## Probability Preliminary Qualifying Exam

1. Consider an experiment consisting of successively selecting a point from the interval $[0, 1]$ (with uniform distribution) until the first time that the result is greater than $\frac{1}{3}$.
a) Construct the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for this experiment.
b) Let $\tau$ be the number of steps in the experiment (i.e. $\tau$ is the step where we first see a value greater than $\frac{1}{3}$). Find the generating function of $\tau$.
c) Find the probability that, on the last step, we get a value greater than $\frac{1}{2}$.

2. Let $X$ and $Y$ be two independent random variables each with a Cauchy distribution, i.e, they both have a joint probability density function (pdf) of the form

$$f(x) = \frac{1}{\pi\,(1 + x^2)}.$$

Find the joint pdf for the random variables $Z_1 = 2\max\{X, Y\}$ and $Z_2 = \min\{X, Y\}$.

3. Consider the random Taylor series

$$F(z) = F(z, \omega) = \sum_{n=0}^{\infty} \frac{X_n(\omega)}{n!} z^n,$$

where $\{X_n : n \geq 0\}$ is a collection of iid $N(0, 1)$ random variables on the probability space $(\Omega, \mathcal{F}, P)$. Prove that, for any complex number $z$, the series converges $P$-a.s.

4. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be independent sub-$\sigma$-algebras.
a) Give the definition of what it means for $\mathcal{F}_1$ and $\mathcal{F}_2$ to be independent.
b) Now let $X$ be a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathrm{E}|X| < \infty$. Let $X_1 = \mathrm{E}[X|\mathcal{F}_1]$ and calculate $\mathrm{E}[X_1|\mathcal{F}_2]$.