1. Suppose that E is a Legesgue measurable subset of $\mathbb{R}$, $1 \leq p < \infty$, and q is the conjugate of p. Further, suppose that

   - $\{f_n\}_{n\geq 1}$ converges weakly to f in $L^p(E)$ (i.e. for each bounded linear functional $\psi : L^p(E) \to \mathbb{R}$, we have $\psi(f_n) \to \psi(f)$), and
   - $\{g_n\}_{n\geq 1}$ converges strongly to g in $L^q(E)$.

   Prove that

$$\lim_{n\to\infty} \int_E g_n \cdot f_n = \int_E g \cdot f.$$

2. Let X be a normed linear space, let $Y \subset X$ be a closed linear subspace, and let $x_0 \in X \setminus Y$. Prove that there exists a bounded linear functional $\psi : X \to \mathbb{R}$ such that $\psi(Y) = 0$ and $\psi(x_0) \neq 0$.

3. Let X be a Banach space, and suppose that $\{x_n\}_{n\geq 1}$ is a sequence in X such that $\sum_n \|x_n\| < \infty$.

   (a) Prove that the series $\sum_n x_n$ converges in X.

   (b) Prove that if $T : X \to X$ is a bounded linear operator, then $T\big(\sum_n x_n\big) = \sum_n T(x_n)$.

4. Let H be a Hilbert space, and let $K : H \to H$ be a compact linear operator from H to itself.
   (a) Prove that if $\{h_n\}_{n\geq 1}$ is bounded, then $\{K(h_n)\}_{n\geq 1}$ has a convergent subsequence.

   (b) Prove that if $\{h_n\}_{n\geq 1}$ is weakly convergent, then $\{K(h_n)\}_{n\geq 1}$ is strongly convergent.