1. [10 points] Let X be a random variable on the probability space $(\Omega, \mathcal{F}, P)$ and let $Y = |X|$. Assume that X has a standard normal distribution, i.e. $X \sim N(0,1)$.

   (a) We can write $\sigma(Y) = \{X^{-1}(B) : B \in \mathcal{A}\}$, where $\mathcal{A}$ is a collection of Borel sets. Describe the sets in $\mathcal{A}$.

   (b) Prove that $P(X = Y|Y) = 0.5$.

2. Let X and Y be *independent* random variables on the probability space $(\Omega, \mathcal{F}, P)$. Assume that $\mathrm{E}|X| < \infty$, $\mathrm{E}|Y| < \infty$, and $\mathrm{E}|XY| < \infty$. Let $\mathcal{F}_1 \subset \mathcal{F}$ is a sub-σ-field and assume that X is independent of $\mathcal{F}_1$. Show that

$$\mathrm{E}[XY|\mathcal{F}_1] = \mathrm{E}[X]\mathrm{E}[Y|\mathcal{F}_1].$$

3. [10 points] Let $X_1, X_2, \ldots$ be iid random variables, whose distribution is absolute continuous with pdf

$$f(x) = 2xe^{-x^2} 1_{[x>0]}.$$

Show that

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{\log n}} \leq \sqrt{2}.$$

4. [10 points] Let $X_1, X_2, \ldots$ be a sequence of random variables, let $c_1, c_2, \ldots$ be a sequence of positive numbers, and let $Y_n = X_n 1_{[|X_n| \leq c_n]}$. Assume that $Y_n \to Y$ a.s. for some random variable Y. Show that if $\sum_{n=1}^{\infty} P(|X_n| > c_n) < \infty$, then $X_n \to Y$ a.s.