ON THE STRUCTURE AND INVARIANTS OF CUBICAL COMPLEXES

by

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ABSTRACT

SARAH J. BIRDSONG. On the structure and invariants of cubical complexes.
(Under the direction of DR. GÁBOR HETYEI)

This dissertation introduces two new results for cubical complexes. The first is a simple statistic on noncrossing partitions that expresses each coordinate of the toric h-vector of a cubical complex, written in the basis of the Adin h-vector entries, as the total weight of all noncrossing partitions. This expression can then be used to obtain a simple combinatorial interpretation of the contribution of a cubical shelling component to the toric h-vector.

Secondly, a class of indecomposable permutations, bijectively equivalent to standard double occurrence words, may be used to encode one representative from each equivalence class of the shellings of the boundary of the hypercube. Finally, an adjacent transposition Gray code is constructed for this class of permutations, which can be implemented in constant amortized time.

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CHAPTER 1: INTRODUCTION

Cubical complexes are the simplest possible generalization of simplicial complexes about which much is known. All possible vectors (f-vectors) of face numbers are give by the Kruskal-Katona Theorem. See [72, Definition 8.32]. The Upper Bound Theorem for face numbers of simplicial polytopes was shown by McMullen [49], and the same upper bound was generalized to all simplicial spheres by Stanley [63]. While McMullen relied on a decomposition called a shelling, shown to exist for all polytopes by Brugeser and Mani [13], Stanley's generalization relied on proving the Cohen-Macaulay property for the face ring of a simplicial sphere. The generalized lower bound for the number of edges of a simplicial polytope was shown by McMullen and Walkup [50]; however, the best lower bound for the number of edges of a simplicial polytope was proven by Barnette [4]. A far-reaching generalization of the Lower Bound Theorem was shown by Stanley [64] using toric varieties.

A central notion in the study of face numbers of simplicial complexes is the h-vector. It is an equivalent re-encoding of the f-vector with smaller numbers which are nonnegative for Cohen-Macaulay simplicial complexes. Several important, apparently independent, quantities may be associated to the h-vector:

- coefficients in the numerator of the Hilbert-Poincaré series of the face ring,
- number of shelling components of a given type if the complex is shellable, and

- invariants of toric varieties associated to rational simplicial polytopes.

Much less is known for cubical complexes. There is no known analogue of the Kruskal-Katona Theorem. Even proving that a d-dimensional cubical complex has at least as many vertices as a d-dimensional cube seemed surprisingly difficult until the recent proof of Klee [43].

There is a cubical analogue of the face ring [34] yielding one h-vector which has nothing to do with the enigmatic toric h-vector proposed by Stanley [64] when he generalized the simplicial h-vector to lower Eulerian posets. The number of possible types of shelling components of a given type is too large to cover with the face numbers. That said, both of these cubical h-vectors are nonnegative for shellable complexes. In fact, they weakly increase after adding each shelling component [35]. A combinatorical model for the contribution of a shelling component to the toric h-vector was given by Chan [15].

An enigmatic new h-vector was proposed by Adin [2]. It was shown by Hetyei [35] that this h-vector is the smallest h-vector that increases on a shelling and the entries of all other cubical h-vectors can be expressed as nonnegative combinations of the Adin h-entries. A combinatorial interpretation of the coefficients connecting the h-vector of the cubical face ring with the Adin h-entries was given by Haglund [33].

This dissertation fills in a blank in the picture described above and opens a new area of research. This blank is a combinatorial interpretation of the coefficients connecting the Adin h-entries and the entries in the toric h-vector. The new area is the study of the structure of all shellings of a cubical complex. Virtually nothing has been done in this area at present, even for simplicial complexes. As a first step in this new direction, this dissertation constructs an adjacent transposition Gray code for all shelling types of a hypercube.

As witnessed by the latest addition to Knuth's classic work [45], Gray codes are widely used in computer science to *enumerate* all words, n-tuples, or permutations of a given type. In particular, there is a significant amount of research devoted to finding Gray codes for classes of permutations, where two permutations are considered adjacent if they differ by an involution of some special kind. For a survey of some key results, see Savage's paper [57, Section 11]. The simplest and most elegant result in this area is the Johnson-Trotter algorithm [41, 70], providing an adjacent transposition Gray code for all permutations of a finite set.

The following chapters of this dissertation are structured as follows. Chapter 2 defines the main terms and properties used throughout the rest of the document. This chapter starts by looking various complexes, face numbers, and h-vectors. Then it discusses types of shellings and cubical shelling component types. At the end of the chapter, the other main structures used are defined and discussed: noncrossing partitions, signed permutations, arc and circular diagrams, and Gray codes.

Chapter 3 finds the coefficients connecting the Adin h-entries and the entries in the toric h-vector, and then gives a combinatorical interpretation of these coefficients. The main result of this chapter is finding the toric contribution of cubical shelling component types and their associated combinatorical interpretation.

Chapter 4 defines equivalence classes for the listings of all of the facets of the hypercube and then represents each with a signed permutation. A Gray code is then found for the set of these listings using the arc diagrams associated to the signed permutations, which are then encoded by words. To find a Gray code for the shellings of the hypercube, all signed permutations which do not represent a shelling are deleted from the first Gray code found. The resulting sublist is a Gray code for the shellings of the n-cube. The main result of this chapter proves that this simple operation actually produces a Gray code.

Finally, Chapter 5 outlines possible directions for future work based off the results found in the first chapters.

CHAPTER 2: DEFINITION OF TERMS

A *partially ordered set* or *poset* P is a set that has a binary relation which is antisymmetric, reflexive, and transitive [68]. The *Möbius function* μ of P is defined as $\mu(x, x) = 1$ for any $x \in P$ and $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$ for all $x < y$ in P.

Example 2.0.1. *Let P be the chain $\mathbb{N}$. Then for $i, j \in P$,*

$$\mu(i,j) = \begin{cases} 1, & i = j \\ -1, & i+1 = j \\ 0, & otherwise. \end{cases}$$

A poset is *graded* if it has a unique minimum element $\hat{0}$, a unique maximum element $\hat{1}$, and a rank function. A *Eulerian poset* is a finite graded poset with $\hat{0}$ and $\hat{1}$ where $\mu(x, y) = (-1)^{\ell(x,y)}$ for $x \leq y$ in the poset [68]. The function $\ell(x, y)$ gives the length of the interval (x, y). For example, the boolean algebra B_3 (see Fig. 2) is an Eulerian poset. In fact, B_n is Eulerian for all $n \geq 1$. Alternatively, a poset is Eulerian if it is graded and for any open interval (x, y) the number of elements at odd and even ranks is the same.

Let $V = \{1, 2, \ldots, \nu\}$ be the vertex set. Then an *abstract complex*, $\mathcal{C}$, is a family of subsets of V, called the faces of $\mathcal{C}$, such that $\emptyset, \{1\}, \ldots, \{\nu\} \in \mathcal{C}$, and for any two faces $F, G \in \mathcal{C}$, then $F \cap G \in \mathcal{C}$. A *polytope* is a convex hull of a finite set, where 1-dimensional faces are edges and maximal proper faces are called *facets* [32].

A (finite) *polyhedral complex* P is a finite family of convex polytopes in d-dimensional Euclidean space, called the faces of P, such that any (geometric) face of a polytope in P also belongs to P, and the intersection of any two polytopes in P is a (geometric) face of both and also contained in P. The dimension of P is the maximum dimension of a polytope in P, and the complex is *pure* if any polytope in P is a (geometric) face of a polytope in P of dim(P) [32].

For a polyhedral complex, define f_i to be the number of i-dimensional faces. These are called the face numbers and form the f-vector $(f_0, \ldots, f_d)$. Since dim $\emptyset = -1$, $f_{-1} = 1$ for any object. See Fig. 1 for an example.

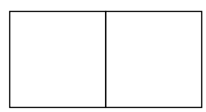


Figure 1: The double-square where $f_0 = 6$, $f_1 = 7$, and $f_2 = 3$

Let $\hat{P}$ be an arbitrary Eulerian poset of rank $n+1$ with $\hat{0}$ and $\hat{1}$. Then the flag f-vector of the complex is defined to be $f_S = |\{\{\hat{0} < x_1 < \ldots < x_k\} \subseteq P : \{\text{rank}(x_1), \ldots, \text{rank}(x_k)\} = S\}|$. Let S be some subset of $\{1, \ldots, \text{rank}(P)\}$. Then the flag h-vector is defined by $h_S = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \cdot f_T$.

Example 2.0.2. *Consider the Boolean algebra B_3. See Fig. 2. Then the flag f- and flag h-vectors are listed in Table 1.*

2.1 Simplicial Complexes

A *simplicial complex* Δ on some vertex set V is a collection of subsets of V such that

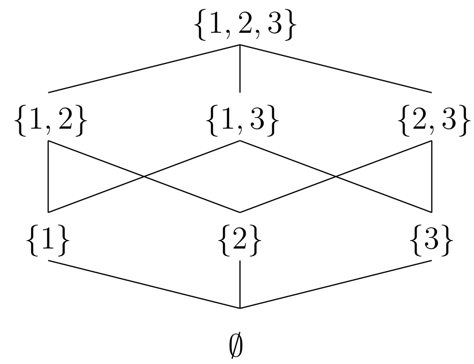


Figure 2: B_3

flag f-vector		flag h-vector	
$f_{\emptyset} = 1$	$f_{12} = 6$	$h_{\emptyset} = 1$	$h_{12} = 1$
$f_1 = 3$	$f_{13} = 6$	$h_1 = 2$	$h_{13} = 3$
$f_2 = 3$	$f_{23} = 3$	$h_2 = 2$	$h_{23} = 0$
$f_3 = 1$	$f_{123} = 6$	$h_3 = 0$	$h_{123} = -3$

Table 1: The flag vectors for B_3

(i) if $\nu \in V$, then $\{\nu\} \in \Delta$; and

(ii) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

The element $F \in \Delta$ is called a face of Δ, and $\dim F = |F| - 1$. Then $\dim \Delta = \max_{F \in \Delta}(\dim F)$ [31, 67].

2.1.1 The Face Ring of a Simplicial Complex

Let Δ be a $(d-1)$-dimensional simplicial complex on an n element vertex set V with f-vector $(f_{-1}, \ldots, f_{d-1})$.

Let K be a field; then $K[\Delta] = K[\nu \in V]/I_{\Delta}$ is defined to be the *face ring* of $K[\Delta]$ [67] where $I_{\Delta} =< x_{i_1} \ldots x_{i_r} | i_1 < \ldots < i_r, \{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta >$, and $\dim K[\Delta] = 1 + \dim \Delta$. Hence, $\dim K[\Delta] = d$.

Example 2.1.1. *Consider Δ given in Fig. 3. Then $I_\Delta =< x_1x_4 >$ since x_1x_4 is not*

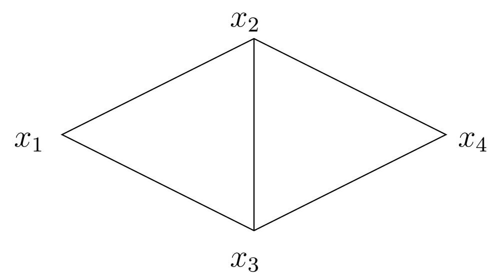


Figure 3: A simplicial complex with $V = \{x_1, x_2, x_3, x_4\}$

a face in Δ. The face ring for this example is $K[x_1, x_2, x_3, x_4]/ < x_1x_4 >$.

If $V = \bigoplus_{i\in N} V_i$ is an N-graded vector space over field K where each subspace V_i of vectors of degree n is finite dimensional, then the *Hilbert-Poincaré series* is $\sum_{i\in N} \dim_k(V_i) \cdot t^i$. For example, the Hilbert-Poincaré series of $K[x]$ is $\mathcal{H}(K[x], t) = \sum_{i\in N} t^i = \frac{1}{1-t}$. Now, since the face ring is a graded ring, its Hilbert-Poincaré series is given by $\mathcal{H}(\Delta, t) = \sum_{i\in N} t^i = \sum_{k=0}^{d} f_{k-1} \left(\frac{t}{1-t}\right)^k = \frac{\sum_{i=0}^{d} h_i t^i}{(1-t)^d}$. The h-vector of Δ is defined by this relation.

2.1.2 Shellability and Shellings

A *shelling* is a particular way of listing the facets of the boundary of a polytope. The following general definition of a shelling was stated in [72, Definition 8.1].

