Problem 1. (10 pt) Let E be a measurable set. Let A and B be two measurable subsets of E such that $m(A \setminus B) = 3$ and $m(B \setminus A) = 5$. Find the $L^3(E)$ norm

$$\|\chi_A - \chi_B\|_3$$

of the difference of the characteristic functions of A and B.

(Here, $X \setminus Y = \{x \in X \big| x \notin Y\}$ denotes the complement of Y in X.)

Problem 2. (10 pt) Let $1 < p < \infty$. Let T be a bounded linear functional on $L^p[1,2]$ having the property

$$T(\chi_{[1,x]}) = x - 1 \qquad \text{for all} \quad x \in [1,2],$$

where $\chi_{[1,x]}$ is the characteristic function of the interval $[1,x]$. Find the norm $\|T\|_*$ of the functional T.

Problem 3. (10 pt) Let $1 < p < \infty$. Let the sequence of functions $\{f_n\}$ on $[0, 1]$ be defined by

$$f_n = n^{1/p}\chi_{[0,1/n]},$$

where $\chi_{[0,1/n]}$ is the characteristic function of the interval $[0, 1/n]$ and $n \in \mathbb{N}$.

a) Prove that the sequence $\{f_n\}$ converges weakly to zero in $L^p([0, 1])$.

b) Prove that the sequence $\{f_n\}$ does not converge strongly in $L^p([0, 1])$.

Problem 4. (10 pt) Let g and h be two vectors in a Hilbert space H. Let $T \in \mathcal{L}(H)$ be the bounded linear operator defined by

$$T(u) = \langle u, h \rangle g \qquad \text{for all} \qquad u \in H.$$

Assume that $\|h\| = 2$ and $\|g\| = 5$.

a) Find the norm of the operator T.

b) Prove that T is a compact operator.