1. Let m^* denote the Lebesgue outer measure, and suppose that $E \subset \mathbb{R}$ satisfies $m^*(E) < \infty$.

   (a) State the definition of $m^*(E)$.

   (b) Prove that for each $\epsilon > 0$, there exists $M > 0$ such that $m^*(E \setminus [-M, M]) < \epsilon$.

   (c) Prove that E is measurable if and only if for each $\epsilon > 0$, there exists a compact set $K \subset E$ such that $m^*(E \setminus K) < \epsilon$.

2. Suppose that $g : E \to \mathbb{R}$ is measurable.

(a) Prove that for all $t > 0$,

$$m\Big(\{x \in E : |g(x)| > t\}\Big) \le \frac{1}{t}\int_E |g|$$

(b) Now suppose that $\int_E |g| = 0$ and prove that $g = 0$ a.e. on E.

3. (a) Suppose that $\{f_n\}_n$ is a sequence of non-negative measurable functions on $[0,1]$ and $f_n \to f$ pointwise a.e. on $[0,1]$, where f is integrable on $[0,1]$. Let $g_n(x) = \min\{f_n(x), f(x)\}$ for all $n \in \mathbb{N}$ and $x \in [0,1]$. Prove that each g_n is integrable and find

$$\lim_n \int_0^1 g_n.$$

(b) Give an example of a sequence $\{f_n\}_n$ of non-negative measurable functions on $[0,1]$ such that $f_n \to 0$ pointwise a.e. on $[0,1]$ and yet $\int_0^1 f_n$ does not converge to 0. Remember to prove that your example has these properties.

4. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be absolutely continuous.

   (a) Show that the product $f \cdot g$ is absolutely continuous.

   (b) Show that the following integration by parts formula holds:

$$\int_a^b f \cdot g' = f(b)g(b) - f(a)g(a) - \int_a^b f' \cdot g.$$