Definition 2.1.2. *A* shelling *of the boundary of a convex polytope $\mathcal{P}$ (i.e., $\mathcal{C}(\partial\mathcal{P})$) is a linear ordering of the facets $F_1, \ldots, F_m$ of the facets of $\mathcal{P}$ such that either* $\dim \mathcal{P} = 0, 1$ *or it satisfies the following two conditions:*

(i) the boundary complex of F_1 has a shelling, and

(ii) for $2 \leq i \leq m$, $(F_1 \cup \ldots \cup F_{i-1}) \cap F_i = G_1 \cup \ldots \cup G_j$, *where* $G_1, \ldots, G_k$ *is a shelling of* F_i *and* $j \leq k$.

Shellings have a number of properties. For example, suppose $F_1, \ldots, F_m$ is a shelling. Then it can be proved by induction that $F_m, \ldots, F_1$ is also a shelling.

In a simplicial complex, facet F_i has type k if $(F_1 \cup \ldots \cup F_{i-1}) \cap F_i$ is the union of k components. Each entry of the h-vector h_k counts the number of type k shelling components of Δ. It is easily seen that the h-vector of a shellable simplicial complex is non-negative.

Example 2.1.3. *Consider the simplicial complex in Fig. 4 with shelling* F_1, F_2, F_3, F_4. *Facet* F_1 *has type 0,* F_2 *and* F_3 *have type 1, and* F_4 *has type 2. Thus,* $h_0 = 1$, $h_1 = 2$, *and* $h_2 = 1$.

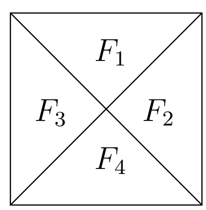


Figure 4: A simplicial complex with shelling: F_1, F_2, F_3, F_4

The Upper Bound Conjecture (UBC) state that cyclic polytopes, which are formed by the convex hull of points on the moment curve, maximize the f-vector of any simplicial convex polytope [44, 63]. The UBC is equivalent to $h_i \leq \binom{h-d+i-1}{i}$ for $0 \leq i \leq d$, where Δ is the boundary complex of a d-dimensional simplicial convex polytope with n vertices [62].

2.1.3 The Dehn-Sommerville Equations

The Dehn-Sommerville equations are a complete set of linear equations satisfied by the face numbers of a simplicial sphere [32]. These equations reduce to $h_i = h_{d-i}$ [64, Theorem 2.4].

$$\sum_{j=0}^{d} f_{j-1}\left(\frac{x}{1-x}\right)^{j} = \frac{\sum_{i=0}^{d} h_i x^i}{(1-x)^d}$$

can be transformed into $\sum_{j=0}^{d} f_{j-1}(1-x)^{d-j} = \sum_{i=0}^{d} h_i x^i$. As a result of the Dehn-Sommerville equations, this is equivalent to

$$\sum_{j=0}^{d} f_{j-1}x^j(1-x)^{d-j} = \sum_{i=0}^{d} h_i x^i.$$

For the order complexes of Eulerian posets, the relations between the flag f- and flag h-vectors (or really their entries) are a generalization of the Dehn-Sommerville equations and are proved using the same techniques. The flag f- and flag h-vectors can be used to compute the simplicial f- and h-vectors: $h_i = \sum_{|S|=i} h_S$ and $f_{j-1} = \sum_{|S|=j} f_S$. It can also be shown that $f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i$.

2.2 Toric Polynomials of an Eulerian Poset

Barnette [4] proved the Lower Bound Conjecture (LBC) for all simplicial d-polytopes:

$$f_k \geq \binom{d}{k} f_0 - \binom{d+1}{k+1} k \quad \text{for} \quad 1 \leq k \leq d-2, \text{ and}$$

$$f_{d-1} \geq (d-1)f_0 - (d+1)(d-2).$$

Define $P = \hat{P}\backslash\{\hat{1}\}$ where $\hat{P}$ has rank $d+1$. Stanley [64] generalized the LBC to $h_0 \leq h_1 \leq \ldots \leq h_{\lfloor \frac{d}{2} \rfloor}$ using toric varieties. Inspired by this model, he defined toric polynomials for all (lower) Eulerian posets using two recursively defined polynomials

$f(P,x)$ and $g(P,x)$ as follows:

1. $f(\emptyset,x) = g(\emptyset,x) = 1$,

2. if $\hat{P}$ has rank $d+1 \geq 1$ and $f(P,x) = k_0 + k_1x + \cdots + k_dx^d$, then $g(P,x) = \sum_{i=0}^{m}(k_i - k_{i-1})x^i$ where $m = \lfloor \frac{\text{rank}(P)}{2} \rfloor$ and $k_{-1} = 0$, and

3. if $\hat{P}$ has rank $d+1 \geq 1$, then $f(P,x) = \sum_{t \in P} g([\hat{0},t),x)(x-1)^{d-\text{rank}(t)}$.

Set $h_i = k_{d-i}$ for each i. Then the toric h polynomial is $h(P,x) = \sum h_i x^i$. Also, applying the Dehn-Sommerville Equations to the toric polynomials yields $h(\mathcal{P},x) = f(\mathcal{P},x)$ for Eulerian posets.

A *lower Eulerian poset* is a finite graded poset with $\hat{0}$ and every interval $[x,y]$ is Eulerian. Stanley extended the definition of the toric f polynomial to lower Eulerian posets. This was possible since the definition uses half open intervals $[\hat{0},t)$ where t is any element in the poset and d is the length of the longest chain in the poset.

The face poset of a *polyhedral complex* is a specific type of lower Eulerian poset. Billera, Chan, and Liu investigated this particular case [9], and their adapted definition is given below.

Definition 2.2.1 (Billera-Chan-Liu)**.** *Let $\mathcal{P}$ be a d-dimensional polyhedral complex and F any face of $\mathcal{P}$. Then the toric f and g polynomials are defined by the following three rules:*

1. $f(\emptyset,x) = g(\emptyset,x) = 1$,

2. $f(\mathcal{P},x) = \sum_{F \in \mathcal{P}} g(\partial F,x)(x-1)^{d-dim(F)}$, *and*

3. $g(\mathcal{P},x) = \sum_{i=0}^{m}(k_i - k_{i-1})x^i$ *where* k_i *is the coefficient of* $f(\mathcal{P},x)$*,* $k_{-1} = 0$*, and* $m = \lfloor \frac{d+1}{2} \rfloor$*.*

Set $h(\mathcal{P},x) = \sum_i h_i x^i = x^{d+1} f(\mathcal{P}, 1/x)$. The toric h polynomial is a degree $d+1$ polynomial, and the toric h-vector is comprised of the coefficients of the toric h polynomial.

If P is the face complex of a convex polytope and ∂P is its boundary complex, then $h(P,x) = g(\partial P, x)$ [38, Corollary 1.1]. For example, the h polynomial of a d-dimensional cube (see Chapter 2.3 for a detailed definition) is the g polynomial of its boundary as noted in [9] and proved by Stanley [64]. Gessel [64] showed that $g(L_d, x) = \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{1}{d-k+1}\binom{d}{k}\binom{2d-2k}{d}(x-1)^k$, and Hetyei [38] showed this was equivalent to $\sum_{k=0}^{\lfloor d/2 \rfloor} C_{d-k}\binom{d-k}{k}(x-1)^k$.

d	$g(L_d, x)$
-1	1
0	1
1	1
2	$1 + x$
3	$1 + 4x$
4	$1 + 11x + 2x^2$

Table 2: The g polynomials of the d-cube for small d

Example 2.2.2. *Let* P *be the object illustrated in Fig. 1, and find the toric* f *polynomial of* P*. Since Definition 2.2.1 sums over components of* P*, one must start by* *finding the toric f and g polynomials of the components of P.*

$$f(\emptyset, x) = g(\emptyset, x) = 1$$

$$f(B_0, x) = g(B_0, x) = 1$$

$$f(B_1, x) = g(\emptyset, x)(x-1) + 2g(B_0, x) = x+1$$

$$g(B_1, x) = 1$$

$$f(B_2, x) = g(\emptyset, x)(x-1)^2 + 4g(B_0, x)(x-1) + 4g(B_1, x) = x^2 + 2x + 1$$

and $g(B_2, x) = x + 1$. *Putting these altogether yields the toric polynomial of P:*

$$f(P, x) = g(\emptyset, x)(x-1)^3 + 6g(B_0, x)(x-1)^2 + 7g(B_1, x)(x-1) + 2g(B_2, x) = x^3 + 3x^2.$$

Then $h(\mathcal{P}, x) = 1 + 3x$, *which implies that* $h_0 = 1$ *and* $h_1 = 3$.

2.2.1 The *cd* Index

For a graded poset P of rank $n+1$, the ab-index $\Psi_P(a, b)$ in the noncommuting variables a and b is defined as $\Psi_P(a, b) = \sum_{S \subseteq [1,n]} h_S \cdot u_S$ where $u_S = u_1 \ldots u_n$ such that for each $1 \leq i \leq n$,

$$u_i = \begin{cases} a, & i \notin S \\ b, & i \in S. \end{cases}$$

For example, consider B_3. Here $S \subseteq [1, 2]$. Then $u_\emptyset = a^2$, $u_1 = u_{\{1\}} = ba$, $u_2 = u_{\{2\}} = ab$, $u_{12} = u_{\{1,2\}} = b^2$. This gives $\Psi_{B_3}(a, b) = a^2 + 2ba + 2ab + b^2$ since the flag h-vector is given by $h_\emptyset = 1 = h_{12}$ and $h_1 = 2 = h_2$.

If the flag f-vector is used instead of the flag h-vector, then the resulting index is $\Upsilon_P(a, b) = \sum_S f_S \cdot u_S = \Psi_P(a + b, b)$ also called the $(a + b)b$-index.

The cd-index, denoted $\Phi_P(c, d)$, of an Eulerian poset P can be calculated from the ab-index using the following conversions: $c = a + b$ and $d = ab + ba$. For example, consider B_3 again. Then $\Psi_{B_3}(a,b) = a^2 + 2ba + 2ab + b^2 = (a+b)^2 + (ab+ba) = c^2 + d = \Phi_{B_3}(c,d)$.

Being able to write the ab-index as a polynomial of c and d is equivalent to a complete set of linear equations for the flag f-vector. Bayer and Billera [5] first found the generalization of the Dehn-Sommerville equations to the flag vectors. Bayer and Klapper [8] then showed that these complicated equations are equivalent simply stating that the cd-index exists.

The ce-index of an Eulerian poset may also be computed from the ab-index by the conversions: $c = a + b$ and $e = a - b$. Stanley [65] noted that the cd-index is equivalent to writing the ce-index as a polynomial of c and e^2. The ce words are given by $v_S = v_1 \ldots v_n$ such that for each $1 \leq i \leq n$,

$$v_i = \begin{cases} c, & i \notin S \\ e, & i \in S. \end{cases}$$

Let L_S be the coefficient of the ce words. Then the ce-index is given by $\sum_S L_S \cdot v_S$. L_S is called the flag L-vector and is related to the flag f-vector by

$$L_S = (-1)^{n-|S|} \sum_{T \supseteq [1,n] \setminus S} \left(-\frac{1}{2}\right)^{|T|} f_T \quad \text{and} \quad f_S = 2^{|S|} \sum_{T \subseteq [1,n] \setminus S} L_T$$

which was shown by Bayer and Hetyei [7].

Definition 2.2.3. *A permutation $\pi \in \mathcal{S}_n$ is an* André Permutation *if*

(i) π has no double descents; i.e., $\pi_{i-1} > \pi_i > \pi_{i+1}$, and

(ii) $\pi_{j-1} = \max(\pi_{j-1}, \pi_j, \pi_{k-1}, \pi_k)$ and $\pi_k = \min(\pi_{j-1}, \pi_j, \pi_{k-1}, \pi_k)$ where $j < k$ in $[1, n]$, then there exists l such that $j < l < k$ and $\pi_l < \pi_k$.

Definition 2.2.4. *An André permutation* $\pi \in \mathcal{S}_n$ *is augmented if* $\pi_n = n$.

Hetyei [36] proved the following proposition.

Proposition 2.2.5 (Hetyei)**.** *A permutation* $\pi \in \mathcal{S}$ *is an André permutation if and only if for* $m := \pi^{-1}(1)$ *the permutations* $\pi\mid_{[1,m]}$ *and* $\pi\mid_{[m+1,n]}$ *are André permutations.*

Letting D_n be the number of augmented André permutations, Foata and Strehl [28] showed that $D_{n+1} = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}D_iD_{n-i}$. See Table 3.

n	André permutations	D_n
1	1	1
2	12	1
3	123, 213	2
4	1234, 1324, 2134, 2314, 3124	5
5	12345, 12435, 13245, 13425, 14235, 21345, 21435, 23145, 23415, 24135, 31245, 31425, 32415, 34125, 41235, 41325	16

Table 3: The André permutations for small n

The *ab-variation* monomial $V_{ab}(\pi)$ of an André permutation $\pi \in \mathcal{S}_n$ is a word $w_1 \dots w_{n-1}$ such that

$$w_i = \begin{cases} a, & \pi_i \text{ is an ascent} \\ b, & \pi_i \text{ is a descent} \end{cases}$$

for $1 \leq i \leq n-1$. The *cd-variation* monomial $V_{cd}(\pi)$ of an André permutation $\pi \in \mathcal{S}_n$ is a word that can be built from $V_{ab}(\pi)$ by replacing every ba with d and replacing every remaining letter with c. Let A_n be the set of André permutations. Then $V_{cd}(A_n) = \sum_{\pi \in A_n} V_{cd}(\pi)$ [36].

