ is preceded by another edge. Hence we may have only one edge $j \to i$ corresponding to $j <_D i$, in contradiction with $i <_D j$.

Assume now that $n_{i,j} = 2$ holds for all $i < j$ and that the incomparability graph of $<_D$ is connected. In this case the edges of negative weight in D are obtained from the incomparability graph of $<_D$ by replacing each undirected edge of the incomparability graph by a pair of opposing directed edges. Since the incomparability graph is connected, there is a directed path using edges of negative weight only from any vertex to any vertex in D. $\square$

Example 3.7. Consider the Linial arrangement and the semiacyclic tournament D containing a directed edge $i \leftarrow j$ of weight -1 for each $i < j$. This is a valid m-acyclic

weighted digraph, it is in fact acyclic. The induced poset of gains is an antichain, the incomparability graph is the complete graph, it is connected. However, D is not strongly connected.

We conclude this section with presenting a class of sparse deformations of the braid arrangement for which the properties of the associated valid m-acyclic weighted digraphs may be used to give an upper bound for the number of regions.

Definition 3.8. *Let $\underline{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_{\geq 0}{}^n$ be a vector of nonnegative real numbers. We define the $\underline{a}$-generalized Linial arrangement as the set of hyperplanes*

$$x_i - x_j = a_i \quad for \ 1 \leq i < j \leq n \ in \ V_{n-1}.$$

Note that setting $a_1 = a_2 = \cdots = a_n = 1$ yields the Linial arrangement. In any valid associated weighted digraph D there is exactly one directed edge for each $i < j$:

(1) either an edge $i \to j$, of weight a_i, corresponding to $x_i - x_j > a_i$: we call such an edge an *ascent*;
(2) or an edge $i \leftarrow j$ of weight $-a_i$: we call such an edge a *descent*.

As a consequence, the weighted digraph D may be uniquely reconstructed from its underlying tournament. It depends on the values of the parameters a_i which tournaments contain no m-ascending cycle, but there is a bijection between the regions of an $\underline{a}$-generalized Linial arrangement and a set of tournaments on $\{1, 2, \ldots, n\}$. That said, certain tournaments may be excluded regardless of the choice of the parameters. Following [14], we call a cycle *alternating* if ascents and descents alternate in it.

Proposition 3.9. *If D is a valid m-acyclic weighted digraph associated to an $\underline{a}$-generalized Linial arrangement, then D contains no alternating cycle.*

Proof. Given a directed cycle in a tournament on $\{1, 2, \ldots, n\}$, let us call a vertex i a *peak* if an descent follows a ascent at i along the cycle, and let us call i a *valley* if ascent follows a descent at i. Peaks and valleys alternate along an alternating cycle, hence we may compute the total weight of its edges by counting the contribution of each edge at the valley incident to it. The incoming edge at a valley i has weight $-a_i$, the outgoing edge has weight a_i. Hence the total weight of all edges is zero, regardless of the values of the parameters a_i. An alternating cycle is m-ascending. $\square$

It has been shown in [14, Theorem 4.4] that the number of alternation acyclic tournaments on the set $\{1, 2, \ldots, n\}$ is the median Genocchi number H_{2n-1}.

Corollary 3.10. *The number of regions in any $\underline{a}$-generalized Linial arrangement in V_{n-1} is less than or equal to the median Genocchi number H_{2n-1}.*

Remark 3.11. It has been shown in [14] that there is a bijection between alternation acyclic tournaments and the regions of the *homogenized Linial arrangement* whose hyperplanes are defined by equations of the form $x_i - x_j = y_j$, where the y_js are also coordinate functions. Hence we may think of an $\underline{a}$-generalized Linial arrangement as the intersection of the homogenized Linial arrangement with the hyperplanes $y_i = a_i$ for $i = 1, 2, \ldots, n$. Corollary 3.10 may also be shown using these observations.

Remark 3.12. The *bigraphical arrangements* introduced in [15] are related to the sparse deformations discussed in this section. Given a simple graph $G = (\{1, 2 \ldots, n\}, E\})$,

for each edge $\{i, j\}$ we choose real parameters $a_{i,j}$ and $a_{j,i}$, such that there is an $x \in \mathbb{R}^n$ satisfying all inequalities of the form $x_i - x_j < a_{i,j}$ and $x_j - x_i < a_{j,i}$. The bigraphical arrangement is the set of $2|E|$ hyperplanes $\{x_i - x_j = a_{i,j} \ : \ \{i, j\} \in E\}$. Since $-a_{j,i} < x_i - x_j < a_{i,j}$ must have a solution, $a_{i,j} + a_{j,i} > 0$ must hold for each $\{i, j\} \in E$. The valid associated weighted digraphs may be described as follows. For each edge $\{i, j\} \in E$, assuming $i < j$, exactly one of the following possibilities hold:

(1) There a directed edge $i \leftarrow j$ of weight $a_{j,i}$ (representing $x_i - x_j < -a_{j,i}$).
(2) There is a directed edge $i \rightarrow j$ of weight $-a_{j,i}$ and a directed edge $i \leftarrow j$ of weight $-a_{i,j}$ (representing $-a_{j,i} < x_i - x_j < a_{i,j}$).
(3) There a directed edge $i \rightarrow j$ of weight $a_{i,}$ (representing $x_i - x_j > a_{i,j}$).

It is part of the definition of a bigraphical arrangement that selecting option (2) on each $\{i, j\} \in E$ should yield an m-acyclic digraph (labeling the *central region*). The definition of a *partial orientation* in [15] is equivalent to the above weighted digraph labeling. The oriented edges in [15] are directed in the opposite way, the unoriented edges correspond to our pairs of opposing directed edges. The definition of A-admissibility is equivalent to our m-acyclic condition (the *score* used in [15] is the negative of our weight) and *potential cycles* are directed cycles using some directed edges whose opposite is also present. The same observations also apply to the *mixed graphs* appearing in [3].

4. SEPARATED DEFORMATIONS AND THE WEAK TRIANGLE INEQUALITY

Definition 4.1. *Let $\mathcal{A}$ be a deformation of the braid arrangement, given by* (1.5). *We call the arrangement $\mathcal{A}$ separated if 0 belongs to the set $\{a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(n_{ij})}\}$ for each $1 \leq i < j \leq n$.*

Proposition 4.2. *In any valid associated weighted digraph of a separated deformation $\mathcal{A}$ of the braid arrangement given by* (1.5) *either $w(i, j) \geq 0$ or $w(j, i) \geq 0$ holds for each $1 \leq i < j \leq n$.*

Indeed, by our assumptions $a_{i,j}^{(1)} \leq 0$ (and hence $-a_{i,j}^{(1)} \geq 0$) and $a_{ij}^{(n_{ij})} \geq 0$ hold, furthermore, for each $k < n_{ij}$, the numbers $a_{ij}^{(k)}$ and $a_{ij}^{(k+1)}$ can not be both nonzero real numbers, having opposite signs. The statement is a direct consequence of Definition 2.14.

Corollary 4.3. *For a separated deformation of the braid arrangement, the induced poset of gains associated to any valid m-acyclic weighted digraph is a totally ordered set.*

Hence there is a unique permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of the coordinates such that $w(\sigma(i), \sigma(j)) \geq 0$ holds for all $i > j$.

Definition 4.4. *We call the permutation σ the* order of gains *associated to the valid m-acyclic weighted digraph of a separated deformation of the braid arrangement. We will also use the notation $i <_{\sigma^{-1}} j$ to indicate that the label i precedes the label j in σ.*

So far we only rephrased the observation that for a separated deformation of the braid arrangement, each region is included in a region $x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n)}$ of the braid arrangement. Using Theorem 2.10 we may refine this observation as follows.

Theorem 4.5. *Let $\mathcal{R}$ be a region of a separated deformation of the braid arrangement and let $x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n)}$ be the unique region of the braid arrangement containing it. Then there is a unique sequence $1 = i_0 \le i_1 < i_2 < \cdots < i_k = n$ such that writing σ as the concatenation*

$$\sigma = (\sigma(i_0) \cdots \sigma(i_1)) \cdot (\sigma(i_1 + 1) \cdots \sigma(i_2)) \cdots (\sigma(i_{k-1} + 1) \cdots \sigma(i_k))$$

of contiguous subwords has the following properties:

(1) *For each $j = -1, 0, \ldots, k - 1$ the intersection of $\mathcal{R}$ with the linear span of $\{e_{\sigma(i_j+1)}, e_{\sigma(i_j+2)}, \ldots, e_{\sigma(i_{j+1})}\}$ is a bounded region.*
(2) *If a subset S of $\{1, 2, \ldots, n\}$ contains indices j_1 and j_2 such that $\sigma(j_1)$ and $\sigma(j_2)$ belong to different subwords in the above decomposition then the intersection of $\mathcal{R}$ with the linear span of $\{e_{\sigma(j)} : j \in S\}$ is unbounded.*

Proof. By Theorem 2.10, for any nonempty subset S of $\{1, 2, \ldots, n\}$ the intersection of $\mathcal{R}$ with the linear span of $\{e_{\sigma(j)} : j \in S\}$ is bounded if the restriction of the weighted digraph $D_{\mathcal{R}}$ associated to $\mathcal{R}$ to the set $\{\sigma(j) : j \in S\}$ is strongly connected. Note that the converse is not necessarily true: inequalities implied by edges not belonging to the restriction may force the intersection to be bounded even if the restriction is not strongly connected.

First we show the existence of such a decomposition. Let i_1 be the largest index such that there is a directed path from $\sigma(i_1)$ to $\sigma(1)$ in $D_{\mathcal{R}}$. In particular we set $i_1 = 1$ if $\sigma(1)1$ has indegree zero in $D_{\mathcal{R}}$. Keep in mind that the definition of V_{n-1} includes the equality $x_1 + \cdots + x_n = 0$, together with $x_{\sigma(2)} = \cdots = x_{\sigma(n)} = 0$ this forces $x_{\sigma(1)} = 0$. Since there is a directed path $\sigma(1) \to \sigma(2) \to \cdots \to \sigma(i_1)$ in $D_{\mathcal{R}}$, the restriction of $D_{\mathcal{R}}$ to the set $[\sigma(1), \sigma(i_1)] = \{\sigma(1), \sigma(2), \ldots, \sigma(i_1)\}$ is strongly connected. Furthermore, given any $j_1 \in [\sigma(1), \sigma(i_1)]$ and any $j_2 \notin [\sigma(1), \sigma(i_1)]$, no directed edge $j_2 \to j_1$ can exist as $\sigma(1)$ is not reachable from j_2. Hence any subset of $[\sigma(1), \sigma(i_1)]$ and any subset of its complement must belong to different strong components: once we leave $[\sigma(1), \sigma(i_1)]$, there is no arrow we could use to return to it. As seen in the proof of Theorem 2.10, this implies that all (implied) differences of the form $x_{\sigma(j_1)} - x_{\sigma(j_2)}$ are only bounded from below in the definition of $\mathcal{R}$ which remains unbounded even after intersecting it with the linear span of $\{e_{\sigma(j)} : j \in S\}$ for any S containing both j_1 and j_2. We may continue finding a suitable $i_2, i_3, \ldots i_k$ in a recursive fashion: for $j = 1, 2, \ldots, k - 1$, we may define i_{j+1} as the largest index such that $\sigma(i_j)$ may be reached from $\sigma(i_{j+1})$.

