1. [10 points] Let f be a Lebesgue (μ) integrable function on $\mathbb{R}$ and $n \in \mathbb{N}$.

   (a) Let $E_n = \{t \in \mathbb{R} : f(t) \geq n\}$. Prove that $\lim_{n \to +\infty} \int_{E_n} f d\mu = 0$.

   (b) Prove that $\lim_{n \to +\infty} n \cdot \mu(E_n) = 0$.

2. [10 points] Let μ be the Lebesgue measure on $\mathbb{R}$ and $f : E \to \mathbb{R}$ be a function with a measurable domain E. Consider the following three statements.

   Statement (a) $f^{-1}((\alpha, \infty))$ is measurable for each $\alpha \in \mathbb{R}$;

   Statement (b) $f^{-1}([\alpha, \infty))$ is measurable for each $\alpha \in \mathbb{R}$;

   Statement (c) $f^{-1}(\{\alpha\})$ is measurable for each $\alpha \in \mathbb{R}$.

   (1) Prove that Statements (a) and (b) are equivalent.

   (2) Does Statement (c) imply that f is a measurable function? Provide your proof (if your answer is 'yes') or a counterexample (if your answer is 'not always true'.)

3. [10 points] Let $f(x)$ be a Lebesgue measurable function on $[0,1]$. For $n \in \mathbb{N}$, define $\varphi_n(x) = \tan^{-1}(nf(x)), x \in [0,1]$.

   (a) Prove that the limit function $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$ is well defined, and that

$$\int_{[0,1]} \varphi d\mu = \frac{\pi}{2}\Big(\mu\big(\{x \in [0,1] : f(x) > 0\}\big) - \mu(\{x \in [0,1] : f(x) < 0\})\Big).$$

   (b) Prove that

$$\lim_{n\to+\infty} \int_{[0,1]} \varphi_n d\mu = \int_{[0,1]} \varphi d\mu.$$

4. [10 points] Do the following.

   (a) State the definition of an absolutely continuous function on a finite interval $[a, b]$.

   (b) Let $f(t)$ be a Lebesgue integrable function on $\mathbb{R}$. Assume that for all rational numbers α, β with $\alpha < \beta$, $\int_{\alpha}^{\beta} f(t)dt = 0$. Prove that $f(t) = 0$ almost everywhere on $\mathbb{R}$.