The cd-index of the Boolean algebra can be computed by $\Phi_{B_n}(c,d) = V_{cd}(A_n)$ [56]. Stanley [65] generalized this to André permutations of the second kind and Simsun permutations.

Hetyei [39] defined the short toric polynomial $t([\hat{0},\hat{1}),x)$ to be a shorter variant of the toric polynomial $f([\hat{0},\hat{1}),x)$ for an Eulerian poset $[\hat{0},\hat{1}]$ of rank $n+1$, where $t([\hat{0},\hat{1}),x)$ is the polynomial formed by discarding all terms of degree less than 0 from $x^n f([\hat{0},\hat{1}),\frac{1}{x^2})$.

2.3 Cubical Complexes

Define the standard n-dimensional *hypercube* or *n-cube* to be $[-1,1]^n \subset \mathbb{R}^n$. As observed by Metropolis and Rota [51], each non-empty face of the standard hypercube may be denoted by a vector $(u_1,\ldots,u_n) \in \{-1,*,1\}^n$, where $u_i = -1$ or $u_i = 1$ indicates that all points in the face have the i^{th} coordinate equal to u_i; whereas, $u_i = *$ indicates the i^{th} coordinate of the points in the face range over the entire set $[-1,1]$. Using this notation, each facet of the boundary of the n-cube is of the form $(u_1,\ldots,u_n)$ such that exactly one u_i belongs to $\{-1,1\}$.

Thus, each facet may be represented by a single number. Namely, if $u_i = 1$, then that face is represented by i, and if $u_i = -1$, then $-i$ represents the facet. An *antipodal* pair of a cubical complex is a pair of facets where there is a pair of parallel (distinct) supporting hyperplanes of the complex such that one of the facets belongs to one of the hyperplanes and the other facet belongs to the other hyperplane [32]. The facets i and $-i$ are an antipodal pair.

For example, consider the 2-cube given by $[-1,1]^2$. Then 1 represents the side of the square where $x = 1$ and y varies, and 2 is the side where $y = 1$ and x varies.

As a result of this notation, a permutation on $\{\pm 1, \ldots, \pm n\}$ represents a list of the facets of the n-cube. This correspondence is a bijection. To continue the above example, the permutation $1, 2, -2, -1$ lists the order in which the facets of the square are numbered. See Fig. 5.

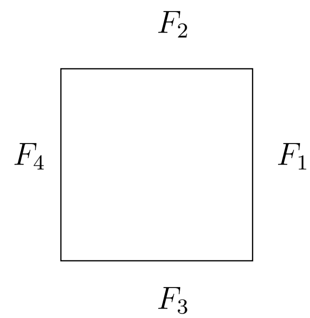


Figure 5: The 2-cube with its facets listed in order: $1, 2, -2, -1$

A *cubical complex* is a polyhedral complex where each face is combinatorially equivalent to a cube. Chan [15] gave the following alternative definition. A cubical d-complex Q is a geometric realization of a finite graded poset P with $\hat{0}$ such that for any $t \in P$ the set $P_t = \{s \in P : \hat{0} \leq s < t\}$ is isomorphic to L_r for some r. L_r is the face poset of an r-dimensional cube. The boundary complex of a cubical polytope is a pure cubical complex.

For cubical complexes, the triangulation, Adin, and toric h-vectors are defined. These are all combinations of the f-vector, where here f_i is the number of i-cubes.

2.3.1 The Face Ring of a Cubical Complex

The triangulation h-vector occurs in the numerator of the Hilbert-Poincaré series of the face ring (or the Stanley ring) of a cubical complex. This is the cubical analogue of the face ring of a simplicial complex. This h-vector also has the property that for cubical polytopes it is the (simplicial) h-vector of the triangulation of the polytope's vertices [34].

2.3.2 Line Shellings

Bruggeser and Mani [13] showed that the boundary complex of any polytope is shellable. They used line shellings to prove this. Bruggeser and Mani [13] defined line shellings in the following way.

Definition 2.3.1 (Bruggeser-Mani)**.** *Let P be a convex polytope. Choose a line L through the interior of P which is not parallel to any facet of P and intersects the affine hulls of different facets of P in distinct points. Let p be a point on L such that p is not in the interior of P. Orient L such that p is in the positive direction relative to P. Starting at the location where L leaves P move along L (towards p) in the positive direction to ∞. Then return to P along L but from the opposite direction. Order the facets of P as they become visible on the journey out to infinity and as they disappear on the horizon on the way back to P. This gives the ordering $F_1, \ldots, F_m$ of the facets of P where this order is called a* line shelling *of P.*

Example 2.3.2. *Consider the convex polytope in Fig. 6. To find a line shelling of this polytope, construct a straight line through the polytope such that the line is not parallel to any side (facet). Extend each side of the polytope until it intersects the line. The extensions of the first four facets will intersect the line in the positive direction. Then, as can be seen, the line shelling constructed from the given line is $F_1, \ldots, F_7$.*

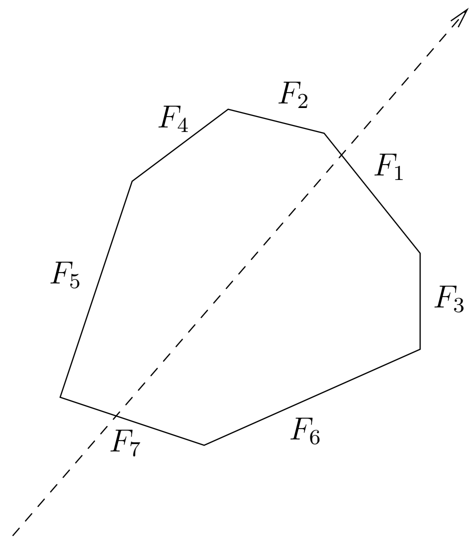


Figure 6: A convex polytope with a line shelling

Now consider the n-cube. From the definition of a line shelling, it is obvious that the n-cube has a line shelling. Stanley [61] found that there are $2^n n!^2$ distinct line shellings of the n-cube. Let $f(n)$ be the number of shellings of the n-cube. Then $\sum_{n\geq 1} f(n)\frac{x^n}{n!} = 1 - \frac{1}{\sum_{n\geq 0}(2n)!\frac{x^n}{n!}}$. For example, the 3-cube has a total of 480 shellings and 288 line shellings. Develin [21] proved that every shelling of the n-cube can be realized as a line shelling of some polytope that is combinatorially equivalently to the n-cube. Also, when traveling along the constructed line, one will never see both the facets denoted by k and $-k$ [61]. This is because $\pm k$ are antipodal pairs, and the constructed line not parallel to either. However, since $\pm k$ are antipodal, exactly one of these two facets will be visible from the positive direction of the line.

2.3.3 Cubical Shelling Components

Let $\mathcal{P}$ be a d-dimensional pure cubical complex. Suppose $\{F_1, \ldots, F_m\}$ are the facets of $\mathcal{P}$. A *shelling* of $\mathcal{P}$ is a way of ordering the facets such that $F_k \cap (F_1 \cup \ldots \cup F_{k-1})$

is a union of $(d-1)$-faces homeomorphic to a ball or sphere for each $1 \leq k \leq m$. This intersection is a *cubical shelling component* whose *type* is (i, j) if it is the union of i antipodally unpaired $(d-1)$-faces and j antipodally paired $(d-1)$-faces. This was stated by Chan [15]. For a proof, see [25, Lemma 3.3].

Lemma 2.3.3 (Ehrenborg-Hetyei)**.** *The ordered pair (i, j) is the type of a shelling component in a shelling of a cubical d-complex if and only if one of the following holds:*

(i) $i = 0$ *and* $j = d$*; or*

(ii) $0 < i < d$ *and* $0 \leq j \leq d - i$*.*

Furthermore, in case (i), the shelling component is homeomorphic to a $(d-1)$-sphere; and in case (ii), the shelling component is homeomorphic to a $(d-1)$-ball.

A direct result of this lemma is that the first shelling component has type $(0, 0)$, and for spheres, the last shelling component has type $(0, d)$.

Let $c_{i,j}$ be the number of type (i, j) shelling components. In particular, $c_{0,0} = 1$. The vector $(\ldots, c_{i,j}, \ldots)$ is called the *c-vector* of the shelling. The c-vector is not unique to the cubical complex; instead, it depends on which shelling is chosen. However, different shellings may have the same c-vector.

Example 2.3.4. *The 3-cube may be written as $[-1, 1]^3$ and has 10 shellings but only two distinct c-vectors. Each side (a_1, a_2, a_3) has exactly one position fixed as 1 or -1. To denote these shellings, let k indicate the facet or side where $a_k = 1$ and $-k$ where $a_k = -1$. Then each shelling is represented by a permutation of $\{\pm 1, \pm 2, \pm 3\}$. See Table 4.*

Shelling 1:	$1,2,3,-1,-2,-3$				
$c_{i,j}$:	$c_{0,0}=1$	$c_{1,0}=1$	$c_{2,0}=2$	$c_{1,1}=1$	$c_{0,2}=1$
Shelling 2:	$1,2,-1,3,-2,-3$				
$c_{i,j}$:	$c_{0,0}=1$	$c_{1,0}=2$	$c_{2,0}=0$	$c_{1,1}=2$	$c_{0,2}=1$

Table 4: Representative shellings, producing the two c-vectors of the 3-cube

2.3.4 The Adin h-Vector

Adin [2] called his h-vector the (long) cubical h-vector and defined it in terms of a short cubical h-vector, denoted $h_i^{(c)}$ and $h_i^{(sc)}$, respectively. Adin's (long) cubical h-vector will be referred to as the cubical h-vector or simply as the Adin h-vector. Let $\mathcal{P}$ be a d-dimensional cubical complex with f-vector $(f_0, \ldots, f_d)$. Then the Adin h-vector is defined by the equations

$$h_i^{(sc)} = \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} 2^j f_j \quad \text{, for } 0 \leq i \leq d \text{ and} \tag{1}$$

$$h_i^{(sc)} = h_i^{(c)} + h_{i+1}^{(c)} \quad \text{for } 0 \leq i \leq d \tag{2}$$

with the initial and last values

$$h_0^{(c)} = 2^d,$$

$$h_1^{(c)} = f_0 - 2^d, \text{ and}$$

$$h_{d+1}^{(c)} = (-2)^d \tilde{\chi}(\mathcal{P})$$

where $\tilde{\chi}(\mathcal{P})$ is the reduced Euler characteristic of $\mathcal{P}$; i.e., $\tilde{\chi}(\mathcal{P}) = \sum_{j=0}^{d+1} (-1)^{j-1} f_{j-1}$. Writing the f-vector in terms of the cubical h-vector yields

$$f_{j-1} = 2^{1-j} \sum_{i=1}^{j} \binom{d+1-i}{d+1-j} [h_i^{(c)} + h_{i-1}^{(c)}] \quad \text{for } 1 \leq j \leq d+1.$$

For example, the boundary complex of the n-dimensional cube has a cubical h-vector of $h_0^{(c)} = \cdots = h_n^{(c)} = 2^{n-1}$.

For ease of computation, normalize the Adin h-vector by dividing each $h_i^{(c)}$ by 2^d. Then drop the superscript (c), resulting in $h_0 = 1$ and the relation

$$f_j = 2^{d-j} \sum_{i=0}^{j} \binom{d-i}{d-j} [h_{i+1} + h_i] \quad \text{for } 0 \leq j \leq d. \tag{3}$$

Example 2.3.5. *Consider the cubical complex shown in Fig. 1. Then the Adin h-vector is* $h_0 = 1$, $h_1 = \frac{1}{2}$, $h_2 = 0$, *and* $h_3 = 1$.

Hetyei [35, Theorem 1] showed that any invariant I of d-dimensional cubical complexes, which can be expressed as a linear combination of the face numbers (or entries of the f-vector f_i), may be rewritten as a nonnegative linear combination of the normalized cubical h-vector if and only if the invariant is nonnegative when applied to the d-cube and adding a facet of type $(1, j)$ or $(0, d)$ in any shelling will not decrease I.