To show the uniqueness of our decomposition, observe first that replacing i_1 with any larger i_1' results in a subset $[\sigma(1), \sigma(i_1')]$ such that the restriction of $D_{\mathcal{R}}$ to $[\sigma(1), \sigma(i_1')]$ is not strongly connected: for any $j_1 \le i_1$ and any $j_2 > i_1$ there is only a directed edge from $\sigma(j_1)$ to $\sigma(j_2)$. As above, using the proof of Theorem 2.10, we may conclude that the differences of the form $x_{\sigma(j_1)} - x_{\sigma(j_2)}$ are only bounded from below in the definition of $\mathcal{R}$ and the intersection of $\mathcal{R}$ with the linear span of $\{e_{\sigma(j)} : 1 \le j \le i_1'\}$ is still unbounded. On the other hand, for any $i_1' < i_1$ the set $[\sigma(1), \sigma(i_1)]$ properly contains $[\sigma(1), \sigma(i_1')]$ and the restriction of $D_{\mathcal{R}}$ to $[\sigma(1), \sigma(i_1)]$ is strongly connected. $\square$

Next we define the *gain function* $g : \{1, 2, \ldots, n\} \to \mathbb{R}$ associated to a valid m-acyclic weighted digraph as follows.

Definition 4.6. *For each $i \in \{1, 2, \ldots, n\}$ we define the* gain function $g(\sigma(i))$ *as the maximum weight of a directed path beginning at $\sigma(1)$ and ending at $\sigma(i)$. In particular, we set $g(\sigma(1)) = 0$. Here σ is the total order of gains.*

Remark 4.7. The word "gain" here is used as the opposite of "cost" in a weighted directed graph setting and it is not related to the notion of a gain graph [5, 30].

Lemma 4.8. *Every gain function has the weakly increasing property*

$$g(\sigma(1)) \leq g(\sigma(2)) \leq \cdots \leq g(\sigma(n)).$$

Indeed, for each $i < j$, every maximum weight directed path from $\sigma(1)$ to $\sigma(i)$ may be extended to a directed walk to $\sigma(j)$ by adding the directed edge $\sigma(i) \to \sigma(j)$ of nonnegative weight. Eliminating cycles from the walk can only increase the total weight, hence we have

$$g(\sigma(j)) \geq g(\sigma(i)) + w(\sigma(i), \sigma(j)) \geq g(\sigma(i)). \tag{4.1}$$

The question naturally arises: could the maximum gain be achieved by using directed edges of nonnegative weight only? In the rest of this section we show that the answer is yes at least in an important special case.

Definition 4.9. *We call a deformation $\mathcal{A}$ of the braid arrangement* integral *if all the numbers $a_{i,j}^k$ appearing in (1.5) are integers. We say that $\mathcal{A}$ satisfies the* weak triangle inequality *if for all triplets (i, j, k), the inequalities $w(i, j) \geq 0$ and $w(j, k) \geq 0$ imply*

$$w(i, k) \leq w(i, j) + w(j, k) + 1$$

in any valid m-acyclic associated weighted digraph.

Theorem 4.10. *Let $\mathcal{A}$ be a separated integral deformation of the braid arrangement satisfying the weak triangle inequality, and let D be an associated m-acyclic weighted digraph. Let σ be the total order of gains associated to D and let g be the gain function. Then, for each $i > 1$ there is a directed path from $\sigma(1)$ to $\sigma(i)$ such that all weights in the path are nonnegative and the total weight of the edges in the path is $g(\sigma(i)) - g(\sigma(1))$.*

Proof. Consider a weighted path $\sigma(1) = i_0 \to i_1 \to \cdots \to i_m = \sigma(i)$ from $\sigma(1)$ to $\sigma(i)$ such that the weight of the path is $g(\sigma(i)) - g(\sigma(1))$ and it has the least possible number of negative weighted edges. We are done if there is no edge of negative weight, otherwise assume that $i_k \to i_{k+1}$ is the first edge satisfying $w(i_k, i_{k+1}) < 0$. All preceding steps being of nonnegative weight, we must have $i_0 <_{\sigma^{-1}} i_1 <_{\sigma^{-1}} \cdots <_{\sigma^{-1}} i_k$. The inequality $w(i_k, i_{k+1}) < 0$ implies $i_{k+1} <_{\sigma^{-1}} i_k$, hence there is a unique $j \leq k - 1$ such that $i_j <_{\sigma^{-1}} i_{k+1} <_{\sigma^{-1}} i_{j+1}$ holds. (Recall that, the m-acyclic property implies that walks revisiting a vertex can only have lower weight than the path obtained by eliminating the closed subwalks, hence the inequalities must be strict.) Applying the weak triangle inequality to he triple (i_j, i_{k+1}, i_{j+1}) we obtain

$$w(i_j, i_{j+1}) \leq w(i_j, i_{k+1}) + w(i_{k+1}, i_{j+1}) + 1.$$

Let us add the weight $\vec{w}(i_{j+1}, i_{j+2}, \ldots, i_k, i_{k+1})$ of the walk $i_{j+1} \to i_{j+2} \to \cdots \to i_k \to i_{k+1}$ to both sides. On the left hand side we obtain the weight of the walk $i_j \to i_{j+1} \to \cdots \to i_k \to i_{k+1}$:

$$\vec{w}(i_j, i_{j+1}, \ldots i_k, i_{k+1}) \leq w(i_j, i_{k+1}) + \vec{w}(i_{j+1}, i_{j+2}, \ldots, i_k, i_{k+1}) + w(i_{k+1}, i_{j+1}) + 1.$$

The sum $\vec{w}(i_{j+1}, i_{j+2}, \ldots, i_k, i_{k+1}) + w(i_{k+1}, i_{j+1})$ on the right hand side is the weight of the closed walk $i_{j+1} \to i_{j+1} \to i_{j+1} \to i_{j+2} \to \cdots \to i_k \to i_{k+1} \to i_{j+1}$, which is negative by the m-acyclic property, and it is at most -1 by the integrality of $\mathcal{A}$. Thus we obtain

$$\vec{w}(i_j, i_{j+1}, \ldots i_k, i_{k+1}) \leq w(i_j, i_{k+1}),$$

which means that we may replace the subpath $i_j \to i_{j+1} \to \cdots \to i_k \to i_{k+1}$ with the nonnegative edge $i_j \to i_{k+1}$, thus decreasing the number of negative edges without decreasing the total weight, in contradiction with our assumptions. □

An example of a saturated integral arrangement in which $g(\sigma(i)) - g(\sigma(1))$ is attained only by a path containing a negative weighted edge is given in Example 5.12. *From now on until the rest of the section we only consider separated integral deformations of the braid arrangement satisfying the weak triangle inequality.* If we know the total order of gains, Theorem 4.10 allows us to compute the gain function in a greedy fashion:

(1) We set $g(\sigma(1)) = 0$.
(2) Once we computed $g(\sigma(1))), g(\sigma(2)), \ldots, g(\sigma(i-1))$, the value of $g(\sigma(i))$ is

$$g(\sigma(i)) = \max_{1 \leq j \leq i-1} (g(\sigma(j) + w(\sigma(j), \sigma(i)). \tag{4.2}$$

The choice of j in equation (4.2) may not be unique. We eliminate the ambiguity by always selecting the rightmost possible label $\sigma(j)$:

Definition 4.11. *We select the largest $j < i$ satisfying (4.2) and call the resulting $\sigma(j)$ the parent of $\sigma(i)$, denoted by $p(\sigma(i))$. We extend the definition to $\sigma(1)$ by setting $p(\sigma(1)) = \sigma(1)$.*

Note that the pairs $(\sigma(i), p(\sigma(i)))$ form the edges of a tree rooted at $\sigma(1)$. We call this rooted tree the *tree of the gain function*. In the study of the properties of this tree the following lemma plays a key role.

Lemma 4.12. *If $i <_{\sigma^{-1}} j$ and $p(j) <_{\sigma^{-1}} i$ hold then we have $w(i,j) = g(j) - g(i) - 1$, $w(p(j), i) = g(i) - g(p(j))$ and $p(j) \leq_{\sigma^{-1}} p(i)$.*

Proof. By our assumption i is to the right of $p(j)$, hence we have

$$w(i, j) \leq g(j) - g(i) - 1.$$

By (4.1) we also have

$$w(p(j), i) \leq g(i) - g(p(j)).$$

The sum of the two inequalities, combined with the weak triangle inequality yields

$$g(j) - g(p(j)) = w(p(j), j) \leq w(p(j), i) + w(i, j) + 1 \leq g(j) - g(p(j)).$$

The left and the right end being equal all inequalities above must be equalities. Hence we have $w(i, j) = g(j) - g(i) - 1$ and $w(p(j), i) = g(i) - g(p(j))$. Finally $p(j) \leq_{\sigma^{-1}} p(i)$ is a direct consequence of $w(p(j), i) = g(i) - g(p(j))$. □

Proposition 4.13. *The edges of the tree of the gain function are noncrossing: there is no $i_1 <_{\sigma^{-1}} i_2$ such that $p(i_1) <_{\sigma^{-1}} p(i_2) <_{\sigma^{-1}} i_1 <_{\sigma^{-1}} i_2$ would hold.*

Proof. Assume the contrary. By Lemma 4.12, $i_1 <_{\sigma^{-1}} i_2$ and $p(i_2) <_{\sigma^{-1}} i_1$ imply $p(i_2) \leq_{\sigma^{-1}} p(i_1)$, in contradiction with $p(i_1) <_{\sigma^{-1}} p(i_2)$. □

Corollary 4.14. *The number of possible types of trees of the gain function is a Catalan number.*

5. CONTIGUOUS INTEGRAL DEFORMATIONS

Definition 5.1. *We call an integral deformation of the braid arrangement in V_{n-1} contiguous if, for every $i < j$, the set $\{a_{i,j}^{(1)}, a_{i,j}^{(2)}, \ldots, a_{i,j}^{(n_{i,j})}\}$ is a contiguous set $[\alpha(i,j), \beta(i,j)] = \{\alpha(i,j), \alpha(i,j) + 1, \ldots, \beta(i,j)\}$ of integers.*

Since the equation $x_i - x_j = c$ is equivalent to the equation $x_j - x_i = -c$, we may consistently extend our notation by setting

$$\alpha(j,i) = -\beta(i,j) \quad \text{and} \quad \beta(j,i) = -\alpha(i,j) \quad \text{for } 1 \le i < j \le n. \qquad (5.1)$$

The truncated affine arrangements $\mathcal{A}_{n-1}^{a,b}$ are contiguous deformations of the braid arrangement: we have $\alpha(i,j) = 1 - a$ and $\beta(i,j) = b - 1$ for $1 \le i < j \le n$.

We may specialize Definition 2.14 to such arrangements as follows.

Proposition 5.2. *Let $\mathcal{A}$ be a contiguous integral deformation of the braid arrangement in V_{n-1}, defined by the intervals $[\alpha(i,j), \beta(i,j)]$. A valid associated digraph on the vertex set $\{1, 2, \ldots, n\}$ satisfies for each (i,j) exactly one of the following:*

(1) *There is no directed edge $i \to j$, and there is a directed edge $i \leftarrow j$ of weight $-\alpha(i,j)$.*

(2) *There is a directed edge $i \to j$ of weight w, and there is a directed edge $i \leftarrow j$ of weight $-w - 1$, for some for some integer $w \in [\alpha(i,j), \beta(i,j) - 1]$.*

(3) *There is a directed edge $i \to j$ of weight $\beta(i,j)$, and there is no directed edge $i \leftarrow j$.*

As we did right after Definition 2.14, we extend the weight function to a function $w : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to \mathbb{R} \cup \{-\infty\}$ by setting the following for all $i < j$:

(1) The weight $w(i,j)$ belongs to the set $[\alpha(i,j), \beta(i,j)] \cup \{-\infty\}$ and the weight $w(j,i)$ belongs to the set $[-\beta(i,j), -\alpha(i,j)] \cup \{-\infty\} = [\alpha(j,i), \beta(j,i)] \cup \{-\infty\}$.

