Theory of Probability, Problem 1 **Name:**

1. Let X and Y be random vectors with joint Lebesgue density $f(x, y)$. Let $g : \mathbb{R}^2 \mapsto \mathbb{R}$ be a Borel function such that $\mathrm{E}[|g(X, Y)|] < \infty$. Show that

$$\mathrm{E}[g(X, Y)|Y] = \frac{\displaystyle\int_{\mathbb{R}} g(x, Y) f(x, Y) \mathrm{d}x}{\displaystyle\int_{\mathbb{R}} f(x, Y) \mathrm{d}x} \text{ a.s.}$$

Theory of Probability, Problem 2 **Name:**

2. Let $\{X_n\}$ be a sequence of independent random variables, and let Y be a random variable measurable $\sigma(X_n, X_{n+1}, X_{n+2}, \dots)$ for every n. Show that there exists a constant $a \in \mathbb{R}$ such that $P(Y = a) = 1$.

3. Let $X_1, X_2, \ldots$ be a sequence of random variables defined on the same probability space and let

$$\bar{X}_n = \frac{1}{n} \sum_{m=1}^{n} X_m.$$

1. Show that if $X_n \to 0$ a.s., then $\bar{X}_n \to 0$ a.s.
2. Give an example to show that $X_n \xrightarrow{p} 0$ does not imply that $\bar{X}_n \to 0$ a.s.