Due to this result, other h-vectors such as the toric h-vector [35], and the triangulation h-vector [33], may be expressed as nonnegative linear combinations of the cubical h-vector. Hence, the cubical h-vector is the smallest h-vector. In Chapter 3, the toric h polynomial is written in terms of the Adin h-vector and several properties of the resulting coefficients are examined.

2.3.5 Adin and the *cd* Index

Using the same notation as Ehrenborg and Hetyei [25], let C_n be the cubical lattice of rank n. Then B_n is the face lattice of the $(n-1)$-dimensional simplex Δ^{n-1} while

C_n is the face lattice of the $(n-1)$-cube. Define $U_n := \Phi_{B_n}(c,d)$ and $V_n := \Phi_{C_n}(c,d)$. If $n = 1$, $U_1 = 1 = V_1$.

Definition 2.3.6 (Ehrenborg-Hetyei)**.** *Given a list of the facets $F_1, \ldots, F_k$ of the boundary of the simplex Δ^{n-1} where $k \leq n-1$, then $B_{n,k}$ denotes the semisuspension of the poset $[\hat{0}, F_1] \cup \ldots \cup [\hat{0}, F_k] \cup \{\hat{1}\} \subset B_n$, and the cd-index of $B_{n,k}$ is called $U_{n,k}$. Similarly, given a list of the facets $F_1, \ldots, F_{i+2j}$ of the boundary of the $(n-1)$-cube of type (i,j) where $i > 0$, then $C_{n,i,j}$ denotes the semisuspension of the poset $[\hat{0}, F_1] \cup \ldots \cup [\hat{0}, F_{i+2j}] \cup \{\hat{1}\} \subset C_n$, and the cd-index of $C_{n,i,j}$ is called $V_{n,i,j}$.*

Proposition 2.3.7 (Ehrenborg-Hetyei)**.** *Let $\mathcal{C}$ be an $(n-1)$-dimensional shellable cubical sphere which has a shelling with c-vector $(\ldots, c_{i,j}, \ldots)$. Then the cd-index of the face poset of $\mathcal{C}$ with $\hat{1}$ attached is given by*

$$V_{n+1,1,0} + \sum_{i>0,j} c_{i,j} \cdot (V_{n+1,i+1,j} - V_{n+1,i,j}).$$

They also found that when P is an Eulerian cubical poset of rank $n+1$ there are several additional results that can be found about the cubical h-vector.

Definition 2.3.8 (Ehrenborg-Hetyei)**.** *Let P be an Eulerian cubical poset of rank $n+1$, with f-vector $(f_{-1}, f_0, \ldots, f_n)$. The cubical h-vector can be defined with the following polynomial:*

$$\sum_{l=0}^{n} h_l \cdot x^l = \frac{1 + x^{n+1} + \sum_{k=0}^{n-1} f_k \cdot x^{k+1} \cdot ((1-x)/2)^{n-1-k}}{1+x}.$$

Theorem 2.3.9 (Ehrenborg-Hetyei)**.** *Let P be an Eulerian cubical poset of rank $n+1$,* *with h-vector* $(h_0,\ldots,h_n)$*. Then*

$$\Phi_P(c,d) = h_0 \cdot V_{n+1,1,0} + \sum_{l=1}^{n-1} h_l \cdot (V_{n+1,1,l} - V_{n+1,1,l-1}).$$

Following the notation in [39], define $t_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} x^{n-2k}$ for $n \geq 0$ where $t_{-1}(x) = 0$. Then obviously, $x^n = t_n(x) - t_{n-2}(x)$ for $n \geq 2$ and $x^n = t_n(x)$ for $n = 0, 1$. Define the functions $\mathcal{C} : \mathbb{Q}[x] \to \mathbb{Q}[x]$ and $\mathcal{D} : \mathbb{Q}[x] \to \mathbb{Q}[x]$ to be $\mathcal{C}(t_n(x)) = t_{n+1}(x) - t_{n-1}(x)$ and $\mathcal{D}(t_n(x)) = \delta_{n,0}$ [39, Corollary 5.3] where $\delta_{n,0}$ is the Kronecker delta function; i.e., $\delta_{n,0}$ equals one if $n = 0$ and zero otherwise.

Reversing the cd-index, denoted by a superscript of rev, means every c is replaced with $\mathcal{C}$ and d with $\mathcal{D}$. Next reverse the order of the $\mathcal{C}$'s and $\mathcal{D}$'s in each term. Since B_n is self-dual, $U_n = U_n^{rev}$. To calculate $U_n^{rev}(1)$ and $V_n^{rev}(1)$, let $x = 1$ in each function $\mathcal{C}$ and $\mathcal{D}$. Using $t_n(x)$, it is easy to see that $U_n^{rev}(1) = t_{n-1}(x)$.

One can write U_n and V_n in terms of the short toric polynomial. Using the fact that $U_1 = 1$ and the formula $U_{n+2} = c \cdot U_{n+1} + \sum_{i=1}^{n} \binom{n}{i} U_i \cdot d \cdot U_{n+1-i}$, all of the U_n can be generated. See Table 5.

n	U_n	U_n^{rev}	$U_n^{rev}(1)$
1	1	1	1
2	c	$\mathcal{C}$	x
3	$c^2 + d$	$\mathcal{C}^2 + \mathcal{D}$	$x^2 + 1$
4	$c^3 + 2cd + 2dc$	$\mathcal{C}^3 + 2\mathcal{D}\mathcal{C} + 2\mathcal{C}\mathcal{D}$	$x^3 + x$
5	$c^4 + 3c^2d + 5cdc + 3dc^2 + 4d^2$	$\mathcal{C}^4 + 3\mathcal{D}\mathcal{C}^2 + 5\mathcal{C}\mathcal{D}\mathcal{C} + 3\mathcal{C}^2\mathcal{D} + 4\mathcal{D}^2$	$x^4 + x^2 + 1$

Table 5: U_n for small n

Using $V_1 = 1$ and $V_{n+2} = V_{n+1} \cdot c + \sum_{i=0}^{n-1} \binom{n}{i} 2^{n-i} V_{i+1} \cdot d \cdot U_{n-i}$ [25], all of the V_n can be generated. See Table 6. $V_n^{rev}(1) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} r_{n-1-2k} \cdot t_{n-1-2k}(x)$ where r_{n-1-2k}

is the coefficient of the polynomials $t_{n-1-2k}(x)$. Notice these coefficients r_{n-1-2k} are equal to $[x^k]g(L_{n-1}, x)$. This is consistent with [39, Theorem 5.4]. See Table 7. Using the formula in [39, Theorem 6.6], $V_{n+1}^{rev}(1) = t(L_n, x)$. This is because for the n-cube $h_i = \binom{n}{i}$.

n	V_n	V_n^{rev}	$V_n^{rev}(1)$
1	1	1	1
2	c	$\mathcal{C}$	x
3	$c^2 + 2d$	$\mathcal{C}^2 + 2\mathcal{D}$	$x^2 + 2$
4	$c^3 + 4cd + 6dc$	$\mathcal{C}^3 + 4\mathcal{D}\mathcal{C} + 6\mathcal{C}\mathcal{D}$	$x^3 + 5x$
5	$c^4 + 6c^2d + 16cdc$ $+14dc^2 + 20d^2$	$\mathcal{C}^4 + 6\mathcal{D}\mathcal{C}^2 + 16\mathcal{C}\mathcal{D}\mathcal{C}$ $+14\mathcal{C}^2\mathcal{D} + 20\mathcal{D}^2$	$x^4 + 12x^2 + 14$

Table 6: V_n for small n

n	$V_n^{rev}(1)$	$t_k(x)$	$t(L_n, x)$
1	1	$t_0(x)$	$t(L_1, x)$
2	x	$t_1(x)$	$t(L_2, x)$
3	$x^2 + 2$	$t_2(x) + t_0(x)$	$t(L_3, x)$
4	$x^3 + 5x$	$t_3(x) + 4t_1(x)$	$t(L_4, x)$
5	$x^4 + 12x^2 + 14$	$t_4(x) + 11t_2(x) + 2t_0(x)$	$t(L_5, x)$

Table 7: $V_n^{rev}(1)$ in terms of the short toric polynomials

2.3.6 The Toric Contribution of $c_{i,j}$

Let $\mathcal{P}$ be a d-dimensional cubical complex. Consider a shelling $F_1, \ldots, F_m$ of $\mathcal{P}$, and let (i, j) be the type of the t^{th} facet in the shelling. According to Adin, $\sum_k h_k x^k = \sum_t \Delta_t h(x)$, where $\Delta_t h(x)$ is the contribution from F_t. Adin [2, Equation (33)] defines this contribution as

$$\Delta_t h(x) = \begin{cases} (\frac{1}{2})^i x^{j+1}(1+x)^{i-1} & \text{if } i \geq 1, \\ 1 & \text{if } i = 0 \text{ and } j = 0 \\ x^{d+1} & \text{if } i = 0 \text{ and } j = d. \end{cases}$$

When $i \geq 1$, Lemma 2.3.3 implies that $1 \leq i \leq d-j$ and $0 \leq j \leq d-1$. Thus,

$$\begin{aligned} \sum_k h_k x^k &= 1 + c_{0,d} \cdot x^{d+1} + \sum_{j=0}^{d-1} \sum_{i=1}^{d-j} c_{i,j} \left(\frac{1}{2}\right)^i x^{j+1}(1+x)^{i-1} \\ &= 1 + c_{0,d} \cdot x^{d+1} + \sum_{j=0}^{d-1} \sum_{i=1}^{d-j} c_{i,j} \left(\frac{1}{2}\right)^i x^{j+1} \sum_{m=0}^{i-1} \binom{i-1}{m} x^m. \end{aligned}$$

Set $k = m + j + 1$, and reorder the summations to get

$$\sum_k h_k x^k = 1 + c_{0,d} \cdot x^{d+1} + \sum_{k=1}^{d} \sum_{j=0}^{k-1} \sum_{i=k-j}^{d-j} c_{i,j} \left(\frac{1}{2}\right)^i \binom{i-1}{k-j-1} x^k.$$

This is equivalent to

$$h_k = \sum_{j=0}^{k-1} \sum_{i=k-j}^{d-j} \left(\frac{1}{2}\right)^i \binom{i-1}{k-1-j} c_{i,j} \quad \text{for } 1 \leq k \leq d. \tag{4}$$

Now consider the f polynomial of $\mathcal{P}$. According to Ehrenborg and Hetyei [25], $\sum_{k=0}^{d} f_k \cdot x^k = \sum_{i,j} c_{i,j} \cdot (x+2)^{d-i-j} \cdot (x+1)^i x^j$. Hence, the contribution of $c_{i,j}$ to the f polynomial is $(x+2)^{d-i-j} \cdot (x+1)^i x^j$. Call this $\tilde{c}_{i,j}(x)$.

Consider the following basis for the polynomials $\tilde{c}_{i,j}(x)$: $\tilde{c}_{0,0}(x)$, $\tilde{c}_{0,d}(x)$, and $\tilde{c}_{1,j}(x)$ for $0 \leq j \leq d-1$. Every $\tilde{c}_{i,j}(x)$ can be written in terms of $\tilde{c}_{1,j}(x)$, namely $\tilde{c}_{i,j}(x) = \left(\frac{1}{2}\right)^{i-1} \sum_{k=0}^{i-1} \binom{i-1}{k} \tilde{c}_{1,j+k}(x)$, which is easily seen from $\tilde{c}_{i,j}(x) = \frac{1}{2}\left[\tilde{c}_{i-1,j+1}(x) + \tilde{c}_{i-1,j}(x)\right]$.

This last equality is easily proven from the definition of $\tilde{c}_{i,j}(x)$. Then

$$\begin{aligned}\sum_{k=0}^{d} f_{k-1}x^k &= \tilde{c}_{0,0}(x) + \tilde{c}_{0,d}(x) + \sum_{j=0}^{d-1}\sum_{i=1}^{d-j} c_{i,j}\cdot \tilde{c}_{i,j}(x)\\ &= \tilde{c}_{0,0}(x) + \tilde{c}_{0,d}(x) + \sum_{j=0}^{d-1}\sum_{i=1}^{d-j} c_{i,j}\cdot \left(\frac{1}{2}\right)^{i-1}\sum_{k=0}^{i-1}\binom{i-1}{k}\tilde{c}_{1,j+k}(x)\\ &= \tilde{c}_{0,0}(x) + \tilde{c}_{0,d}(x) + \sum_{k=0}^{d-1}\tilde{c}_{1,k}(x)\sum_{j=0}^{k}\sum_{i=k+1-j}^{d-j} c_{i,j}\left(\frac{1}{2}\right)^{i-1}\binom{i-1}{k-j}.\end{aligned}$$

Let $a_{d,k}$ be the coefficient of $\tilde{c}_{1,k}$. Then $a_{d,k} = \sum_{j=0}^{k}\sum_{i=k+1-j}^{d-j} c_{i,j}\left(\frac{1}{2}\right)^{i-1}\binom{i-1}{k-j}$. Comparing this to the Adin h-vector found above yields $a_{d,k} = 2\cdot h_{k+1}$, which should not be surprising since this is an explicit case of [35, Theorem 1] and was commented on in the proof of that theorem.