(2) If $w(i,j) = -\infty$ then $w(j,i) = -\alpha(i,j) = \beta(j,i)$ and if $w(i,j) = \beta(i,j)$ then $w(j,i) = -\infty$.

(3) For all other values of $w(i,j)$ we have $w(j,i) = -1 - w(i,j)$.

In other words, regardless of the order of i and j, $w(i,j)$ belongs to the set $[\alpha(i,j), \beta(i,j)] \cup \{-\infty\}$, and $w(i,j) = -\infty$ holds if and only if $w(j,i) = \beta(j,i)$.

As we will see in Theorem 5.13 below, the m-acyclic property is especially easy to check for truncated affine arrangements satisfying $|a - b| \le 1$. The following lemma still holds for all contiguous integral deformations.

Lemma 5.3. *Let $C = (i_0, i_1, \ldots, i_k)$ be an m-ascending cycle of minimum length in a valid weighted digraph associated to a contiguous integral deformations of the braid arrangement. Then for any diagonal pair of vertices of C, that is, for any $\{i_r, i_s\}$ such that i_r and i_s are not cyclically consecutive, the associated weight function satisfies $-\infty \in \{w(i_r, i_s), w(i_s, i_r)\}$.*

Proof. Assume by contradiction that there is a pair $\{i_r, i_s\}$ of cyclically not consecutive vertices such that $-\infty \notin \{w(i_r, i_s), w(i_s, i_r)\}$. In this case the real numbers $w(i_r, i_s)$

and $w(i_s, i_r)$ must satisfy $w(i_r, i_s) + w(i_s, i_r) = -1$. The cyclic lists $(i_r, i_{r+1}, \ldots, i_s)$ and $(i_s, i_{s+1}, \ldots, i_r)$ represent shorter closed walks in the associated weighted digraph, which can not be m-ascending. Since all weights are integers we must have

$$w(i_r, i_{r+1}, \ldots, i_s) \leq -1 \quad \text{and} \quad w(i_s, i_{s+1}, \ldots, i_r) \leq -1.$$

Taking the sum of these inequalities, after subtracting $w(i_r, i_s) + w(i_s, i_r) = -1$ on both sides yields

$$w(C) = w(i_r, i_{r+1}, \ldots, i_s) + w(i_s, i_{s+1}, \ldots, i_r) - (w(i_r, i_s) + w(i_s, i_r)) \leq -1$$

in contradiction with the closed walk represented by C being m-ascending. □

The next theorem is a generalization of [19, Theorem 8.6].

Theorem 5.4. *Consider a contiguous integral deformation of the braid arrangement in V_{n-1} satisfying*

$$\beta(i, k) \leq \beta(i, j) + \beta(j, k) + 1 \quad \text{for all triplets } \{i, j, k\}. \tag{5.2}$$

Then any valid associated weighted digraph is m-acyclic if and only if it contains no m-ascending cycle of length at most four.

Proof. It suffices to show that there is no valid associated weighted digraph in which a shortest m-ascending cycle of length $k \geq 5$ would exist. Assume by contradiction that $C = (i_0, i_1, \ldots, i_{k-1})$ is such an is an m-ascending cycle. Recall that, by Lemma 5.3, only one of the directed edges $i_r \to i_s$ and $i_r \leftarrow i_s$ is present for each diagonal pair $\{i_r, i_s\}$. Let us call $i_s \in \{i_0, i_1, \ldots, i_{k-1}\}$ a *diagonal tail (diagonal head) vertex* if $i_s \to i_r$ $(i_s \leftarrow i_r)$ holds for some diagonal pair $\{i_r, i_s\}$. Clearly, for $k \geq 4$, each vertex i_s is either a diagonal tail or a diagonal head vertex, some of them may be both. If each $i_s \in \{i_0, i_1, \ldots, i_{k-1}\}$ is either only a diagonal tail or only a diagonal, then we may reach a contradiction as follows. At least one i_r must be a diagonal head, without loss of generality we may assume $r = 0$, hence we must have $i_2 \to i_0$, $i_3 \to i_0$, $\ldots$, $i_{k-2} \to i_0$, as a consequence i_2 and i_{k-2} are diagonal tails and by $k \geq 5$ the pair $\{i_2, i_{k-2}\}$ is a diagonal pair for which neither $i_2 \to i_{k-2}$ nor $i_2 \leftarrow i_{k-2}$ can hold, in contradiction with Lemma 5.3. Hence there is at least one vertex $i_r \in \{i_0, i_1, \ldots, i_{k-1}\}$ which is simultaneously a diagonal head and a diagonal tail: $i_r \leftarrow i_s$ and $i_r \to i_{s'}$ hold for some $s, s' \in \{0, 1, \ldots, k-1\} - \{r-1, r, r+1\}$. (Additions and subtractions in the subscripts are performed modulo k.) Without loss of generality we may assume that s and s' are cyclically consecutive.
Case 1: $s' = s + 1$ holds. The cycles $(i_r, i_{r+1}, \ldots, i_s)$ and $(i_{s'}, i_{s'+1}, \ldots, \ldots i_{r-1}, i_r)$ are shorter than C, hence they can not be m-ascending:

$$w(i_r, i_{r+1}, \ldots, i_s) \leq -1 \quad \text{and} \quad w(i_{s'}, i_{s'+1}, \ldots, i_r) \leq -1 \quad \text{must hold.}$$

Adding $w(i_s, i_{s'})$ on both sides to the sum of the above inequalities we obtain

$$w(i_s, i_r) + w(i_r, i'_s) + w(C) \leq -2 + w(i_s, i_{s'}).$$

Since $w(C)$ is nonnegative, we obtain

$$w(i_s, i_r) + w(i_r, i_{s'}) \leq -2 + w(i_s, i_{s'}). \tag{5.3}$$

Since $\{i_s, i_r\}$ and $\{i_r, i_{s'}\}$ are diagonals, we have $w(i_s, i_r) = \beta(i_s, i_r)$ and $w(i_r, i_{s'}) = \beta(i_r, i_{s'})$. Furthermore $w(i_s, i_{s'})$ is at most $\beta(i_s, i_{s'})$. Equation (5.3) implies

$$\beta(i_s, i_r) + \beta(i_r, i_{s'}) \leq -2 + \beta(i_s, i_{s'}),$$

in contradiction with (5.2).

Case 2: $s' = s - 1$ holds. This case is analogous to the previous one. Now the cycles $(i_r, i_{r+1}, \ldots, i_s)$ and $(i_{s'}, i_{s'+1}, \ldots, \ldots i_{r-1}, i_r)$ both contain the directed edge $i_{s'} \to i_s$, and, in analogy to (5.3) we obtain the inequality

$$w(i_s, i_r) + w(i_r, i'_s) \leq -2 - w(i_{s'}, i_s). \tag{5.4}$$

Once again, $\{i_s, i_r\}$ and $\{i_r, i_{s'}\}$ are diagonals, hence the left hand side of (5.4) is $\beta(i_s, i_r) + \beta(i_r, i_{s'})$ whereas the right hand side is at most $-2 - \alpha(i_{s'}, i_s) = -2 + \beta(i_s, i'_s)$. Once again, we obtain a contradiction with (5.2).

$\square$

Theorem 5.5. *If the truncated affine arrangement $\mathcal{A}_{n-1}^{a,b}$ satisfies $a, b \geq 0$, then a valid associated weighted digraph is m-acyclic if and only if it contains no m-ascending cycle of length at most four.*

Proof. As noted in Remark 1.5, without loss of generality we may assume that $a \leq b$ holds. By Theorem 5.4, it suffices to verify the validity of the inequality (5.2). We distinguish several cases depending on the relative order of i, j and k. If $i < j < k$ holds then $\beta(i, j) = \beta(j, k) = \beta(i, k) = b - 1$, and Equation (5.2) is equivalent to $b - 1 \leq 2(b - 1) + 1$, which is equivalent to $b \geq 0$.

If $i > j > k$ holds then $\beta(i, j) = \beta(j, k) = \beta(i, k) = a - 1$, and Equation (5.2) is equivalent to $a - 1 \leq 2(a - 1) + 1$, which is equivalent to $a \geq 0$.

In all other subcases at least one of $\beta(i, j)$ and $\beta(j, k)$ is $b - 1$, the right hand side of (5.2) is at least $(a - 1) + (b - 1) + 1$, and the left hand side of (5.2) is at most $b - 1$. The inequality $b - 1 \leq (a - 1) + (b - 1) + 1$ is equivalent to $a \geq 0$. $\square$

It may be possible to extend Theorem 5.5 to further cases, but this can not be done by inspecting the value of a alone. The next two examples illustrate this fact and indicate some of the difficulties in discussing the truncated affine arrangements which do not fall under the validity of Theorem 5.5.

Example 5.6. In the case when $a = -1$ and $b = 4$ there is a valid weighted digraph associated to the truncated affine arrangement $\mathcal{A}_4^{-1,4}$ whose shortest m-ascending cycle has length 5. An example is shown in Figure 1. This weighted digraph has 5 vertices and for each i, j we have either $i < j$, $w(i, j) = b - 1 = 3$ and $w(j, i) = -\infty$, or we have $i > j$, $w(i, j) = a - 1 = -2$ and $w(j, i) = -\infty$. In other words, only one of the arrows $i \to j$ or $j \to i$ is present, the digraph is a tournament. The weight of the directed cycle $C = (1, 3, 5, 4, 2)$ is $2 \cdot 3 + 3 \cdot (-2) = 0$, this cycle is m-ascending. To show that all other directed cycles have negative weight, consider the following "potential function" ϕ on the vertices: $\phi(1) = 0$, $\phi(2) = 2$, $\phi(3) = 3$, $\phi(4) = 4$ and $\phi(5) = 6$. The values of this function are the circled numbers in Figure 1. For any arrow $i \to j$ that belongs to C, we have $w(i, j) = \phi(j) - \phi(i)$ and for all other arrows $i \to j$ we have $w(i, j) < \phi(j) - \phi(i)$. Hence any cycle containing at least one directed edge that does not belong to C must have negative weight.
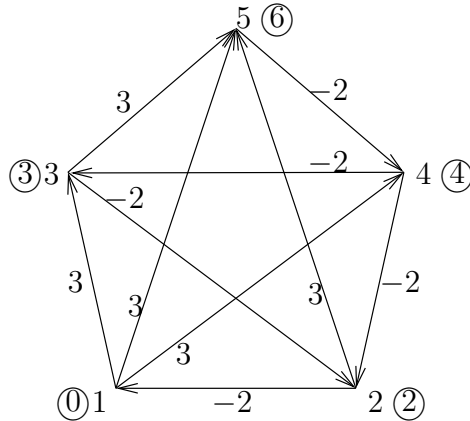
FIGURE 1. A shortest m-ascending cycle of length 5

Example 5.7. In the case when $a = -1$ and $b = 3$, the truncated affine arrangement $\mathcal{A}_{n-1}^{-1,3}$ consists of the hyperplanes $x_i - x_j = 2$ for $1 \leq i < j \leq n$. A dilation by a factor of $1/2$ puts the regions of this hyperplane arrangement in bijection with the regions of the Linial arrangement $\mathcal{A}_{n-1}^{0,2}$ to which Theorem 5.5, as well as [19, Theorem 8.6], are applicable.

