1. Let $1 < p < \infty$ and $q = p/(p-1)$. Prove that for any $\lambda \in (0, 1/q)$ and any $f \in L^p([0,1])$,

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^\lambda} \int_0^\epsilon f = 0.$$

2. Let X be a normed linear space, and let $T : X \to X$ be a linear operator. Recall that the kernel of T is

$$\ker(T) = \big\{x \in X : T(x) = 0\big\}.$$

(a) Prove that $\ker(T)$ is closed in X.

(b) Prove that T is injective if and only if $\ker(T) = \{0\}$.

3. Let X be a Banach space and $T : X \to X$ a linear operator such that $\|T\| < 1$. For $n \geq 1$, let T^n denote the composition of T with itself n times: $T^n(x) = T \circ \cdots \circ T(x)$. Let $S : X \to X$ be defined by

$$S(x) = \sum_{n=1}^{\infty} T^n(x).$$

Prove that S is a well-defined bounded linear operator.

4. Let H be a Hilbert space.

(a) Let $v \in H$. Define $T : H \to \mathbb{R}$ by $T(u) = < u, v >$. Prove that T is a bounded linear functional and $\|T\| = \|v\|$.

(b) Suppose that $\mathcal{S} \subset H$ and for all $u \in H$, there exists $M_u > 0$ such that for all $v \in \mathcal{S}$,

$$| < u, v > | \leq M_u.$$

Prove that $\mathcal{S}$ is bounded.