1. Let $a < b$, and let $p \in [1, \infty)$.
   (a) Prove that there exists $C > 0$ such that for all continuous $f : [a, b] \to \mathbb{R}$,

$$\|f\|_p \leq C\|f\|_\infty.$$

   (b) Prove that there is no constant $c > 0$ such that for all continuous $f : [a, b] \to \mathbb{R}$,

$$\|f\|_\infty \leq c\|f\|_p.$$

2. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{L}(X, Y)$, where X and Y are Banach spaces. Prove that the sequence $\{\|T_n\|\}_{n=1}^{\infty}$ is bounded if and only if for each $x \in X$, the sequence $\{T_n(x)\}_{n=1}^{\infty}$ is bounded in Y.

3. Let $H = L^2([0,1])$, and let $\mathcal{F}$ be an orthonormal subset of H.

(a) Prove that for any $f, g \in \mathcal{F}$,

$$\|f - g\|_2 = \sqrt{2}.$$

(b) Prove that $\mathcal{F}$ must be countable.

4. Let H be a Hilbert space, and let $K : H \to H$ be a linear operator from H to itself. Prove that if $K(H)$ is a finite dimensional subspace, then K is compact.