Looking at the formulas for the number of regions and the characteristic polynomial $\mathcal{A}_{n-1}^{ab}$ in [19], counting regions in a combinatorial way promises to be easier in the cases when $|a - b| \leq 1$ and $a + b \geq 2$ hold for the parameters a and b. As we will see in Proposition 5.11 below, these are exactly the cases when the truncated affine arrangements is separated and satisfies the weak triangle inequality (as defined in Section 4). As a direct consequence of the definitions we obtain:

Corollary 5.8. *A contiguous integral deformation of the braid arrangement in V_{n-1} is separated if and only if 0 belongs to $[\alpha(i, j), \beta(i, j)]$ for all $1 \leq i < j \leq n$.*

Corollary 5.9. *The truncated affine arrangement $\mathcal{A}_{n-1}^{ab}$ satisfying $a \leq b$ and $a + b \geq 2$ is separated if and only if $a \geq 1$ holds.*

Next we characterize the cases when the weak triangle inequality is satisfied by a separated contiguous integral deformation of the braid arrangement.

Theorem 5.10. *Consider a separated contiguous integral deformation of the braid arrangement in V_{n-1}. This satisfies the weak triangle inequality if and only if*

$$\beta(i, j) \leq \beta(i, k) + 1 \quad and \quad \beta(i, j) \leq \beta(k, j) + 1 \tag{5.5}$$

hold for all $\{i, j, k\} \subseteq \{1, 2, \ldots, n\}$.

Proof. Assume first that (5.5) is satisfied and let us compare $w(i, k)$ with $w(i, j) + w(j, k) + 1$, where we assume that the weights $w(i, j)$, (j, k) and $w(i, k)$ are nonnegative. If the reverse arrows $i \leftarrow j$ (of weight $-1 - w(i, j)$) and $j \leftarrow k$ (of weight $-1 - w(j, k)$) both exist, then the fact that the cycle (i, k, j) is m-acyclic implies

$$-1 - w(i, j) - 1 - w(j, k) + w(i, k) \leq -1,$$

which is equivalent to the weak triangle inequality

$$w(i,k) \leq w(i,j) + w(j,k) + 1.$$

We are left to consider the cases when at least one of the arrows $i \leftarrow j$ and $j \leftarrow k$ does not exist. In this case either $w(i,j) = \beta(i,j)$ or $w(j,k) = \beta(j,k)$. In either case the sum $w(i,j) + w(j,k) + 1$ is at least $\min(\beta(i,j), \beta(j,k)) + 1$, whereas $w(i,k)$ is at most $\beta(i,k)$. The weak triangle inequality follows from (5.5).

Next we prove the contrapositive of the converse. Assume that (5.5) fails for some $\{i,j,k\}$, without loss of generality we may assume that $\beta(i,j) > \beta(i,k) + 1$ holds. We construct a valid associated weighted digraph violating the weak triangle inequality as follows.We set $w(i,j) = \beta(i,j)$ (hence there is no arrow $i \leftarrow j$) and $w(i,k) = \beta(i,k)$ (hence there is no arrow $i \leftarrow k$). We set $w(j,k) = 0$ and $w(k,j) = -1$. Note that there is no m-ascending cycle on the restriction of our weighted digraph to $\{i,j,k\}$ as the only cycle is (j,k), which has weight (-1). We fix a linear order on $\{1,2,\ldots,n\}$ in such a way that i, j, k are the smallest vertices in this order. For any pair of vertices $\{i',j'\}$ not contained in $\{i,j,k\}$ we set $w(i',j') = \beta(i',j')$ where i' is the smaller vertex in our order. This choice guarantees that there is no arrow $i' \leftarrow j'$, hence our weighted digraph does not contain any other cycle than (j,k). $\square$

Proposition 5.11. *Assume the truncated affine arrangement $\mathcal{A}_{n-1}^{ab}$ satisfies $1 \leq a \leq b$ and $n \geq 3$. This integral and separated arrangement satisfies the weak triangle inequality if and only of $b \leq a + 1$ holds.*

Proof. The arrangement is separated by Corollary 5.9. Assume first that $b \leq a + 1$ holds. For all $\{i,j,k\}$ the value of $\beta(i,j)$ is at most $b - 1$, whereas the value of $\beta(i,k)$ and of $\beta(k,j)$ is at least $a - 1$. The inequality (5.5) is a direct consequence of $b \leq a + 1$.

Conversely, if $a \leq b - 2$ holds, consider $i = 2$, $j = 3$ and $k = 1$. Then $\beta(i,j) = \beta(2,3) = b - 1$, $\beta(i,k) = \beta(2,1) = a - 1$ and $\beta(2,3) \leq \beta(2,1) + 1$ fails as $b - 1 \not\leq (a-1) + 1$. $\square$

Example 5.12. Consider the truncated affine arrangement $\mathcal{A}_2^{13}$ and the associated m-acyclic weighted digraph given by $w(2,1) = 0$, $w(1,3) = 0$, $w(3,1) = -1$ and $w(2,3) = 2$. The arrangement is saturated, the total order of gains is $\sigma = 213$. Not only the weak triangle inequality fails because of $w(2,3) > w(2,1) + w(1,3) + 1$, but the largest weight path form $\sigma(1) = 2$ to $\sigma(2) = 1$ is $2 \to 3 \to 1$, which contains an edge of negative weight.

For truncated affine arrangements, a combinatorial approach to the case $1 \leq a \leq b \leq a + 1$ is further facilitated by the following result.

Theorem 5.13. *Consider a truncated affine arrangement $\mathcal{A}_{n-1}^{ab}$ satisfying $1 \leq a \leq b \leq a + 1$. A valid associated weighted digraph is m-acyclic if and only if in contains no m-ascending cycle of length three.*

Proof. As a consequence of Theorem 5.5, it suffices to show that there is no ab-weighted digraph in which a shortest m-ascending cycle of length 4 would exist. Assume by contradiction that (i_0, i_1, i_2, i_3) is such a cycle. By Lemma 5.3, for each diagonal pair $\{i_r, i_s\}$, only one of the directed edges $i_r \to i_s$ and $i_r \leftarrow i_s$ is present. After cyclic rotation of the labels, if necessary, we may assume that $i_0 \to i_2$ and $i_1 \to i_3$ are the

directed edges present in our weighted digraph. By our assumption, the directed cycles (i_0, i_2, i_3) and (i_1, i_3, i_0) are not m-acyclic, we have

$$w(i_0, i_2, i_3) \leq -1 \quad \text{and} \quad w(i_1, i_3, i_0) \leq -1.$$

In analogy to the derivation of (5.3), adding $w(i_1, i_2) - w(i_3, i_0)$ on both sides of the sum of the two inequalities yields

$$w(i_0, i_2) + w(i_1, i_3) + w(i_0, i_1, i_2, i_3) \leq -2 + w(i_1, i_2) - w(i_3, i_0).$$

Since (i_0, i_1, i_2, i_3) is m-ascending, the last inequality implies

$$w(i_0, i_2) + w(i_1, i_3) \leq -2 + w(i_1, i_2) - w(i_3, i_0).$$

The left hand side is at least $2(a-1)$, the right hand side is at most $-2+(b-1)+(b-1)$. Hence we obtain

$$2(a - 1) \leq -2 + (b - 1) + (b - 1)$$

which is equivalent to $a \leq b - 1$. Together with $a \geq b - 1$ we obtain $a = b - 1$ and all inequalities used must be equalities. But this implies $i_0 > i_2 > i_1 > i_3 > i_0$, a contradiction. $\square$

Theorem 5.13 may be extended to contiguous integral deformations of the braid arrangement using the following auxiliary terminology.

Definition 5.14. *A separated contiguous integral deformation of the braid arrangement in V_{n-1} satisfying the weak triangle inequality has* a crossing β-pattern *if there is a 4-element subset $\{i_1, i_2, j_1, j_2\}$ of $\{1, 2, \ldots, n\}$ such that*

$$\beta(i_1, j_1) < \beta(i_1, j_2) \quad \text{and} \quad \beta(i_2, j_1) > \beta(i_2, j_2) \quad \text{hold.} \tag{5.6}$$

Theorem 5.15. *Let $\mathcal{A}$ be a separated contiguous integral deformation of the braid arrangement in V_{n-1} satisfying the weak triangle inequality. Then $\mathcal{A}$ has a valid associated weighted digraph containing a minimal m-ascending cycle of length 4, if and only if $\mathcal{A}$ has a crossing β-pattern.*

Proof. Assume first that $\mathcal{A}$ has a crossing β-pattern, that is, a set $\{i_1, i_2, j_2, j_3\}$ satisfying (5.6). Introducing $\beta = \beta(i_1, j_1)$, as a consequence of Theorem 5.10 and (5.6) we obtain $\beta(i_1, j_2) = \beta + 1$. Similarly, introducing $\beta' = \beta(i_2, j_1)$, Theorem 5.10 and (5.6) yield $\beta(i_1, j_2) = \beta' + 1$. Theorem 5.10 also implies $\beta(i_1, j_2) - \beta(i_2, j_2) \leq 1$, that is, $\beta + 1 - \beta' \leq 1$ and $\beta(i_2, j_1) - \beta(i_1, j_1) \leq 1$, that is, $\beta' + 1 - \beta \leq 1$. We obtain that $\beta - \beta'$ and $\beta' - \beta$ are both at most zero, hence β must equal β'. Consider the weighted digraph shown in Figure 2. The weights $w(i_1, j_1) = \beta(i_1, j_1)$, $w(i_2, j_2) = \beta(i_2, j_2)$ and $w(i_2, j_1) = \beta(i_2, j_1)$ are largest possible, hence there is no arrow $i_1 \leftarrow j_1$, $i_2 \leftarrow j_2$ or $i_2 \leftarrow j_1$. The weight $w(i_1, j_1) = \beta(i_1, j_1) - 1$ is less than $\beta(i_1, j_1$ hence there is also an arrow $i_1 \leftarrow j_1$ of weight $(j_1, i_1) = -\beta - 1$. The choices $w(i_1, i_2) = 0$ and $w(j_1, i_2) = 0$ are valid since the arrangement is separated, (dashed) the reverse arrows $i_1 \rightarrow i_2$, respectively $j_1 \rightarrow j_2$ are only present if $\beta(i_1, i_2) > 0$, respectively $\beta(j_1, j_2) > 0$ hold. The weight of each directed cycle of length 2 is (-1) (as it should), and the same holds for the 3-cycles (i_1, j_1, j_2) and (i_1, i_2, j_2). The 4-cycle (i_1, i_2, j_1, j_2) has weight zero. Just like in the proof of Theorem 5.10, we may select a linear order on $\{1, 2, \ldots, n\}$ in such a way that i_1, i_2, j_1, j_2 are the least elements, and for any pair of vertices $\{i', j'\}$ not contained in $\{i_1, i_2, j_1, j_2\}$ we set $w(i', j') = \beta(i', j')$ where i' is the smaller vertex in
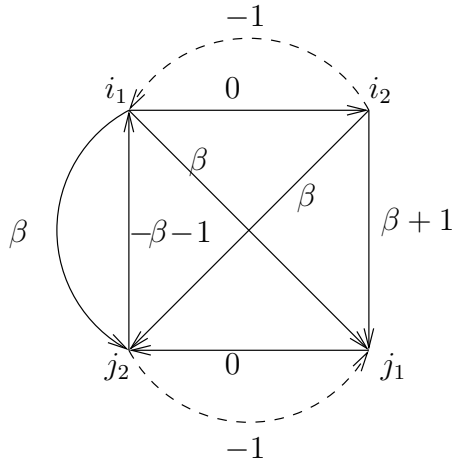
FIGURE 2. A valid weighted digraph with a minimal m-ascending 4-cycle

our order. The resulting weighted digraph is valid, and its only m-ascending cycle is (i_1, i_2, j_1, j_2).

