Real Analysis I, Problem 1 **UNCC ID:**

1. [10 points] Do the following.

   (a) Let $E \subset \mathbb{R}$. State the definition of the outer measure $m^*(E)$ of E.

   (b) Let $E \subset \mathbb{R}$ and $0 < m^*(E) < \infty$. Prove that there exists an open interval $I = (a, b)$ with the property that

$$m^*(E \cap I) > 0.99\ m^*(I).$$

2. [10 points] Let $\{f_n, n \in \mathbb{N}\}$ be Lebesgue measurable functions on $[0, 1]$.

   (a) Prove that the function $f(x) \equiv \sup\{f_n(x), n \in \mathbb{N}\}$ is a measurable function.

   (b) Let $E \subset [0, 1]$ be the set of points where the sequence $\{f_n(x), n \in \mathbb{N}\}$ converges. Prove that E is Lebesgue measurable.

3. [10 points] Define

$$f_n(x) \equiv \frac{3 + \exp(n \sin x)}{2 + \exp(n \sin x)}, \ \ x \in [0, 2\pi].$$

(a) Prove that

$$\lim_{n \to +\infty} \int_{[0,2\pi]} f_n = \int_{[0,2\pi]} \lim_{n \to +\infty} f_n.$$

(b) Find the value of the above limit.

4. [10 points] Do the following.

   (a) State the definition of a *function of bounded variation* on $[a, b]$.

   (b) Let f be the function defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0; \\ x \sin\left(\dfrac{1}{x}\right) & \text{if} \quad x \neq 0. \end{cases}$$

   Prove that this function is *not* of bounded variation on the interval $\left[0, \dfrac{2}{\pi}\right]$.