1. Suppose that $E$ is a Lebesgue measurable subset of $\mathbb{R}$ with $0 < m(E) < \infty$ and $f \in L^p(E)$ for all $1 \leq p \leq \infty$. Prove that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

2. Suppose $X$ and $Y$ are Banach spaces. Let $\{\ell_n\}_n$ be a sequence of bounded linear functionals on $X$, and let $\{y_n\}_n$ be a sequence in $Y$ such that for each $x \in X$, the following series converges in $Y$:

$$\sum_{n=1}^{\infty} \ell_n(x) y_n.$$

Let $S : X \to Y$ be the map defined by

$$S(x) = \sum_{n=1}^{\infty} \ell_n(x) y_n.$$

Prove that $S$ is a bounded linear operator.

3. Let $X$ be a Banach space. Let $\{y_j\}_j$ be a subset of $X$, and let $Y = \overline{\text{span}}\{y_j\}$ (the closed linear span of $\{y_j\}_j$). Let $x_0 \in X$. Prove that the following statements are equivalent:
   (i) $x_0$ is in $Y$
   (ii) for every bounded linear functional $\ell : X \to \mathbb{R}$, if $\ell(y_j) = 0$ for all $j$, then $\ell(x_0) = 0$.

4. Let $\{\varphi_k\}_k$ be an orthonormal basis of the Hilbert space $H$, and let $\{u_n\}_n$ be a bounded sequence in $H$. Prove that $\{u_n\}_n$ converges to 0 weakly in $H$ if and only if for each $k$,

$$\lim_n < u_n, \varphi_k >= 0.$$