Assume next that a valid weighted digraph associated to $\mathcal{A}$ has a minimal m-ascending cycle (i_0, i_1, i_2, i_3) of length four. We will adapt the proof of Theorem 5.13 to tackle this case. As before, we may use Lemma 5.3, and after a cyclic rotation we may assume that $i_0 \to i_2$ and $i_1 \to i_3$ are the diagonals of finite weight present. Using the fact that the directed cycles (i_0, i_2, i_3) and (i_1, i_3, i_0) are not m-acyclic, the same calculation yields

$$w(i_0, i_2) + w(i_1, i_3) \leq -2 + w(i_1, i_2) - w(i_3, i_0).$$

The left hand side is exactly $\beta(i_0, i_2) + \beta(i_1, i_3)$, as there is no arrow $i_0 \to i_2$ or $i_1 \to i_3$. The right hand side is at most $-2 + \beta(i_1, i_2) - \alpha(i_3, i_0) = -2 + \beta(i_1, i_2) + \beta(i_0, i_3)$. We obtain the inequality

$$\beta(i_0, i_2) + \beta(i_1, i_3) \leq -2 + \beta(i_1, i_2) + \beta(i_0, i_3). \tag{5.7}$$

By Theorem 5.10, we have $\beta(i_1, i_2) \leq \beta(i_1, i_3) + 1$ and $\beta(i_0, i_3) \leq \beta(i_0, i_2) + 1$. Using these as upper bounds for the terms on the right hand side of (5.7) we obtain

$$\beta(i_0, i_2) + \beta(i_1, i_3) \leq -2 + \beta(i_1, i_2) + \beta(i_0, i_3) \leq \beta(i_0, i_2) + \beta(i_1, i_3).$$

All inequalities must be equalities, in particular we get $\beta(i_1, i_2) = \beta(i_1, i_3) + 1$ and $\beta(i_0, i_3) = \beta(i_0, i_2) + 1$. As a consequence $\beta(i_1, i_2) > \beta(i_1, i_3)$ and $\beta(i_0, i_3) < \beta(i_0, i_2)$ hold, that is, we have a crossing β-pattern. $\square$

6. EXTENDED SHI ARRANGEMENTS

In this section we show how labeled weighted digraphs may be used to fill in the omitted details in Stanley's proof of the injectivity of the *Pak-Stanley labeling* of the regions of the extended Shi arrangement in [24, 2.1 Theorem]. We also revisit and generalize the Athanasiadis-Linusson labeling [2] of its regions.

By Corollary 5.9 the extended Shi arrangement is separated: for each region there is a unique permutation $\sigma(1)\sigma(2)\ldots\sigma(n)$ such that $w(\sigma(i), \sigma(j)) \geq 0$ holds for all $i < j$.

Equivalently, every region that is represented by a weighted digraph whose total order of gains is σ is a subset of the cone $x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n)}$. For $i <_{\sigma^{-1}} j$, there is always a directed edge $i \to j$, and there is no arrow $i \leftarrow j$ exactly when $w(i,j)$ is as large as possible, that is, $w(i,j) = \beta(i,j)$ where

$$\beta(i,j) = \begin{cases} a & \text{when } i < j, \\ a-1 & \text{when } i > j. \end{cases}$$

By Theorem 5.13 we may verify the m-acyclic condition in a valid associated weighted digraph by only checking the directed cycles of length three. Consider a set $\{i,j,k\} \subseteq \{1,2,\ldots,n\}$, without loss of generality we may assume $i <_{\sigma^{-1}} j <_{\sigma^{-1}} k$. There is no m-ascending cycle (i,j,k) if and only if either there is no arrow $i \leftarrow k$ or $w(i,j) + w(j,k) - 1 - w(i,k) \leq -1$ holds. This condition may be equivalently rewritten as

$$w(i,k) \geq \min(\beta(i,j), w(i,j) + w(j,k)) \quad \text{for } i <_{\sigma^{-1}} j <_{\sigma^{-1}} k. \tag{6.1}$$

There is no m-ascending cycle (i,k,j) exactly when either one of the arrows $k \to j$, $j \to i$ is missing, or we have $w(i,k) - 1 - w(i,j) - 1 - w(j,k) \leq -1$. The last inequality is equivalent to

$$w(i,k) \leq w(i,j) + w(j,k) + 1 \quad \text{for } i <_{\sigma^{-1}} j <_{\sigma^{-1}} k. \tag{6.2}$$

Note that (6.2) also holds when one of the arrows $k \to j$, $j \to i$ is missing, as $\beta(i,k) \leq \beta(i,j) + 1$ and $\beta(i,k) \leq \beta(j,k) + 1$ always hold, regardless of the relative order of the numbers i, j and k. To summarize, given a valid associated digraph satisfying $w(i,j) \geq 0$ for all $i <_{\sigma^{-1}} j$, the weighted digraph is m-acyclic if and only if (6.1) and (6.2) hold.

Recall that the *Pak-Stanley labeling* associates to each region R a vector $\lambda(R) = (f(1), f(2), \ldots, f(n))$ in the following way:

(1) The label of the central region $x_1 > x_2 > \cdots > x_n - 1$ is $(0, 0, \ldots, 0)$.
(2) Suppose R is an already labeled region, and the region R' separated from R by the unique hyperplane $x_i - x_j = m$ for some $i < j$. Then $\lambda(R') = \lambda(R) + e_j$ if $m \leq 0$ and $\lambda(R') = \lambda(R) + e_i$ if $m > 0$. Here $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$.

In [24] Stanley gives a detailed equivalent definition of the function $f(i)$ in the case when $a = 1$. We now extend this equivalent definition to all extended Shi arrangements.

Definition 6.1. *Consider a valid m-acyclic associated weighted digraph of $\mathcal{A}_{n-1}^{a,a+1}$ with weight function w. We define the* Pak-Stanley label *$(f(1), \ldots, f(n))$ of the corresponding region as*

$$f(i) = \sum_{i <_{\sigma^{-1}} j} w(i,j) + |\{(i,j) \ : \ i <_{\sigma^{-1}} j \text{ and } i > j\}|.$$

We let the reader verify that the two definitions are equivalent. For example, in the case when $i < j$ and $0 \leq w \leq a - 1$ hold, $w(i,j) = w$ is equivalent to stating $w < x_i - x_j < w + 1$, hence $w(i,j)$ is the number of hyperplanes of the form $x_i - x_j = m$ crossed while reaching our region from the central region. Following [24] we call the sum $\sum_{i <_{\sigma^{-1}} j} w(i,j)$ the number of *separations*, whereas $|\{(i,j) \ : \ i <_{\sigma^{-1}} j \text{ and } i > j\}|$ is the number of *inversions*. We do not need to prove the equivalence of the above

definitions, as we may easily show the following statement directly. Recall that an a-*parking function* is a sequence $(f(1), \ldots, f(n)) \in \mathbb{N}^n$ whose monotonic rearrangement $\tilde{f}(1) \leq \tilde{f}(2) \leq \cdots \leq \tilde{f}(n)$ satisfies $0 \leq \tilde{f}(i) \leq a(i-1)$.

Lemma 6.2. *The Pak-Stanley labels $(f(1), \ldots, f(n))$ are a-parking functions.*

Proof. This is a direct consequence of the fact that, for each $i <_{\sigma^{-1}} j$, the number $w(i, j)$ is at most a if (i, j) is not an inversion and at most $a - 1$ if (i, j) is an inversion. Hence the total number of inversions and separations contributing to $f(i)$ is at most a times the number of labels j succeeding i in σ. $\square$

Next we rephrase Stanley's key lemma [24, p. 363] to match our terminology.

Lemma 6.3. *Given $i <_{\sigma^{-1}} j$, if $i > j$ or $w(i, j) > 0$ holds then we have $f(i) > f(j)$.*

Proof. Assume first $i > j$ holds, i.e., (i, j) is an inversion. Then for each inversion (j, k) the pair (i, k) is also inversion, hence we may identify the set of all inversions of j with those inversions (i, k) of i which satisfy the stronger inequality $j > k$. Regarding separations, note that (6.1) implies $w(i, k) \geq w(j, k)$ in almost all cases: either we have $w(i, k) \geq w(i, j) + w(j, k) \geq w(j, k)$ or we have $w(i, k) \geq \beta(i, j)$ forcing $w(i, k) \geq a - 1$ and $w(j, k) \leq a$. The inequality $w(i, k) < w(j, k)$ can only hold when we have $j < k < i$, $w(j, k) = a$ and $w(i, k) = a - 1$. In this exceptional case $w(i, k)$ contributes one less to the separations of i than $w(j, k)$ to the separations of j, but this lag is offset by the presence of the inversion (i, k) which has no corresponding inversion (j, k). Hence we have $f(i) \geq f(j)$, and the presence of the additional inversion (i, j) (contributing to $f(i)$ only) makes the inequality strict.

Assume next $w(i, j) > 0$ holds. Once again, we may use (6.1) to state $w(i, k) \geq w(j, k)$ almost always. In the exceptional case $j < k < i$, $w(j, k) = a$ and $w(i, k) = a - 1$ must hold, and (i, k) is an additional inversion. Regarding an inversion (j, k), we may match it to (i, k) if (i, k) is an inversion, the unmatched inversions (j, k) satisfy $j < k < i$. For these (6.1) implies

$$w(i, k) \geq \min(a, w(i, j) + w(j, k)) > w(j, k),$$

as $w(j, k) \leq a - 1$ and $w(i, j) > 0$ hold, hence $w(i, k)$ contributes more separations to $f(i)$ than $w(j, k)$ to $f(j)$. Therefore we have $f(i) \geq f(j)$, and the presence of the additional separations contributed by $w(i, j) > 0$ to $f(i)$ makes the inequality strict. $\square$

Now we are ready to fill in the omitted details in the proof of [24, 2.1 Theorem].

Theorem 6.4 (Stanley). *The labels of the regions of the extended Shi arrangement are the a-parking functions of length n, each occurring exactly once.*

Proof. The fact that the labels are a-parking functions have been shown in Lemma 6.2 above. As in the proof of [24, 2.1 Theorem], we will only show the injectivity of the labeling, and then we will rely on existing results in the literature to confirm that the number of regions equals the number of a-parking functions.

Given an a-parking function $(f(1), \ldots, f(n))$, we insert the labels i into σ one by one and show the uniqueness of the place and of the function values $w(i, j)$ one step at a time. As in the proof of [24, 2.1 Theorem], we insert the labels i in increasing order of the value $f(i)$, and we insert labels with the same f-value in decreasing order of their

numerical value. Suppose i is the most recently inserted label. If this label is preceded by any label i' in σ, then by Lemma 6.3 we must have $w(i', i) = 0$ and $i' < i$, otherwise we get $f(i') > f(i)$ in contradiction with our defined order of insertion. Note that we must also have $f(i') = f(i)$ in this case. We also obtained that the label i can not contribute any inversion or separation to any preceding i'. This guarantees that at the insertion of i we only need to make sure that the instances of (6.1) and (6.2) involving i are satisfied and that the values $w(i, j)$ for all previously inserted j succeeding i in σ are consistently defined.

