1. [10 points] Let $\mathcal{D}(T)$ be the set of functions defined on the closed bounded interval $[a, b]$ $(a < b)$ whose derivatives are continuous, and $X = C[a, b]$ be the set of continuous functions defined on $[a, b]$.

   (a) Show that the differential operator $T : \mathcal{D}(T) \longrightarrow X$ defined by

   $$T(f(t)) = \frac{df}{dt}$$

   is a linear operator but is unbounded.

   (b) Let $\kappa(t, \xi)$ be a continuous function defined on $[a, b] \times [a, b]$. Show that the operator $S : C[a, b] \longrightarrow C[a, b]$ defined by

   $$S(f) = g(t), \text{ where } g(t) = \int_a^b \kappa(t, \xi) f(\xi) d\xi$$

   is a bounded and linear operator.

2. [10 points] For $1 \leq p < \infty$, $q$ the conjugate of $p$ and $f \in L^p(E)$, show that $f = 0$ a. e. if and only if

$$\int_E f \cdot g = 0 \text{ for all } g \in L^q(E).$$

3. [10 points] Do the following.

   (a) Let $X$ and $Y$ be normed linear spaces and $T : X \longrightarrow Y$ a linear operator. Prove that if $T$ is compact, then it is continuous.

   (b) Let $X$ be an inner product space. Suppose $y$ and $z$ are two fixed elements of $X$. Show that an operator $T : X \longrightarrow X$ defined by

$$T(x) = < x, y > z \text{ for every } x \in X$$

   is bounded, linear, and compact.

4. [10 points] Let $X$ be a normed linear space and $X^*$ its dual space (the set of all real valued continuous linear functions defined on $X$). A sequence $\{x_n\}$ in $X$ is a weak Cauchy sequence provided that for every $f \in X^*$, the sequence $f(x_n)$ is Cauchy in $\mathbb{R}$.

   (a) Show that a weak Cauchy sequence is bounded (Hint: the Uniform Boundedness Principle by noting that $X^*$ is a Banach space).

   (b) Suppose $T : X \longrightarrow Y$ be a bounded linear operator between two normed linear spaces and $\{x_n\}$ is a sequence in $X$. If $\{x_n\} \longrightarrow x_0$ weakly, show that $\{T(x_n)\} \longrightarrow T(x_0)$ weakly.