2.4 Noncrossing Partitions

Suppose π is a partition of $[1,d] := \{1,\ldots,d\}$. If whenever $1 \le a < b < c < e \le d$ where a and c are in the same block of the partition and b and e are also in one block implies that all four elements must be in the same block of π, the partition π is called noncrossing. Let $NC(d)$ denote the set of all noncrossing partitions of $[1,d]$. Then it is known that $|NC(d)| = C_d = \frac{1}{d+1}\binom{2d}{d}$. See [60] or [69].

There are several ways to represent (noncrossing) partitions on $[1,d]$ visually. The two representations used here are linear and circular. To create a linear representation or an arc diagram of a noncrossing partition, see Chapter 2.5.

To create a circular representation (see Fig. 7), place d points around a circle. Label the points 1 through d, increasing clockwise. Connect any two (cyclically) consecutive elements in a block of the partition with a chord. In the circular representation, nonsingleton blocks are chords or polygons.

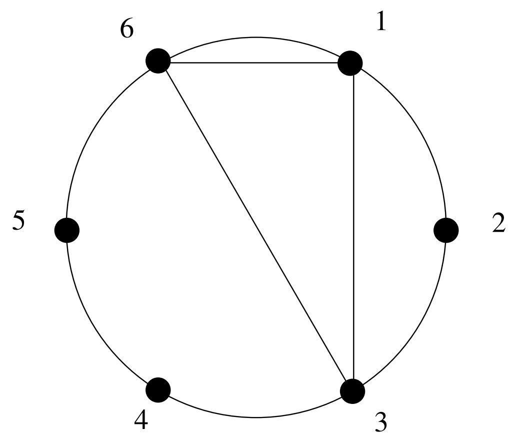


Figure 7: The circular representation of $\pi = (136)(2)(4)(5)$

A partition of $[1, d]$ is noncrossing if and only if no chord crosses another in the diagram. For example, let $\pi = (1)(24)(356)$. This partition is not noncrossing since (24) and (356) are in separate blocks. The 3 and 4 cause the noncrossing condition to fail and, in the circular representation, cause the chord representing (24) to cross the triangle representing (356). See Fig. 8.

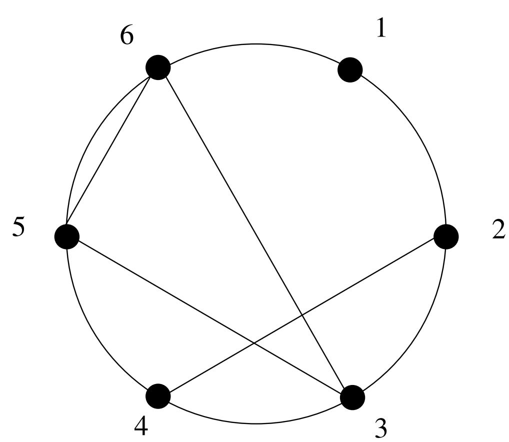


Figure 8: The circular representation of $\pi = (1)(24)(356)$

In a noncrossing partition, consider three special types of elements. The first are *singleton* elements. In a nonsingleton block, call the largest element of the block the *last element*. Obviously, the number of last elements in a partition is exactly the number of its nonsingleton blocks. Lastly, suppose both k and $k+1$ are in the same block of a partition; in this case, call the element k an *antisingleton* element, and call the element d an antisingleton if and only if both d and 1 are in the same block of the partition. This naming convention follows from the duality of the lattice of $NC(d)$. See Lemma 2.4.6 and its proof for more details.

Let $\text{block}(\pi)$ equal the number of nonsingleton blocks in the partition π and $\text{sing}(\pi)$ equal the number of singleton elements in π. Then the total number of components of the partition is $|\pi| = \text{block}(\pi) + \text{sing}(\pi)$. For example, if $\pi = (136)(2)(4)(5)$, then $\text{block}(\pi) = 1$, $\text{sing}(\pi) = 3$, and $|\pi| = 4$.

2.4.1 The Kreweras Complement and the Simion-Ullman Involution

Kreweras [46] showed that $NC(d)$ forms a lattice ordered by refinement. Suppose $\pi, \sigma \in NC(d)$. Then $\pi \leq \sigma$ under refinement if every block of π is contained in a block of σ. For example, $(12)(347)(56) \leq (1256)(347)$. The rank function of $NC(d)$ is $\text{rank}(\pi) = d - |\pi|$. Noncrossing partitions are also related to the Narayana numbers. For $1 \leq k \leq d$, $NC_d(k) := |\{\pi \in NC(d) : \text{rank}(\pi) = d - k\}| = \frac{1}{d}\binom{d}{k}\binom{d}{k-1}$. This is consistent with $|NC(d)| = C_d$ since

$$
\begin{aligned}
|NC(d)| = \sum_{k=1}^{d} NC_d(k) = \sum_{k=1}^{d} \frac{1}{d}\binom{d}{k}\binom{d}{k-1} &= \frac{1}{d}\sum_{k=1}^{d}\binom{d}{k}\binom{d}{d+1-k} \\
&= \frac{1}{d}\binom{2d}{d+1} = \frac{1}{d+1}\binom{2d}{d} = C_d.
\end{aligned}
$$

Kreweras also showed the lattice of $NC(d)$ to be self-dual. See Fig. 9 for the lattice when $d = 4$. The Kreweras complement has the important property that for $\pi \in NC(d)$, $|\pi| + |K(\pi)| = d + 1$ [46]. He used what later became known as the Kreweras complement to show this property.

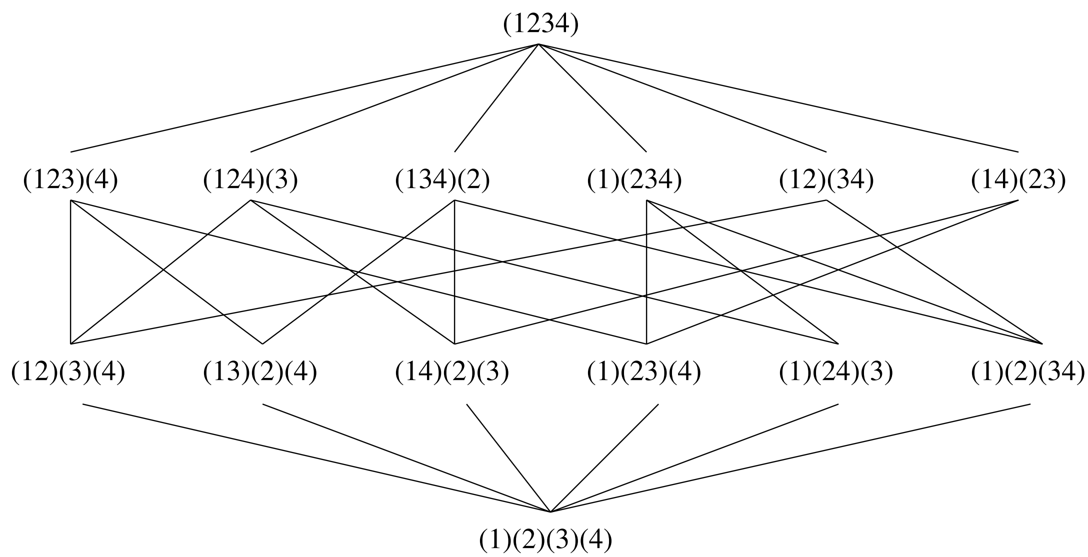


Figure 9: The lattice of $NC(4)$

Given some $\pi \in NC(d)$, define its Kreweras Complement as follows. Consider the circular representation of π. On the same circle, label the midpoints between each existing point $\bar{1}$ through $\bar{d}$, where $\bar{i}$ is the midpoint of the arc determined by $(i, i+1)$. $K(\pi)$ is the coarsest noncrossing partition of $[\bar{1}, \bar{d}]$ whose chords do not cross any chord of π [46, 53]. See Fig. 10.

Set $\tau = (12 \ldots d)$ to be the maximum element of $NC(d)$ and $\rho = (1)(2) \ldots (d)$ to be the minimum element of $NC(d)$. McCammond [48] showed the following proposition.

Proposition 2.4.1 (McCammond)**.** *If $\rho < \sigma < \tau$ in $NC(d)$ and $(\alpha_1, \ldots, \alpha_n)$ is a sequence of transpositions labeling a saturated chain from ρ to σ, then there is a unique element σ' in $NC(d)$ and a saturated chain from σ' to τ with the exact same*

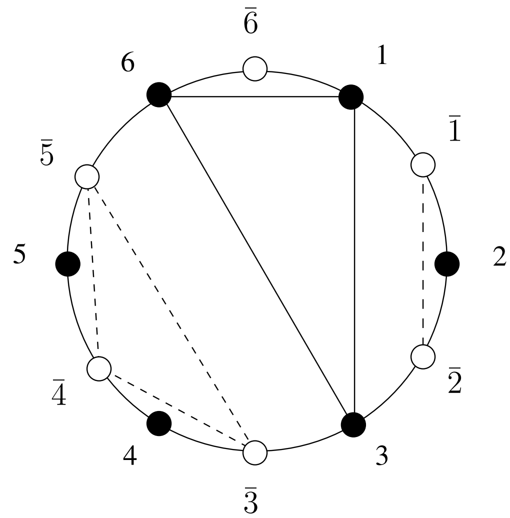


Figure 10: The circular representation of $\pi = (136)(2)(4)(5)$ with $K(\pi) = (12)(345)(6)$

sequence of labels.

Similarly, if $(\beta_1, \ldots, \beta_n)$ is a sequence of transpositions labeling a saturated chain from σ to τ, then there is a unique element σ'' in $NC(d)$ and a saturated chain from ρ to σ'' with the exact same sequence of labels.

In Proposition 2.4.1, σ'' will be $K(\sigma)$ and σ' will be $K(\sigma)$ adjusted such that one is added to each element of $K(\sigma)$ in circular order. If the original element in $K(\pi)$ is d, then the associated element in σ' is 1.

Example 2.4.2. *Let $\sigma = (1)(235)(4)(6)$. Then $K(\sigma) = (156)(2)(34)$, and there is a sequence of transpositions $(\alpha_1, \alpha_2) = ((25), (23))$. See Fig. 11.*

Illustrating the second half of the proposition, there is a sequence is $(\beta_1, \beta_2, \beta_3) = ((56), (34), (16))$. See Fig. 12.

While the Kreweras Complement is an order-reversing isomorphism of $NC(d)$ which is quite useful, it is not an involution [53]. For example, if $\pi = (136)(2)(4)(5)$, then

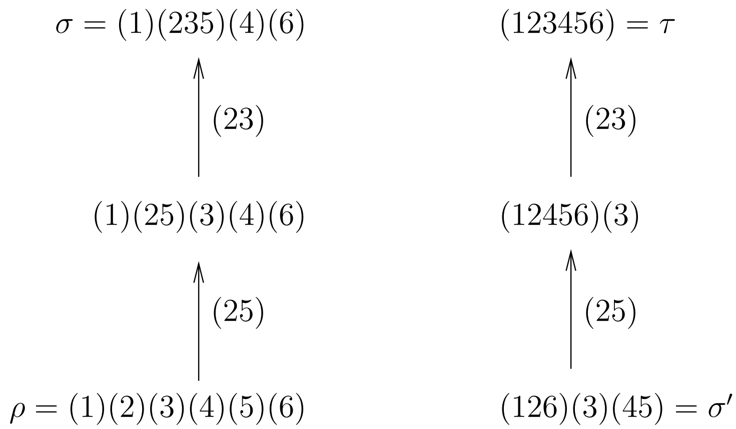


Figure 11: Applying the transpositions $(\alpha_1, \alpha_2) = ((25), (23))$

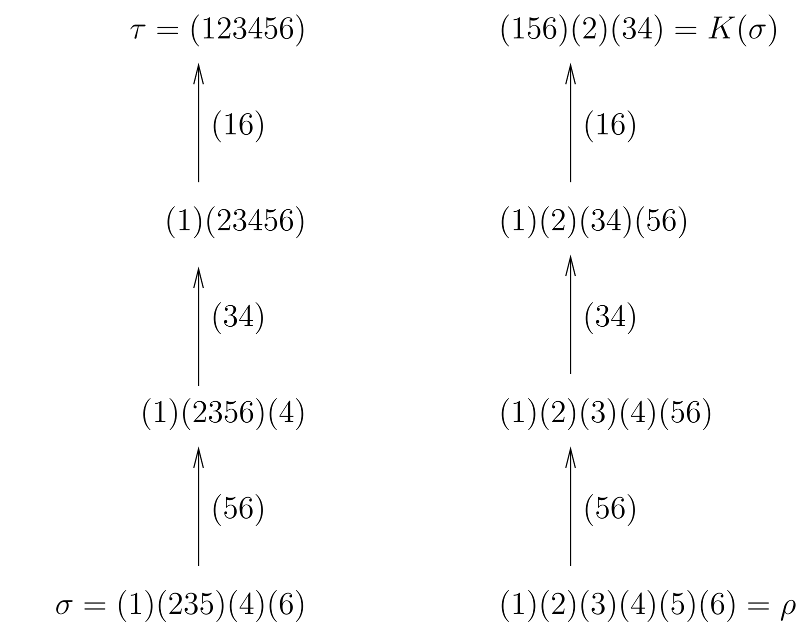


Figure 12: Applying the transpositions $(\beta_1, \beta_2, \beta_3) = ((56), (34), (16))$

$K(K(\pi)) = (1)(256)(3)(4)$. Simion and Ullman [59] gave an alternative proof for the self duality of $NC(d)$, which used the circular representation of partitions, and the involution $\alpha : NC(d) \to NC(d)$ that they defined. This involution is the Kreweras Complement relabeled, and it will be used throughout the rest of the dissertation instead of $K(\pi)$.