First we show that i can not be inserted at two different places into σ. Assume that the word representing the currently inserted labels is $\widetilde{\sigma} = i_1 \cdots i_m$ and i could be inserted either right after i_r or right after i_s for some $r < s$ (we set $r = 0$ if i is inserted as the first letter). Let w_r respectively w_s be a weight function arising with the insertion of i right after i_r, respectively i_s. As in the proof of Lemma 6.3, (6.1) implies $w_r(i, k) \geq w_r(i_s, k)$ for almost all k satisfying $i_s <_{\sigma^{-1}} k$, the conclusion $w_r(i, k) \geq w_s(i, k)$ fails to hold only when we have $w_r(i_s, k) = a$, $w_r(i, k) = a - 1$ and $i_s < k < i$. On the other hand, (6.1) implies

$$w_s(i_s, k) \geq w_s(i_s, i) + w_s(i, k) \geq w_s(i, k)$$

for all k satisfying $i_s <_{\sigma^{-1}} k$: the exceptional case $w_s(i_s, k) = a - 1$, $w_s(i, k) = a$, $i < k < i_s$ cannot occur, as it would create the inversion (i_s, i), in contradiction with the possibility of inserting i right after i_s. Combining the last two observations and using the fact that $w_r(i_s, k) = w_s(i_s, k)$ (it is the same weight function created before the insertion of i), we obtain

$$w_r(i, k) \geq w_s(i, k) \tag{6.3}$$

for almost all k satisfying $i_s <_{\sigma^{-1}} k$. We only need to check the case when $w_r(i_s, k) = a$, $w_r(i, k) = a - 1$ and $i_s < k < i$ hold. In this exceptional case $i > k$ implies that the maximum value of $w_s(i, k)$ is $a - 1 = w_r(i, k)$, hence (6.3) holds in this exceptional case as well. As a consequence the total number of separations of i can not be greater when we insert it right after i_s than in the case when we insert it right after i_r. The same holds for the number of inversions, finally the inversion (i, i_s) is only present when we insert i right after i_r. In conclusion we can not obtain the same value $f(i)$ in both scenarios.

We are left to show that inserting i at the same place can not be continued by extending the weight function in two different ways. By Lemma 6.3 we must set $w(i', i) = 0$ for all previously inserted $i' <_{\sigma^{-1}} i$, variation may only occur in the definition of $w(i, j)$ for some $i <_{\sigma^{-1}} j$. Assume there are two different extensions: w_1 and w_2 of the weight function. Then there is a leftmost j in the order $<_{\sigma^{-1}}$ for which $w_1(i, j) \neq w_2(i, j)$, without loss of generality we may assume $w_1(i, j) < w_2(i, j)$. By definition, for all k satisfying $i <_{\sigma^{-1}} k <_{\sigma^{-1}} j$ we have $w_1(i, k) = w_2(i, k)$, these k's contribute the same separation to $f(i)$ in either case. On the other hand, for all k satisfying $j <_{\sigma^{-1}} k$, the inequality (6.1) implies

$$w_2(i, k) \geq \min(\beta(i, k), w_2(i, j) + w_2(j, k)).$$

Using $w_2(i, j) \geq w_1(i, j) + 1$ and $w_2(j, k) = w_1(j, k)$ (since j and k were inserted before i) we may rewrite this as

$$w_2(i, k) \geq \min(\beta(i, k), w_1(i, j) + w_1(j, k) + 1)$$

The inequality (6.2) implies

$$w_2(i, k) \geq \min(\beta(i, k), w_1(i, k)).$$

Since $\beta(i, k)$ is the maximum value of $w_1(i, k)$, we obtain $w_2(i, k) \geq w_1(i, k)$ for all k satisfying $j <_{\sigma^{-1}} k$. Keeping in mind $w_1(i, j) < w_2(i, j)$, we obtain a strictly larger separation in the computation of $f(i)$ when we use w_2 instead of w_1. The set of inversions being the same, we can not obtain the same $f(i)$ using either extension of the weight function. $\qquad\square$

Remark 6.5. Mazin [18] has shown that the Pak-Stanley labeling of the regions of the extended Shi arrangement is surjective. Together with Stanley's above result we have a self-contained proof of the fact that the Pak-Stanley labeling is a bijection between the regions of the regions of the extended Shi arrangement and the a-parking functions.

We conclude this section by revisiting and generalizing the construction of Athanasiadis and Linusson [2].

Definition 6.6. *We say that the regions of a contiguous, separated and integral deformation of the braid arrangement in V_{n-1} given by the equations*

$$x_i - x_j = m \quad m \in [-\beta(j, i), \beta(i, j)] \quad \text{for } 1 \leq i < j < n$$

have Athanasiadis-Linusson diagrams if for each $j \in \{1, 2, \ldots, n\}$ the set $\{\beta(i, j) : i \neq j\}$ has at most two elements and these elements are consecutive nonnegative integers. We set $\beta(j) = \min_{i \neq j} \beta(i, j)$ for all j.

Note that for the extended Shi arrangement $\mathcal{A}_{n-1}^{a,a+1}$ we have $\beta(1) = \cdots = \beta(n) = a$. More generally, as a consequence of Theorem 5.10, the regions of every contiguous, separated and integral deformation of the braid arrangement satisfying the weak triangle inequality have Athanasiadis-Linusson diagrams. When a contiguous, separated and integral deformation of the braid arrangement in V_{n-1} has Athanasiadis-Linusson diagrams, we define one for each of its regions essentially the same way as it is done in [2]:

(1) We fix a representative $\underline{x} = (x_1, x_2, \ldots, x_n)$ of the region. These appear on the reversed number line in the order $x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n)}$. (The arrangement being separated forces σ to be independent from the choice of $\underline{x}$.)

(2) For each $j \in \{1, 2, \ldots, n\}$ satisfying $\beta(j) > 0$ we also mark $x_j + \beta(j), x_j + \beta(j) - 1, \ldots, x_j + 1$ on the reversed number line and we draw an arc connecting $x_j + k + 1$ with $x_j + k$ for $k = 0, 1, \ldots, \beta(j) - 1$. We label all of these points with j

(3) For each $\{i, j\} \subseteq \{1, 2, \ldots, n\}$ we also draw an arc between x_i and $x_j + \beta(j)$ if $\beta(i, j) = \beta(j) + 1$ $x_i - x_j > \beta(i, j)$ holds.

(4) We remove all nested arcs, that is, all arcs that contain another arc.

In terms of weighted digraphs, the permutation σ is the total order of gains, and in step (3) we add an arc between x_i and $x_j + \beta(j)$ exactly when $w(i, j) = \beta(i, j) = \beta(j) + 1$ holds. In such a case $w(i, j) > 0$ implies $i <_{\sigma^{-1}} j$ regardless of the choice of $\underline{x}$. In all other cases, assuming $i <_{\sigma^{-1}} j$, we have $w(i, j) = w$ if and only if the point representing x_i is to the left of the point $x_i + w$, but not to the left of $x_i + w + 1$ (if $w < \beta(i, j)$ and there is a point representing $x_i + w + 1$. Since the valid m-acyclic weighted

digraphs bijectively represent the regions of our hyperplane arrangement and they can be uniquely reconstructed from the associated Athanasiadis-Linusson diagrams, there is always a bijection between the regions and their Athanasiadis-Linusson diagrams. It seems hard, however, to characterize which diagrams may occur.

Example 6.7. The diagram shown in Figure 3 is obtained from [2, Fig. 6] by adding a single point 5 and an arc between the rightmost copy of 3 and this newly added point. Without this addition, the original diagram represents a region in $\mathcal{A}_3^{1,2}$, we have
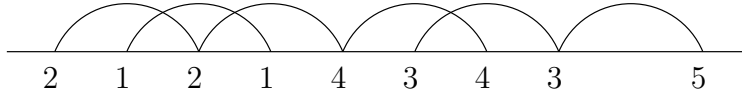

FIGURE 3. An Athanasiadis-Linusson diagram

$\beta(i,j) = 2$ for $i < j$ and $\beta(i,j) = 1$ for $i > j$ for all pairs $\{i,j\} \subset \{1,2,3,4\}$. We add $\beta(i,5) = \beta(5,i) = 0$ for $i = 1,2,4$, and we add $\beta(3,5) = 1$ and $\beta(3,5) = 0$.

As in [2], for each $i \in \{1,2,\ldots,n\}$ we define $f(i)$ as the position of the leftmost element of the continuous component of i. We call the resulting $(f(1), f(2), \ldots, f(n))$ the β-parking function of the region. For example, in Example 6.7 we have $f(1) = 2$, $f(2) = f(4) = 1$ and $f(3) = f(5) = 6$. The proof of Athanasiadis and Linusson showing that we may reconstruct the diagram from its β-parking function still applies: we insert the components in increasing order of the position of the left end of their components and we interlace the rest of the inserted component with the already inserted components in such a way that no pair of nested arcs is formed. It seems hard to characterize the resulting β-functions in general, but they can be easily visualized, using an idea implicit in [21, Ch1. Exercise (32)(c)].
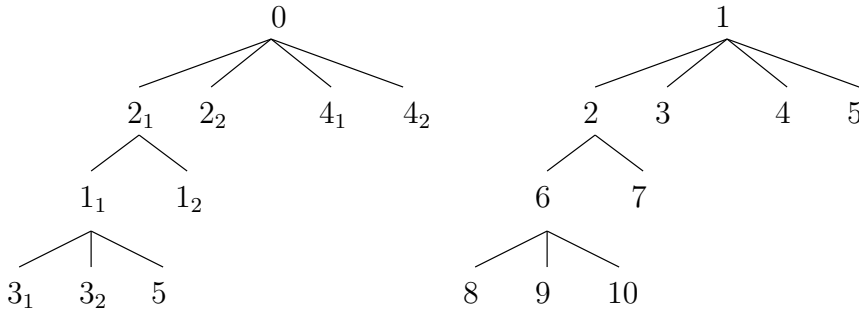

FIGURE 4. A rooted tree encoding an Athanasiadis-Linusson diagram

Definition 6.8. *Given an Athanasiadis-Linusson diagram, we define the* parking tree *representing it as follows:*
  (1) *Replace the labels j with j_1, j_2,..., $j_{\beta(j)+1}$, numbered left to right, so that we can distinguish the copies.*
  (2) *The copies of the labels satisfying $f(j) = 1$ become the children of the root 0.*
  (3) *We number the nodes in the tree level-by-level and in increasing order of the labels (breadth-first-search order).*

(4) *Once we inserted the copies of all labels j satisfying $f(j) < i$, all copies of the labels j satisfying $f(j) = i$ will be the children of the node whose number is i.*

The parking tree associated to the Athanasiadis-Linusson diagram shown in Figure 3 is the tree on the left hand side in Figure 4. The tree on the right hand side shows the numbering of the positions in a breadth-first-search order: we call this the *position numbering*. The description of such a parking tree depends on the characterization of the set of labels which can be siblings. This seems hard in general, but easy in the following special case.

Definition 6.9. *For a sequence $\underline{\beta} = (\beta(1), \beta(2), \ldots, \beta(n)) \in \mathbb{N}^n$ we define the $\underline{\beta}$-extended Shi arrangement as the hyperplane arrangement*

$$x_i - x_j = -\beta(j), -\beta(j) + 1, \ldots, \beta(j) + 1 \quad 1 \le i < j \le n \quad in \ V_{n-1}. \qquad (6.4)$$

In particular, setting $\beta(1) = \cdots = \beta(n) = a - 1$ yields the extended Shi arrangement $\mathcal{A}_{n-1}^{a,a+1}$.

