1. Let A and B be subsets of $\mathbb{R}$, and suppose that $A \subset (-\infty, 0)$ and $B \subset (0, \infty)$. Let m^* denote the Lebesgue outer measure. Prove that

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

2. Suppose that f is a measurable function on E, $m(E) < \infty$, and f is finite almost everywhere.

   (a) For $n \in \mathbb{N}$, let

   $$A_n = \{x \in E : |f(x)| > n\}.$$

   Find $\lim_n m(A_n)$, and justify your answer.

   (b) Prove that for each $\epsilon > 0$, there exists a measurable set $F \subset E$ such that f is bounded on F and $m(E \setminus F) < \epsilon$.

3. Compute the following limit and justify your answer:

$$\lim_{n\to\infty} \int_1^\infty \frac{\sin^n(x)}{x^2}\, dx.$$

4. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of nonnegative numbers, and let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} (-1)^n a_n, & \text{if } \frac{1}{n+1} < x \le \frac{1}{n} \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that f has bounded variation if and only if $\sum_{n=1}^{\infty} a_n < \infty$.