The involution $\alpha(\pi)$ may be defined in the following way. Take any $\pi \in NC(d)$, and represent it circularly. Label the midpoint of the arc determined by $(d-1, d)$ as $1'$. Subdivide each arc $(i, i+1)$ by placing a new point between the existing points on the circle. Label this new point $(d-i)'$. $1'$ through d' increase counterclockwise, and d' is the midpoint of the arc $(d, 1)$. Then $\alpha(\pi)$ can be represented on the same circle as π by the coarsest noncrossing partition on $[1', d']$, which does not intersect any of the chords of π. In Fig. 13, π is represented by the solid lines and points and $\alpha(\pi)$ by the dashed lines and open points.

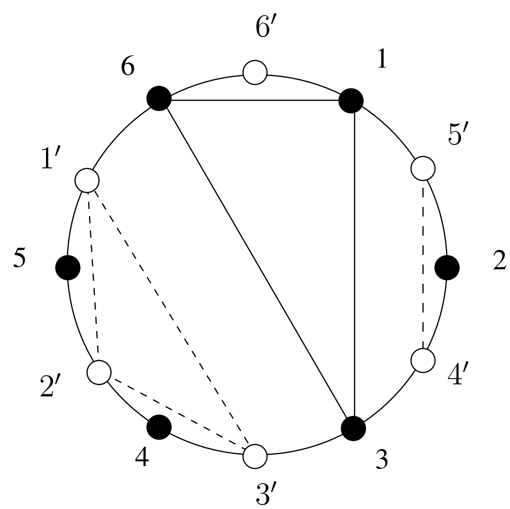


Figure 13: The circular representation of $\pi = (136)(2)(4)(5)$ and $\alpha(\pi) = (123)(45)(6)$

For the rest of this dissertation, k will be used to represent k' as an element of $\alpha(\pi)$ unless it is unclear in the context as to whether the element k is in $\alpha(\pi)$ or π.

The involution α may also be defined without using the construction above [59]. Pick any $\pi \in NC(d)$, and let $k = |\pi|$. Define $\alpha(\pi)$ to be the noncrossing partition of $[1, d]$ such that the following condition holds. Let $i < j$. Then i and j are in the same block of $\alpha(\pi)$ if and only if no block of π contains elements $l < k$ which have the property $i \leq d-k < j \leq d-l$ or the property $d-k < i \leq d-l < j$. This is equivalent to saying that no chord of $\alpha(\pi)$ may cross a chord of π.

Proposition 2.4.1 can be rewritten in terms of α. Since α is an involution, both σ' and σ'' will be become α. Define an involution $\phi : (a,b) \to (c,e)$ where $a < b$ are elements of $[1,d]$. If $b \neq d$, set $c = d-b$ and $e = d-a$. If $b = d$, then $c = d-a$ and $e = d$.

Proposition 2.4.3. *If $\rho < \pi < \tau$ in $NC(d)$ and $(\beta_1, \ldots, \beta_n)$ is a sequence of transpositions labeling a saturated chain from ρ to π where $\beta_n \ldots \beta_1 \pi = \rho$, then the sequence $(\phi(\beta_1), \ldots, \phi(\beta_n))$ of transpositions labels a saturated chain from $\alpha(\pi)$ to τ where $\tau = \phi(\beta_n) \ldots \phi(\beta_1)\alpha(\pi)$.*

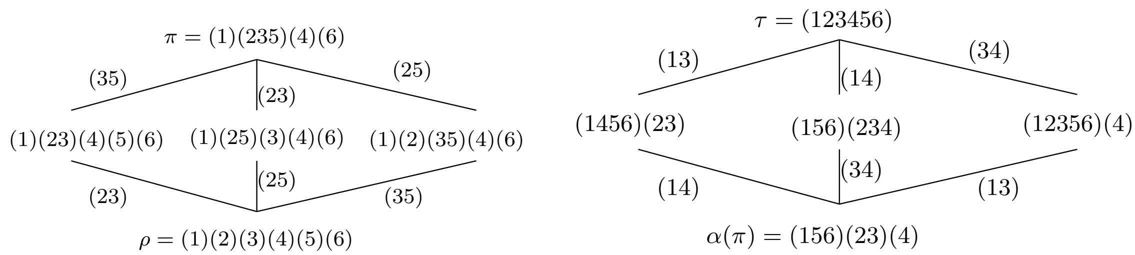
Example 2.4.4. *Suppose $\pi = (134)(2)(56)$; then $\alpha(\pi) = (1)(26)(3)(45)$. There is a sequence $(\beta_1, \beta_2) = ((23),(25))$ such that $(25)(23)\pi = \rho$. Then $(\phi(\beta_1), \phi(\beta_2)) = ((34),(14))$ where $(14)(34)\alpha(\pi) = \tau$.*

The sequence $(\beta_1, \ldots, \beta_n)$ is not unique. For example, see Fig. 14 where the prefix labels along the paths in Fig. 14(a) correspond to the dual prefix labels along the paths in Fig. 14(b) by way of the involution ϕ. See Table 8.

prefix	dual prefix
(23)	(34)
(25)	(14)
(35)	(13)

Table 8: The prefixes β_i and the corresponding dual prefixes $\phi(\beta_i)$

In Simion and Ullman's [59] proof of the self-duality of the lattice of $NC(d)$, the



(a) The sublattice from ρ to π

(b) The sublattice from $\alpha(\pi)$ to τ

Figure 14: Multiple options for $(\beta_1, \ldots, \beta_n)$

involution α was defined such that if $\pi \in NC(d)$ has k blocks (either singleton or nonsingleton blocks) then $\alpha(\pi)$ has $d+1-k$ blocks. Lemma 2.4.5 follows immediately.

Lemma 2.4.5. *Let $\pi \in NC(d)$ for some $d > 0$. Then $|\pi| + |\alpha(\pi)| = d + 1$.*

Lemma 2.4.6. *Let $\pi \in NC(d)$. Then $k \in [1, d-1]$ is an antisingleton element of π if and only if $d - k$ is a singleton element of $\alpha(\pi)$.*

Proof. If k is an antisingleton of π, $k+1$ must be in the same block of π as k. Thus, in the circular representation of π, k and $k+1$ are connected by a chord. By definition, $\alpha(\pi)$ is the coarsest noncrossing partition on $[1, d]$ whose chords do not cross any chords of π. Since $d - k$ is the element of $\alpha(\pi)$ that is located between k and $k+1$ of π, $d - k$ must be a singleton of $\alpha(\pi)$.

The converse is proved similarly. □

Remark 2.4.7. *Following the same method as in the proof of Lemma 2.4.6, one can show that d is a singleton element of $\alpha(\pi)$ if and only if d an antisingleton element of π.*

For more information on noncrossing partitions and their applications, see [16], [58], [68], and [66].

2.4.2 Special Elements for Noncrossing Partitions

Definition 2.4.9 defines a weight function in terms of a family $\mathcal{S}$ of i pairwise disjoint subsets of $[1, d]$ such that each element in $\mathcal{S}$ is a set of consecutive integers of $[1, d]$ in circular order. An element of $\mathcal{S}$ is denoted by $[k, l] := \{k, \ldots, l\}$ and is called an interval of $\mathcal{S}$. In particular, if $k = l$, the interval consists of a single element. If $1 \leq k < l \leq d$, then $[k, l]$ is a set of increasing consecutive integers. Call $[k, l] = \{k, \ldots, d, 1, \ldots, l\}$ a *wrapped* interval when $k > l$. Notice the last element of a wrapped interval, namely l, will not be the largest element of the interval. Let $[1, d]^*$ denote the special wrapped interval consisting of all of the elements 1 through d; whereas, $[1, d]$ is the non-wrapped interval.

The family of sets $\mathcal{S}$ is defined in the following way. Let $\mathcal{S} = \{[k_1, l_1], \ldots, [k_i, l_i]\}$ where $1 \leq k_1 \leq l_1 < k_2 \leq l_2 < \ldots < k_i \leq d$ and either $k_i \leq l_i \leq d$ or $1 \leq l_i < k_1$. If $1 \leq l_i < k_1$, then the last interval in $\mathcal{S}$ is wrapped. Also, only the last interval may be wrapped. Define j to be the number of elements of $[1, d]$ that are not contained in any interval of $\mathcal{S}$.

$\mathcal{S}$ may be represented visually using the same circular representation that was defined earlier. The intervals of $\mathcal{S}$ correspond to sets of consecutive elements; see Fig. 15. Let B be the union of the regions bounded by the arcs corresponding to the intervals of $\mathcal{S}$. Call the remaining region A. In other words, for each $[k, l] \in \mathcal{S}$, draw an arc such that the elements of $[k, l]$ are on one side of the arc and all other elements of $[1, d]$ are on the opposite side of the arc.

Define $[1, d] - \mathcal{S}$ to be the set of intervals determined by the longest sequence of

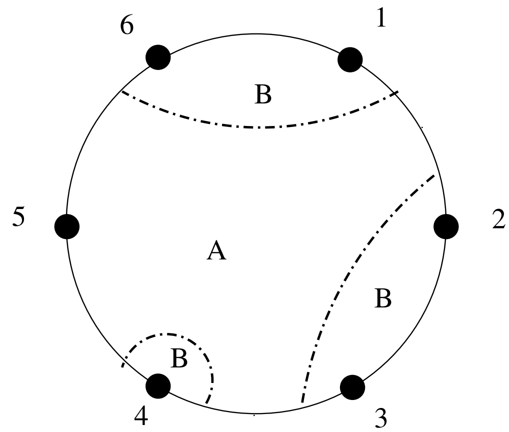


Figure 15: The circular representation of $\mathcal{S} = \{[2,3],[4],[6,1]\}$ and $[1,6] - \mathcal{S} = \{[5]\}$

consecutive elements in $[1,d]$ located between the intervals of $\mathcal{S}$ in circular order. If $\mathcal{S}$ is defined as above, then $[1,d]-\mathcal{S} = \{[l_1+1,k_2-1],[l_2+1,k_3-1],\ldots,[l_i+1,k_1-1]\}$, where $[l_n+1,k_{n+1}-1]$ exists for $1 \leq n < i$ if and only if $l_n < k_{n+1}-1$, and $[l_i+1,k_1-1]$ exists if and only if $l_i + 1 \not\equiv k_1$ in circular order. Also, if $l_i = d$, then $l_i + 1 \equiv 1$, and if $k_1 = 1$, then $k_1 - 1 \equiv d$. If $d \leq l_i < k_1 - 1$, then $[l_i + 1, k_1 - 1]$ is the first interval of $[1,d] - \mathcal{S}$ in the same sense as in the ordering of the intervals of $\mathcal{S}$; otherwise, it is considered the last interval in the set $[1,d] - \mathcal{S}$.

Notice that $[1,d]-\mathcal{S}$ has the same number of intervals as $\mathcal{S}$ exactly when, in circular order, each interval of $\mathcal{S}$ is directly followed by an element of $[1,d]$ that is not in any interval of $\mathcal{S}$; i.e., $l_n + 1 < k_{n+1}$ for $n \geq 1$. Otherwise, $[1,d] - \mathcal{S}$ will contain fewer intervals than $\mathcal{S}$. See Fig. 15. How j is defined implies that $[1,d] - \mathcal{S}$ will always contain a total of j elements from $[1,d]$ in its intervals.

