Probability Preliminary Qualifying Exam

Solve any 3 of the following.

1. Let $X, X_1, X_2, \ldots$ with $\mathrm{E}|X| < \infty$ and $\mathrm{E}|X_i| < \infty$ for $i = 1, 2, \ldots$. We say that $X_1, X_2, \ldots$ converges to X in L^1 and write $X_i \to X$ in L^1 if

$$\lim_{i\to\infty} \mathrm{E}|X_i - X| = 0.$$

a) Show that if $X_1, X_2, \ldots$ converge to X in L^1 then $X_1, X_2, \ldots$ converge in probability to X.
b) Give an example of a sequence of random variables $X_1, X_2, \ldots$ with $\mathrm{E}|X_i| < \infty$, such that $X_i \to 0$ almost surely, but the sequence does not converge to 0 in L^1.
c) Give an example of a sequence of random variables $X_1, X_2, \ldots$ such that $\mathrm{E}|X_i| < \infty$ for $i = 1, 2, \ldots$, $X_i \to 0$ in L^1, but the sequence does not converge to 0 almost surely.

2. a) State the two Borel-Cantelli lemmas.
b) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with distribution function F. Let $\{\lambda_n\}$ be a deterministic sequence of numbers with $\lambda_n \to \infty$. Show that, with probability 1,

$$\limsup_{n\to\infty} \frac{X_n}{\lambda_n} > 1$$

if and only if $\sum_{n=1}^{\infty} (1 - F(\lambda_n)) = \infty$.

3. Let X and Y be bounded random variables on a common probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G} \subset \mathcal{F}$ be a sub σ-algebra.
a) Give the definition of the conditional expectation $\mathrm{E}[X|\mathcal{G}]$
b) Prove that

$$\mathrm{E}[X\mathrm{E}[Y|\mathcal{G}]] = \mathrm{E}[Y\mathrm{E}[X|\mathcal{G}]].$$

4. a) Define a π-system, a λ-system, and state the $\pi - \lambda$-theorem.
b) Prove that, if A is independent of a π-system $\mathcal{P}$ and $A \in \sigma(\mathcal{P})$, then $P(A)$ is either 0 or 1.