Theorem 6.10. *The number of regions in a $\underline{\beta}$-extended Shi arrangement $\mathcal{A}$ is*

$$r(\mathcal{A}) = \left(\sum_{j=1}^{n} (\beta(j) + 1) + 1 \right)^{n-1}.$$

Proof. Obviously, the regions of a $\underline{\beta}$-extended Shi arrangement have Athanasiadis-Linusson diagrams. When we build these diagrams, $\beta(i,j) = \beta(j) + 1$ holds exactly when $i < j$, hence we may draw an arc between the points representing x_i and $x_j + \beta(j)$ if and only if $i < j$ and $x_i - x_j > \beta(j) + 1$ hold, and these are all the potential arcs added in step (3). As a consequence, a set of labels can be a set of siblings exactly when for each j it contains either all copies of j, or neither of them. We may count all such rooted trees using a colored variant of the Prüfer code algorithm as follows. We consider the elements of $\{1, 2, \ldots, n\}$ as colors, and we say that a color j is *exposed* if all labels j_i are leaves in the tree. In each step we remove the vertices of the least exposed color and record the common parent $p(j_i)$ in our Prüfer code. We never remove the root 0: we consider it the highest numbered vertex. We never remove $p(j_i)$ before all copies of j_i as our algorithm removes leaves only in each step, hence the unique path between any not yet removed vertex and the root must still be present. For example, for the tree on the left hand side of Figure 4, the least exposed color is 3. The vertex 1_2 is also a leaf, but 1_1 is not, hence the color 1 is not yet exposed. There is always at least one exposed color, because a leaf whose distance from the root is maximum has only leaf siblings. We stop when all remaining non-root vertices have the same color. For the tree on the left hand side of Figure 4, we remove the colors $3, 4, 5, 1$ in this order and we record the Prüfer code $(1_1, 0, 1_1, 2_1)$. The tree can be uniquely reconstructed from its colored Prüfer code, as follows. The least nonzero color not present in the code is 3: this must be the first removed color, and the common parent of all vertices of color 3 is the first coordinate of the code, that is, 1_1. We remove the first coordinate from the Prüfer code and record 3 as a color in a separate list of already reinserted vertices: we obtain the pair of lists $((0, 1_1, 2_1), (3))$. The least nonzero color not present in the current pair of lists is 4, hence the vertices of color 4 have parent 0, and we get the pair

of lists $((1_1, 2_1), (3, 4))$. Now the least nonzero color not present in our pair of lists is 5 and its parent is 1_1. We continue the reconstruction of our tree in a similar fashion. Our colored Prüfer code has $n - 1$ coordinates and each coordinate can be either the root or any of the labeled vertices. ☐

7. The a-Catalan arrangements

In this section we present two labelings of the regions of the a-Catalan arrangement $\mathcal{A}_{n-1}^{a,a}$ for $a \geq 1$: one using the Athanasiadis-Linusson diagrams (defined in Section 6), the other provides a simple direct definition of the weighted digraphs using labeled a-Catalan paths. These labelings are different from each other and also from the labeling of Duarte and Guedes de Oliveira [8] which encode the Pak-Stanley labeling using labeled rational Dyck paths. It seems an interesting question for future research to find direct bijections between these labelings.

The structure of the Athanasiadis-Linusson diagrams is the simplest possible for the a-Catalan arrangement $\mathcal{A}_{n-1}^{a,a}$ for $a \geq 1$. In this case $\beta(i, j) = a - 1 = \beta(j)$ holds for all $i \neq j$ and we may omit step (3) in constructing the diagrams. Each continuous
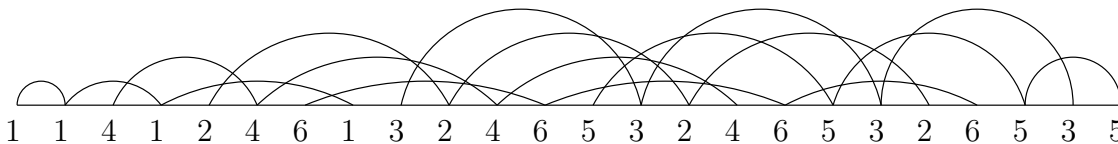

FIGURE 5. Athanasiadis-Linusson diagram of a region in $\mathcal{A}_5^{4,4}$

component consist of all a copies of a single $j \in \{1, 2, \ldots, n\}$ as shown in Figure 5. In this figure we have $x_1 > x_4 > x_2 > x_6 > x_3 > x_5$, but we can freely permute the labels as these components can never be "glued" together. By the same reason, the diagram may be reconstructed from the *Athanasiadis-Linusson word* 11412461324653246532653 of the labels in the diagram. (It is helpful but not necessary to underline the last appearance of each label.) After fixing the order $x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n)}$, the parking trees associated to these diagrams are in bijection with the rooted incomplete a-ary trees on $(a-1)n+1$ vertices. Their number is the a-Catalan number $\frac{1}{(a-1)n+1}\binom{an}{n}$. Multiplying it with $n!$, which is the number of possible choices of σ, we get

$$r(\mathcal{A}_{n-1}^{a,a}) = an(an - 1) \cdots ((a - 1)n + 2), \tag{7.1}$$

which is the same as [19, Corollary 9.2].

We may also represent the regions of the a-Catalan arrangement $\mathcal{A}_{n-1}^{a,a}$ using all pairs (π, Λ), where π is a permutation of the set $\{1, 2, \ldots, n\}$ and Λ is an a-*Catalan path* with n up steps.

Definition 7.1. *An a-Catalan path with n up steps is a lattice path containing n up steps $(1, (a - 1))$ and $(a - 1)n$ down steps $(1, -1)$ from $(0, 0)$ to $(an, 0)$ that stays weakly above the horizontal axis.*

It is well-known that the number of a-Catalan paths with n up steps is the a-Catalan number $\frac{1}{(a-1)n+1}\binom{an}{n}$ [13, Eq. (7.67)]. This representation is easier to visualize but somewhat mysterious at this time. We use the permutation π to number the up steps

of Λ from left to right. In the example shown in Figure 6 the permutation π is simply 123456, but we may reuse the same lattice path with any other permutation. As it was also the case for the Athanasiadis-Linusson diagrams, the set of valid representations is invariant under permuting the labels.
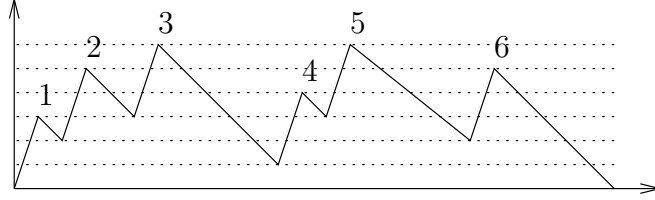

FIGURE 6. a-Catalan path corresponding to the Athanasiadis-Linusson diagram shown in Figure 5

We identify the ith up step from the right with the variable $x_{\pi(i)}$ and denote the level of the lower end of the up step with $\ell(\pi(i))$. We use this *level function* $\ell : \{1, 2, \ldots, n\} \to \mathbb{N}$ to define the weight function, as follows.

$$w(\pi(i), \pi(j)) = \begin{cases} \ell(\pi(j)) - \ell(\pi(i)) & \text{if } \ell(\pi(j)) - \ell(\pi(i)) \in [1 - a, a - 1] \\ -\infty & \text{if } \ell(\pi(j)) - \ell(\pi(i)) < 1 - a \\ a - 1 & \text{if } \ell(\pi(j)) - \ell(\pi(i)) > a - 1 \end{cases} \tag{7.2}$$

Recall that the notation $w(\pi(i), \pi(j)) = -\infty$ indicates the case when there is no arrow $\pi(i) \to \pi(j)$ in the corresponding weighted digraph, only an arrow $\pi(i) \leftarrow \pi(j)$ of weight $a - 1$. Equation (7.2) defines exactly one of $w(i, j)$ and $w(j, i)$ for each pair $\{i, j\}$, we may uniquely extend this definition to all ordered pairs using the rule

$$w(i, j) = \begin{cases} -1 - w(j, i) & \text{if } w(j, i) \in [1 - a, a - 2] \\ -\infty & \text{if } w(j, i) = a - 1 \\ a - 1 & \text{if } w(j, i) = -\infty \end{cases} \tag{7.3}$$

Clearly we obtain the weight function of a valid weighted digraph.

Proposition 7.2. *Equations* (7.2) *and* (7.3) *define an m-acyclic weighted digraph.*

Proof. Consider a directed cycle $(\pi(i_1), \pi(i, 2), \ldots, \pi(i_k)))$. In the proof we add the indices modulo k, that is we set $i_{k+1} = i_1$. For $i_s < i_{s+1}$ we have $w(\pi(i_s), \pi(i_{s+1}) \le \ell(\pi(i_{s+1})) - \ell(\pi(i_{s+1}))$. For $i_s > i_{s+1}$ we must have $w(\pi(i_{s+1}), \pi(i_s)) < a - 1$ otherwise $w(\pi(i_s), \pi(i_{s+1})) = -\infty$ holds. Hence we have

$$w(\pi(i_s), \pi(i_{s+1}) = -1 - w(\pi(i_{s+1}), \pi(i_s)) = -1 - (\ell(\pi(i_s)) - \ell(\pi(i_{s+1}))$$
$$< \ell(\pi(i_{s+1}) - \ell(\pi(i_s).$$

To summarize $w(\pi(i_s), \pi(i_{s+1}) \le \ell(\pi(i_{s+1}) - \ell(\pi(i_s)$ always holds, hence the total weight of the cycle is at most $\sum_{j=1}^{k} (\ell(\pi(i_{j+1})) - \ell(\pi(i_j))) = 0$. The sum is strictly negative because there is at least one s such that $i_s > i_{s+1}$ holds. $\square$

Definition 7.3. *We call the m-acyclic valid weighted digraph associated to (π, Λ) by the Equations* (7.2) *and* (7.3) *the* weighted digraph encoded by (π, Λ).

Lemma 7.4. *Consider weighted digraph encoded by (π, Λ). The total order of gains $\sigma = \gamma \circ \pi$ is the order of the labels $\pi(1), \ldots, \pi(n)$ in increasing order of their levels, where $\pi(i)$ is listed before $\pi(j)$ if $\ell(\pi(i)) = \ell(\pi(j))$ and $i < j$ hold.*

For example, for the labeled lattice path shown in Figure 6 we get $\sigma = 142635$. It is an immediate consequence of Equation (7.2) that $w(\sigma(i), \sigma(j)) \geq 0$ holds for all $i < j$, and there is exactly one permutation with this property.