The set $\mathcal{S}$ consists of i intervals, which contain a total of $d - j$ elements. Define a related family of intervals $\mathcal{S}' = \{[d - k_i + 1, d - l_{i-1}], \ldots, [d - k_2 + 1, d - l_1], [d - k_1 + 1, d - l_i]\}$. Since $k_1 \leq l_1 < k_2 \leq l_2 < \ldots < k_i$, then $d - k_i + 1 \leq d - l_{i-1} <$

$d - k_{i-1} + 1 \leq d - l_{i-2} < \ldots < d - k_1 + 1$. $\mathcal{S}'$ contains i pairwise disjoint intervals and a total of $i + j$ elements. See below for a justification. In particular, if $\mathcal{S} = \emptyset$, then $\mathcal{S}' = \{[1, d]^*\}$ and vice versa.

Let $\mathcal{S}$ be as above. Then $\mathcal{S}$ could also be defined as a subset of $\{1, 1', \ldots, d, d'\}$, which consists of strings of consecutive elements where the first and last element of each string must be from $\{1, 2, \ldots, d\}$. Considered in this light, $\mathcal{S}'$ is exactly the complement of $\mathcal{S}$. Then each string of elements in $\mathcal{S}'$ will begin and end with an element from $\{1', 2', \ldots, d'\}$.

Define β to be the operation that takes a given $\mathcal{S}$ and transforms it into the associated $\mathcal{S}'$; i.e., $\beta(\mathcal{S}) = \mathcal{S}'$. Then β is an involution since given $\mathcal{S}$ and $\mathcal{S}'$ as defined above $\beta(\mathcal{S}') = \{[d - (d - k_1 + 1) + 1, d - (d - l_1)], \ldots, [d - (d - k_{i-1} + 1) + 1, d - (d - l_{i-1})], [d - (d - k_i + 1) + 1, d - (d - l_i)]\} = \mathcal{S}$.

$\mathcal{S}$ and $\mathcal{S}'$ may be represented visually on the same circle. Place $[1, d]$ and $[1', d']$ on a circle as before with π and $\alpha(\pi)$. For each $[k_n, l_n] \in \mathcal{S}$, insert an arc, whose endpoints are adjacent to k_n and l_n, in the way described above. Call the union of the regions determined by these arcs B. Let the remaining region be A. $\mathcal{S}'$ is the set of intervals determined by the longest list of consecutive elements of $[1', d']$ in region A, which do not skip over any region of B. See Fig. 16.

Obviously, $\mathcal{S}'$ consists of i intervals. To see that $\mathcal{S}'$ contains a total of $i+j$ elements, consider $[1, d] - \mathcal{S}'$ where each interval in this set corresponds to the elements of $[1', d']$ contained in a single arc of region B. For $1 \leq n \leq i$, the region determined by $[k_n, l_n] \in \mathcal{S}$ is associated to $[d - l_n + 1, d - k_n] \in [1, d] - \mathcal{S}'$ when $k_n < l_n$. If $k_n = l_n$, there is no associated interval in $[1, d] - \mathcal{S}'$. If $k_i \leq l_i \leq d$, then $[d - l_i + 1, d - k_i]$

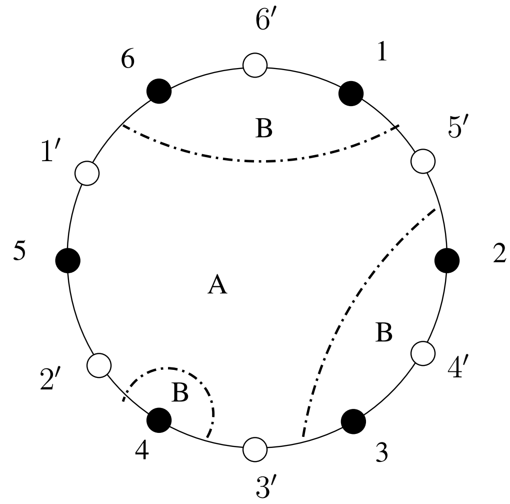


Figure 16: The circular representation of $\mathcal{S} = \{[2,3],[4],[6,1]\}$ with $\mathcal{S}' = \{[1,2],[3],[5]\}$

becomes the first interval in $[1,d]-\mathcal{S}'$ in the same sense as in the order of the intervals of $\mathcal{S}$; otherwise, it is the last.

Since the intervals in $[1,d]-\mathcal{S}'$ correspond to the elements of $[1',d']$ within the region B, these intervals have exactly one less element than their associated intervals in $\mathcal{S}$. Because there is a total of $d-j$ elements contained in the intervals of $\mathcal{S}$, there must be a total of $d-j-i$ elements in the intervals of $[1,d]-\mathcal{S}'$. Thus, there is a total of $i+j$ elements in the intervals of $\mathcal{S}'$. For example, in Fig. 16, $[1,6]-\mathcal{S}' = \{[4],[6]\}$.

Lemma 2.4.8. *Let $\mathcal{S}$ be as defined above and $\mathcal{S} \neq \{[1,d]^*\}$ or $\emptyset$. If $l_i \neq d$, then exactly one interval of $\mathcal{S}$ and $\mathcal{S}'$ is wrapped. If $l_i = d$, then no interval of $\mathcal{S}$ or $\mathcal{S}'$ is wrapped.*

Proof. By the numbering convention of the intervals of $\mathcal{S}$, only the last interval $[k_i,l_i]$ may be wrapped. This is also true for $\mathcal{S}'$. Let $l_i \neq d$. Suppose $[k_i,l_i]$ is not wrapped. Then $1 \leq k_1 < l_i < d$, which implies that $1 \leq d-l_i < d-k_1+1 \leq d$. This is equivalent to the last interval of $\mathcal{S}'$ $[d-k_1+1, d-l_i]$ being wrapped. These implications are all reversible; hence, $[k_i, l_i]$ is not wrapped if and only if $[d-k_1+1, d-l_i]$ is wrapped. The result that $[k_i, l_i]$ is wrapped if and only if $[d-k_1+1, d-l_i]$ is not wrapped is proved similarly.

If $l_i = d$, then the last interval of $\mathcal{S}$ is $[k_i, d]$, and the last interval of $\mathcal{S}'$ is $[d-k_1+1, d]$. Neither of which is wrapped. □

If $\mathcal{S} = \{[1, d]^*\}$, then $\mathcal{S}' = \emptyset$. Obviously, in this situation, exactly one of the intervals of $\mathcal{S}$ and $\mathcal{S}'$ is wrapped. The situation is similar when $\mathcal{S} = \emptyset$ and $\mathcal{S}' = \{[1, d]^*\}$. This lemma can also be proved using the properties of the circular representations of $\mathcal{S}$ and $\mathcal{S}'$. See below for the alternative proof.

alternative proof. By the definition of $\mathcal{S}$, only the last interval $[k_i, l_i]$ may be wrapped. This is also true of $\mathcal{S}'$. Let $l_i \neq d$. Suppose $[k_i, l_i]$ is wrapped. This only occurs if, in the circular representation, d' is within the arc determined by $[k_i, l_i]$ since $\{d, 1\} \subseteq [k_i, l_i]$. Hence, d' is in $[1, d] - \mathcal{S}'$. This is equivalent to no interval in $\mathcal{S}'$ being wrapped. Note, in the circular notation, $l_i \neq d$ is equivalent to $d - l_i \not\equiv d$. Thus, each of the above implications are reversible; hence, $[k_i, l_i] \in \mathcal{S}$ is wrapped if and only if $[d-k_1+1, d-l_i] \in \mathcal{S}'$ is not wrapped. The result that $[k_i, l_i]$ is not wrapped if and only if $[d-k_1+1, d-l_i]$ is wrapped is proved similarly.

If $l_i = d$, then the last interval of $\mathcal{S}$ is $[k_i, d]$, which is not wrapped. Then the last interval of $\mathcal{S}'$ is $[d-k_1+1, d]$ which is also not wrapped. □

Given some partition of $[1, d]$ and an $\mathcal{S}$ as defined above, a singleton element or a last element is called *in* $\mathcal{S}$ if the element is contained in some interval of $\mathcal{S}$. An antisingleton element k is called *in* $\mathcal{S}$ when the pair $\{k, k+1\}$ is contained in a single interval of $\mathcal{S}$. Also, if d is an antisingleton element of the partition, it is considered *in* $\mathcal{S}$ if and only if the last interval of $\mathcal{S}$ is wrapped. Note, the only difference between the intervals $[1, d]^*$ and $[1, d]$ is that d may be an antisingleton in $[1, d]^*$ but not in $[1, d]$.

2.4.3 Weight Functions

Definition 2.4.9. *Let $\pi \in NC(d)$ and $\mathcal{S}$ as defined above. Define the weight function $wt_{\mathcal{S}}(\pi)$ as follows. Singleton and antisingleton elements have a weight of x if they are in $\mathcal{S}$ and a weight of 1 otherwise. Nonsingleton blocks have a weight of x.*

Example 2.4.10. *Let $\pi = (136)(2)(4)(5)$ and $\mathcal{S} = \{[2,3],[4],[6,1]\}$. Then $\alpha(\pi) = (123)(45)(6)$ and $\mathcal{S}' = \{[1,2],[3],[5]\}$. See Fig. 17. Hence, using the definition of the weight function, $wt_{\mathcal{S}}(\pi) = x^4$ and $wt_{\mathcal{S}'}(\alpha(\pi)) = x^3$.*

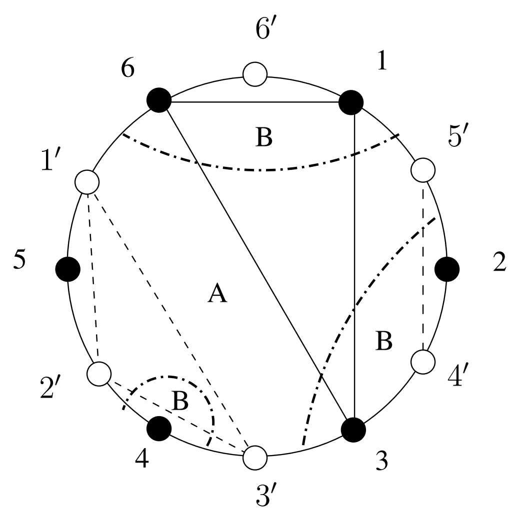


Figure 17: The circular representation of $\pi = (136)(2)(4)(5)$ and $\mathcal{S} = \{[2,3],[4],[6,1]\}$ with $wt_{\mathcal{S}}(\pi) = x^4$ and $wt_{\mathcal{S}'}(\alpha(\pi)) = x^3$

If $\mathcal{S}$ consists of a single interval, namely $\mathcal{S} = \{[k, d]\}$ for some k, the following abbreviated notation for the weight function is used.

Definition 2.4.11. *Let* $\pi \in NC(d)$*. If* $1 \leq k \leq d$*, define* $wt_k(\pi) = wt_{\{[k,d]\}}(\pi)$*. Define* $wt_0(\pi) = wt_{\{[1,d]^*\}}(\pi)$ *and* $wt_{d+1}(\pi) = wt_{\emptyset}(\pi)$*.*

A direct consequence of this definition is that $wt_{d+1}(\pi) = x^{\text{block}(\pi)}$ and $wt_0(\pi) = x^{|\pi|+\text{sing}(\alpha(\pi))}$. By Lemma 2.4.5, the second weight function may be rewritten as $w_0(\pi) = x^{d+1-\text{block}(\alpha(\pi))}$.

Using [38, Lemma 5.4], given below, $\sum_{\pi \in NC(d)} w_{d+1}(\pi) = g(L_d, x)$.

Lemma 2.4.12 (Hetyei)**.** $[x^k]g(L_d, x)$ *is the number of noncrossing partitions on* $[1, d]$ *with exactly* k *nonsingleton blocks.*

Let $\text{sing}_{\mathcal{S}}(\pi)$ denote the number of singleton elements of π in $\mathcal{S}$. By Lemma 2.4.6, $\text{sing}_{[1,d]-\mathcal{S}'}(\alpha(\pi))$ is the number of antisingletons of π in $\mathcal{S}$. Thus, Definition 2.4.9 yields the explicit formula

$$wt_{\mathcal{S}}(\pi) = x^{\text{block}(\pi)+\text{sing}_{\mathcal{S}}(\pi)+\text{sing}_{[1,d]-\mathcal{S}'}(\alpha(\pi))}, \tag{5}$$

and for $1 \leq k \leq d$, Definition 2.4.11 gives the formula

$$wt_k(\pi) = x^{\text{block}(\pi)+\text{sing}_{\{[k,d]\}}(\pi)+\text{sing}_{\{[1,d-k]\}}(\alpha(\pi))}. \tag{6}$$

2.5 Arc Diagrams

To create a linear representation, or an arc diagram, of a partition, place d consecutive points in a horizontal row. Label the points 1 through d, and connect consecutive elements in a block of the partition with an arc. If there are k elements in the block, there will be $k - 1$ linked arcs representing the block in the diagram. A singleton block will have no arcs. Fig. 18 shows the arc diagram of a partition of $[1, 6]$ which is not noncrossing.

