Probability Preliminary Qualifying Exam

Solve any 3 of the following.

1. Consider an experiment consisting of successively selecting a point from the interval $[0,1]$ (with uniform distribution) until the first time that the result is greater than $\frac{1}{2}$.
a). Construct the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for this experiment.
b) Let Z be the result on the final step of the experiment (i.e. the step where we first see a value greater than $\frac{1}{2}$). Find the distribution of Z and evaluate $\mathrm{E}[Z]$ and $\mathrm{Var}(Z)$.

2. Let X and Y be two independent random variables each with an exponential distribution, i.e, $P(X > a) = e^{-a}$ and $P(Y > a) = e^{-a}$, $a > 0$. Find the probability density function of $Z = \frac{X}{X+Y}$.

3. Let $X_1, X_2, \ldots$ be an infinite sequence of independent random variables such that for some sequence of real numbers $a_n \geq 0$, $n = 1, 2, \ldots$ we have

$$P\{X_n = a_n\} = p_n, \qquad P\{X_n = -a_n\} = p_n, \qquad P\{X_n = 0\} = 1 - 2p_n, \qquad 0 \leq p_n \leq \frac{1}{2}.$$

Give the necessary and sufficient condition(s) for the P-a.s. convergence of the series $\sum_n X_n$.

4. Let X be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and assume that $E[X] = a$ for some finite a.
a) Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and assume that X is independent of $\mathcal{G}$. Prove that $\mathrm{E}[X|\mathcal{G}] = a$.
b) Now let $\mathcal{F}_1$,$\mathcal{F}_2$ be two independent sub-σ-algebras of $\mathcal{F}$. Let $X_1 = E[X|\mathcal{F}_1]$ and calculate $E[X_1|\mathcal{F}_2]$.

5. Let X and Y be two independent $\mathcal{N}(0,1)$ random variables. Calculate:

$$\mathrm{E}[2X + Y + 3 | X - 3Y + 2 = a].$$