Proposition 7.5. *For the weighted digraph encoded by (π, Λ) the gain function is the level function: we have $g(\sigma(i)) = \ell(\sigma(i))$.*

Proof. We proceed by induction on i. For $i = 1$ we have $g(\sigma(1)) = \ell(\sigma(1)) = 0$. Assume that $g(\sigma(i)) = \ell(\sigma(i))$ holds for all $i < j$ and let us compute $g(\sigma(j))$. By Theorem 4.10 there is a $j' < j$ satisfying $g(\sigma(j)) = g(\sigma(j')) + w(\sigma(j'), \sigma(j))$. By the induction hypothesis $g(\sigma(j')) = \ell(\sigma(j'))$ holds, hence we have

$$g(\sigma(j)) = \ell(\sigma(j')) + w(\sigma(j'), \sigma(j)) \leq \ell(\sigma(j')) + \ell(\sigma(j)) - \ell(\sigma(j')) = \ell(\sigma(j)).$$

On the other hand, in an a-Catalan path, between two consecutive up steps the level increases by at most $a - 1$ (it may also decrease all the way to zero). If we project all up steps to the horizontal axis, we obtain a set of intervals whose union is an interval of integers. Therefore, between any two up steps there is such a sequence of up steps that the level function can increase by at most $a - 1$ between two subsequent up steps $(1, a - 1)$, hence there is at least one up step labeled with $\sigma(j'')$ preceding the up step labeled with $\sigma(j)$ whose level is in the interval $[-a+1+\ell(\sigma(j)), \ell(\sigma(j))]$. By Lemma 7.4 $j'' < j$ holds, hence by Equation (7.2) we have $w(\sigma(j''), \sigma(j)) = \ell(\sigma(j)) - \ell(\sigma(j''))$. Combining this with the induction hypothesis we also obtain

$$g(\sigma(j)) \geq g(\sigma(j'')) + w(\sigma(j''), \sigma(j)) = \ell(\sigma(j'')) + \ell(\sigma(j)) - \ell(\sigma(j'')) = \ell(\sigma(j)).$$

$\square$

Theorem 7.6. *The correspondence between the pairs (π, Λ) and the valid weighted m-acyclic digraphs encoded by them is a bijection.*

Proof. It suffices to show that the correspondence is injective: by (7.1) the number of regions (and of valid m-acyclic weighted digraphs) is the same as the number of labeled a-Catalan paths. Consider a labeled a-Catalan path (π, Λ) and the weighted digraph (described by the weight function $w(i, j)$) it encodes. By the results of Section 4, we can compute the total order of gains $\sigma(i)$ and the gain function $g(i)$ from the function $w(i, j)$. By Lemma 7.4 and Proposition 7.5 the gain function is the same as the level function and σ lists the up steps level by level, and left to right within the same level. Using this information we can determine the level of each labeled up step, and also the relative order of two up steps at the same level. Consider finally a pair of up steps, labeled with i and j, respectively, whose level is different. If $|\ell(i) - \ell(j)| \leq a - 1$ holds then, by Equations (7.2) and (7.3), $w(i, j) = \ell(j) - \ell(i)$ holds exactly when the up step labeled i precedes the up step labeled j in Λ. As noted in the proof of Proposition 7.5 the difference of levels between any two subsequent up steps in the sequence is at most $a - 1$. As a consequence, we can determine the relative position of any two labeled up steps whose level is different. $\square$

It is not hard to create the Athanasiadis-Linusson diagram of a region given by its labeled a-Catalan path (π, Λ): the weight function is easily computed, the linear order of gains σ can be read off the labeled lattice path, and a table such as Table 1 is easily created. Using the table of the weight function, we can insert the letters into the Athanasiadis-Linusson word in the order of σ. Finding the inverse of this map appears to be a much more difficult task.

$w(\sigma(i), \sigma(i+5))$				3		
$w(\sigma(i), \sigma(i+4))$			3	2		
$w(\sigma(i), \sigma(i+3))$		2	1	1		
$w(\sigma(i), \sigma(i+2))$		2	1	1	0	
$w(\sigma(i), \sigma(i+1))$	1	0	0	0	0	
vertices $\sigma(1)\sigma(2)\cdots\sigma(6)$	1	4	2	6	3	5
$g(\sigma(i))$	0	1	2	2	3	3

TABLE 1. Weight function table associated to the Athanasiadis-Linusson diagram shown in Figure 5

We conclude this section with a description of the bounded regions.

Proposition 7.7. *A region of $\mathcal{A}_{n-1}^{a,a}$ is bounded if and only if the total order of gains σ satisfies $w(\sigma(i), \sigma(i+1)) < a - 1$ for $1 \le i \le n - 1$.*

Proof. By definition we have $w(\sigma(i), \sigma(i+1)) \ge 0$ for $1 \le i \le n-1$. If $w(\sigma(i), \sigma(i+1)) < a - 1$ for $1 \le i \le n - 1$, then the directed edges $\sigma(i) \to \sigma(i+1)$ and $\sigma(i) \leftarrow \sigma(i+1)$ are both present, and the associated weighted digraph is strongly connected. Assume now that $w(\sigma(i_0), \sigma(i_0 + 1)) = a - 1$ holds for some $1 \le_0 \le n - 1$. In analogy to the case of the extended Shi arrangement the fact that there is no m-ascending cycle implies (6.1), that is,

$$w(i_1, i_3) \ge \min(a - 1), w(i_{1,2}) + w(i_2, i_3)) \quad \text{holds for } i_1 <_{\sigma^{-1}} i_2 <_{\sigma^{-1}} i_3.$$

Using this inequality it is easy to show that $w(i_1, i_2) = a - 1$ holds for $i_1 \le_{\sigma^{-1}} i <_{\sigma^{-1}} i_2$. In other words, there is no directed edge from the set $\{i_2\ i <_{\sigma^{-1}} i\}$ into the set $\{i_1\ i_1 \le <_{\sigma^{-1}} i\}$. The associated weighted digraph is not strongly connected. $\square$

Using Lemma 7.4, Proposition 7.7 may be rephrased as follows.

Corollary 7.8. *A region of $\mathcal{A}_{n-1}^{a,a}$, encoded by the labeled lattice path (π, Λ), is unbounded if and only there are two successive levels ℓ and $\ell + a - 1$ whose difference is $a - 1$ and the last up step of Λ at level ℓ precedes the first up step at level $\ell + a - 1$.*

As noted in Corollary 4.14 the number of possible types of the trees of the gain function is a Catalan number.

Conjecture 7.9. *For a fixed n and a fixed tree of gain functions, the number of regions of $\mathcal{A}_{n-1}^{a,a}$ associated to it is a polynomial of a.*

Conjecture 7.9 implies that the n-th a-Catalan number, considered as a polynomial of a, could be written as a sum of C_n polynomials, where C_n is the n-th Catalan number.

References

[1] C. A. Athanasiadis, Characteristic polynomials of subspace arrangements and finite fields, *Adv. Math.* **122** (1996), 193–233.

[2] C. A. Athanasiadis and S. Linusson, A simple bijection for the regions of the Shi arrangement of hyperplanes, *Discrete Math.* **204** (1999), 27–39.

[3] M. Beck, A. Berrizbeitia, M. Dairyko, C. Rodriguez, A. Ruiz, S. Veeneman, Parking functions, Shi arrangements, and mixed graphs, *Amer. Math. Monthly* **122** (2015), 660–673.

[4] O. Bernardi, Deformations of the braid arrangement and trees, *Adv. Math.* **335** (2018), 466–518.

[5] P. Berthomé, R. Cordovil, D. Forge, V. Ventos and T. Zaslavsky, An elementary chromatic reduction for gain graphs and special hyperplane arrangements, *Electron. J. Combin.* **16** (2009), no.1, Research Paper 121, 31 pp.

[6] W. B. Carver, Systems of linear inequalities. *Ann. of Math.* **(2) 23** (1922), 212–220.

[7] B. Drake, An Inversion Theorem for Labeled Trees and Some Limits of Areas under Lattice Paths, PhD thesis, Brandeis University, 2008.

[8] R. Duarte, A. Guedes de Oliveira, Pak-Stanley labeling of the m-Catalan hyperplane arrangement, *Adv. Math.* **387** (2021), Paper No. 107827, 17 pp.

[9] R. Flórez and D. Forge, Activity from matroids to rooted trees and beyond, *J. Combin. Theory Ser. A* **198** (2023), Paper No. 105755, 21 pp.

[10] D. Forge and T. Zaslavsky, Lattice point counts for the Shi arrangement and other affinographic hyperplane arrangements, *J. Combin. Theory Ser. A* **114** (2007), 97–109.

[11] T. Gallai, Maximum-minimum Sätze über Graphen, *Acta Math. Acad. Sci. Hungar.* **9** (1958), 395–434.

[12] I. M. Gessel, S. T. Griffin and V. Tewari, Labeled binary trees, subarrangements of the Catalan arrangements, and Schur positivity, *Adv. Math.* **356** (2019), 106814, 67 pp.

[13] R. Graham, D. E. Knuth and O. Patashnik, "Concrete mathematics. A foundation for computer science, Second Edition," Addison–Wesley Publishing Company, Reading, MA, 1994.

[14] G. Hetyei, Alternation acyclic tournaments, *European J. Combin.* **81** (2019), 1–21.

[15] S. Hopkins and D. Perkinson, Bigraphical arrangements, *Trans. Amer. Math. Soc.* **368** (2016), 709–725.

[16] L. H. Kalikow, Symmetries in trees and parking functions, LaCIM 2000 Conference on Combinatorics, Computer Science and Applications (Montreal, QC) *Discrete Math.* **256** (2002), 719–741.

[17] B, Korte and J. Vygen, Combinatorial optimization, Theory and algorithms, Sixth edition *Algorithms Combin.* , **21** Springer, Berlin, 2018. xxi+698 pp.

[18] M. Mazin, Multigraph hyperplane arrangements and parking functions, *Ann. Comb.* **21** (2017), 653–661.

[19] A. Postnikov and R. P. Stanley, Deformations of Coxeter hyperplane arrangements, *J. Combin. Theory Ser. A* **91** (2000), 544–97.

[20] K. Roos, Linear Optimization: Theorems of the Alternative, In: Floudas, C., Pardalos, P. (eds) Encyclopedia of Optimization. Springer, Boston, MA. `https://doi.org/10.1007/978-0-387-74759-0_334`

[21] B. E. Sagan, Combinatorics: the art of counting, Grad. Stud. Math., **210** American Mathematical Society, Providence, RI, [2020], ©2020. xix+304 pp.

[22] D. Scott and P. Suppes, Foundational aspects of theories of measurement, *J. Symbolic Logic* **23** (1958), 113–128.

[23] R. P. Stanley, Hyperplane arrangements, interval orders, and trees, *Proc. Nat. Acad. Sci. U.S.A.* **93** (1996), 2620–2625.

[24] R. P. Stanley, Hyperplane arrangements, parking functions and tree inversions, in: Mathematical essays in honor of Gian-Carlo Rota (Cambridge, MA, 1996), 359–375. **Progr. Math., 161** Birkhäuser Boston, Inc., Boston, MA, 1998

[25] R. P. Stanley, An introduction to hyperplane arrangements, Geometric combinatorics, 389–496. IAS/Park City Math. Ser., **13**, American Mathematical Society, Providence, RI, 2007

[26] R. P. Stanley, Enumerative combinatorics, Volume 1, Second edition, Cambridge Stud. Adv. Math., **49** Cambridge University Press, Cambridge, 2012. xiv+626 pp.

[27] V. Tewari, Gessel polynomials, rooks, and extended Linial arrangements, *J. Combin. Theory Ser. A* **163** (2019), 98–117.

[28] H. Whitney, A logical expansion in mathematics, *Bull. Amer. Math. Soc.* **38**(1932), 572–579.

[29] T. Zaslavsky, Facing up to arrangements: face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.* **1** (1975), issue 1, no. 154, vii+102 pp.

[30] T. Zaslavsky, Perpendicular dissections of space, *Discrete Comput. Geom.* **27** (2002), 303–351.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNC CHARLOTTE, CHARLOTTE NC 28223-0001. WWW: `http://webpages.uncc.edu/ghetyei/`.