Real Analysis I Qualify Exam

Student ID ________________

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1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, $G \subset \mathbb{R}$ is open, and $F \subset \mathbb{R}$ is closed.

   a) Prove from the definition that $f^{-1}(G)$ is open.

   b) Prove from the definition that $f^{-1}(F)$ is closed.

   c) Give an example of a bounded open set G and a continuous function $g : G \to \mathbb{R}$ such that $g(G)$ is closed.

2. Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) > 0$, and let $c \in (0,1)$. Prove that there exists a nonempty open interval (a,b) with the property that

$$m(E \cap (a,b)) \geq c \cdot m((a,b)).$$

3. a) State the Lebesgue Dominated Convergence Theorem (LDCT).

b) Let

$$f_n(x) \equiv \frac{1}{n} \cdot \frac{1}{\frac{1}{n^2} + x^2} = \frac{n}{1 + (nx)^2}, x \in [0, \infty), n \in \mathbb{N}.$$

Prove that

$$\int_{[0,\infty)} \lim_{n \to +\infty} f_n \neq \lim_{n \to +\infty} \int_{[0,\infty)} f_n.$$

c) Explain why in the example b) one can not use the LDCT as you stated in a).

4. a) Let $f : [0, 1] \to \mathbb{R}$ be an increasing, continuous function such that f is absolutely continuous on the interval $[\frac{1}{n}, 1]$ for each $n \in \mathbb{N}$. Prove that f is absolutely continuous on $[0, 1]$.

b) Give an example of a continuous function $f : [0, 1] \to \mathbb{R}$ such that f is absolutely continuous on the interval $[\frac{1}{n}, 1]$ for each $n \in \mathbb{N}$, and yet f is not absolutely continuous on $[0, 1]$. (You should prove that your example has these properties.)