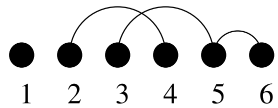


Figure 18: The arc diagram of $\pi = (1)(24)(356)$

Now, suppose π is a noncrossing partition of $[1, d]$ as defined in Chapter 2.4. Recall that this means that whenever $1 \leq a < b < c < e \leq d$ with a and c in the same block of the partition and b and e contained in a single block, then all four elements must be in the same block of π. See Fig. 19.

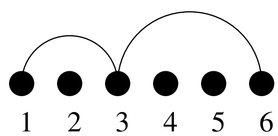


Figure 19: The arc diagram of $\pi = (136)(2)(4)(5)$

As can be seen from Fig. 18 and 19, a partition is noncrossing exactly when none of the arcs in its diagram intersect.

2.5.1 Standard Permutations

Definition 2.5.1. *A* standard permutation *is defined as a permutation of* $\{\pm 1, \ldots, \pm n\}$ *such that the following two properties hold:*

(i) i occurs before $-i$ for all i in the list, and

(ii) the negative numbers in the list appear in the following order: $-1, \ldots, -n$.

Since there are 2^n ways to assign the negative sign to the pair k and $-k$ and $n!$ ways to order $\{-1, \ldots, -n\}$, there is a total of $\dfrac{(2n)!}{2^n \cdot n!}$ standard permutations.

Definition 2.5.2. *Call a permutation π of $\{\pm 1, \ldots, \pm n\}$* sign-connected *if and only if for all $1 \le m < 2n$ there is at least one $j \in \{1, \ldots, n\}$ such that $|\{\pi(1), \ldots, \pi(m)\} \cap \{-j, j\}| = 1$. If a permutation of $\{\pm 1, \ldots, \pm n\}$ is not sign-connected, then call it* sign-disconnected*.*

The following characterization of standard sign-connected permutations is an immediate consequence of Definitions 2.5.1 and 2.5.2.

Lemma 2.5.3. *A standard permutation π of $\{\pm 1, \ldots, \pm n\}$ is sign-disconnected if and only if there exists an i such that $|\pi(j)| \le i$ for all $j \le 2i$ (and $|\pi(j)| \ge i + 1$ for all $j > 2i$).*

Applying the definition of a standard permutation to the previous notion of a sign-connected permutation yields the following characterization.

Lemma 2.5.4. *A standard permutation π is sign-connected if and only if for all $m < 2n$, the sum $\pi(1) + \cdots + \pi(m)$ is positive.*

Proof. By definition of a standard permutation, each j appears before $-j$; thus, $\pi(1) + \cdots + \pi(m) \ge 0$ for every $m \le 2n$. Equality occurs exactly when the set $\{\pi(1), \ldots, \pi(m)\}$ is the union of pairs of the form $\{j, -j\}$. Thus, the existence of an $m < 2n$ satisfying $\pi(1) + \cdots + \pi(m) = 0$ is equivalent to the standard permutation being sign-disconnected. □

Remark 2.5.5. *The proof of the sufficiency part of Lemma 2.5.4 may be restated in the following, stronger from: π is sign-connected if for all $m < 2n$ then $\pi(1) + \cdots + \pi(m) \ne 0$.*

A standard permutation $\pi \in \mathcal{S}_{\{\pm 1,\ldots,\pm n\}}$ can be represented visually by an arc diagram. The arc diagram is constructed as follows: put $2n$ vertices in a row; label them left to right with $\pi(1), \ldots, \pi(2n)$; then for each $k \in \{1, \ldots, n\}$, create an arc connecting the vertices labeled $-k$ and k. See Fig. 20 for an example when $n = 3$.

The labels of the vertices in the arc diagram are uniquely determined by the underlying complete matching. This matching is represented by arcs whose right endpoints must be labeled left to right by $-1, \ldots, -n$, in this order, and whose left endpoint is labeled k if its right endpoint is $-k$. Using these rules, the standard permutation can be uniquely reconstructed from its arc diagram. Thus, the arc diagram provides a one-to-one correspondence between standard permutations in $\mathcal{S}_{\{\pm 1,\ldots,\pm n\}}$ and complete matchings of a $2n$ element set.

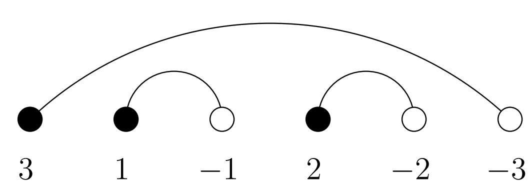


Figure 20: The arc diagram associated to $(3, 1, -1, 2, -2, -3)$

Almost the same bijection appears in the work of Drake [24]; the only difference being that Drake encodes complete matchings with *standard double occurrence words* in the letters $1, \ldots, n$. A double occurrence word is a word in which each letter occurs exactly twice. Ossona de Mendez and Rosenstiehl [55] call a double occurrence word *standard* if the first occurrence of the letters happens in increasing order. See Fig. 21. (Note that Drake [24] omits the adjective 'standard,' but he adopts his terminology from [55], and the words he uses to encode complete matchings are the standard double occurrence words.)

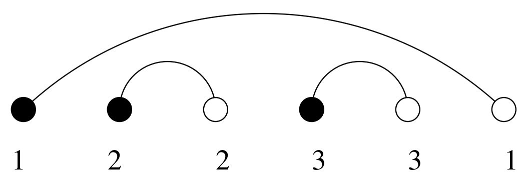


Figure 21: The arc diagram associated to 122331

Ossona de Mendez and Rosenstiehl [55] call double occurrence words *connected* if the word cannot be factored into two nonempty double occurrence words. For example, 122331 is connected, but 121233 is not since it can be factored into 1212 and 33, which are both double occurrence words.

Taking the *reverse complement* of such a double occurrence word and changing the second occurrence of each k to $-k$ results in a standard permutation. This correspondence is a bijection. (As usual, the complement of a word in the letters $1, \ldots, n$ is the word obtained by replacing each letter i with $n+1-i$, and the reverse of a word $w_1 \cdots w_{2n}$ is the word $w_{2n} \cdots w_1$.)

For example, the standard permutation $(3, 1, -1, 2, -2, -3)$ corresponds to the same matching as the double occurrence word 122331 in Drake [24] as can be seen in Fig. 20 and 21. Using the this bijection, it is easy to see that connected standard double occurrence words correspond exactly to sign-connected standard permutations.

Remark 2.5.6. *Associate each standard permutation π of $\{\pm 1, \ldots, \pm n\}$ to the fixed-point-free involution $\widehat{\pi}$ of $\{\pm 1, \ldots, \pm n\}$ that exchanges the elements $\pi^{-1}(-i)$ and $\pi^{-1}(i)$ for $i \in \{1, \ldots, n\}$. This correspondence is a bijection. Under this bijection, sign-connected standard permutations correspond to indecomposable fixed-point-free* *involutions, as defined by Cori [19].*

For $k \in \{1, \ldots, n\}$, the definition of a standard permutation forces the vertex labeled k to be to the left of the vertex labeled $-k$ in the arc diagram. Any arc diagram associated to a standard permutation can be encoded by a word $a_1 \cdots a_n$ as follows. Let a_n be the position of the left endpoint of the rightmost arc, that is, a_n is the position of the vertex labeled n. Remove the rightmost arc from the diagram, and repeat the process until all arcs are removed. The word $a_1 \cdots a_n$ is obtained.

Example 2.5.7. $(3, 1, -1, 2, -2, -3) \cong 131$.

Definition 2.5.8. *The word $a_1 \cdots a_n$ described above is the* encoding *of a standard permutation of* $\{\pm 1, \ldots, \pm n\}$.

Equivalently, $a_1 = 1$, and for $1 < i \leq n$, the letter a_i is 1 plus of the number of elements proceeding i whose absolute value is less than i. As a consequence of the definition, $1 \leq a_i \leq 2i - 1$ holds for each a_i. Conversely, each word $a_1 \cdots a_n$ of this form corresponds to exactly one standard permutation as every such word will encode some arc diagram.

Definition 2.5.9. *Let $\sigma(a_1 \cdots a_n)$ be the standard permutation encoded by $a_1 \cdots a_n$.*

Lemma 2.5.10. *A standard permutation $\sigma(a_1 \cdots a_n)$ is sign-disconnected if and only if there exists k with $k \geq 2$ such that $a_k = 2k - 1$ and $a_j \geq a_k$ for $j > k$.*

Proof. Suppose $a_1 \cdots a_n$ encodes a sign-disconnected standard permutation called $\pi(1) \cdots \pi(2n)$. By Lemma 2.5.3, there exists i such that $|\pi(j)| \geq i + 1$ for all $j > 2i$, meaning that in the associated arc diagram both ends of the $(i+1)^{st}$ through n^{th} arcs are located at vertex positions $2i+1$ or greater. Hence, $a_{i+1} = 2i+1$ and $a_j \geq 2i+1$ for $j > i$.

Conversely, let $a_1 \cdots a_n$ encode a standard permutation $\pi(1) \cdots \pi(2n)$. Suppose k is the smallest integer such that $k \geq 2$, $a_k = 2k-1$, and $a_j \geq a_k$ for all $j > k$. In the associated arc diagram, the right endpoint of the $(k-1)^{st}$ arc is located at vertex position $2k-2$. Since the right endpoint of any arc is labeled with the negative number of the antipodal pair, then $\pi(2k-2) = -(k-1)$. Because $\pi(1) \cdots \pi(2n)$ is a standard permutation, $|\pi(j)| \leq k-1$ for all $j \leq 2k-2$. When $a_j \geq 2k-1$ for all $j \geq k$, the endpoints of the j^{th} through n^{th} arcs in the associated arc diagram are located at position $2k-1$ or greater. Hence, $|\pi(j)| \geq k$ for all $j \geq 2k-1$. □

Clearly, a standard permutation is sign-disconnected if and only if a vertical line can be drawn between the first and last vertices of the associated arc diagram, which does not intersect any of the arcs and that separates two adjacent vertices. In other words, the arc diagram is comprised of more than one component of overlapping arcs. Drake [24] noted that this occurs if there exists a vertex k with $1 < k < 2n$ that is not nested under any arc in the diagram.

The arc diagram of a sign-connected standard permutation will be referred to as *connected* and the arc diagram of a sign-disconnected standard permutation as *disconnected*. An arc diagram is connected exactly when it represents a connected matching as defined by Drake [24]. In terms of standard double occurrence words, connected arc diagrams correspond to the connected standard double occurrence words in the work of Ossona de Mendez and Rosenstiehl [55].

Call a word $a_1 \cdots a_n$ *arc-connected* if and only if the arc diagram of $\sigma(a_1 \cdots a_n)$ is connected; otherwise call the word *arc-disconnected.*

Example 2.5.11. *The arc diagram of* $(1, 2, -1, -2, 3, -3)$, *shown in Fig. 22, is disconnected. This diagram is encoded as* 125.

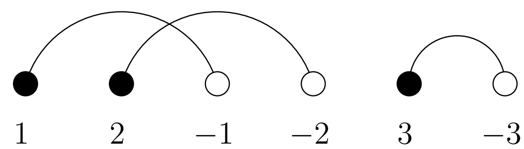


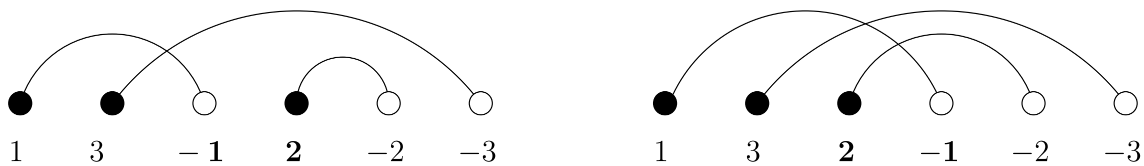
Figure 22: The arc diagram associated to $(1, 2, -1, -2, 3, -3)$

Corollary 2.5.12. *A word is arc-disconnected if and only if there exists k such that $a_k = 2k - 1$ and $a_j \geq a_k$ for $j > k$. Similarly, a word is arc-connected if and only if no such k exists.*

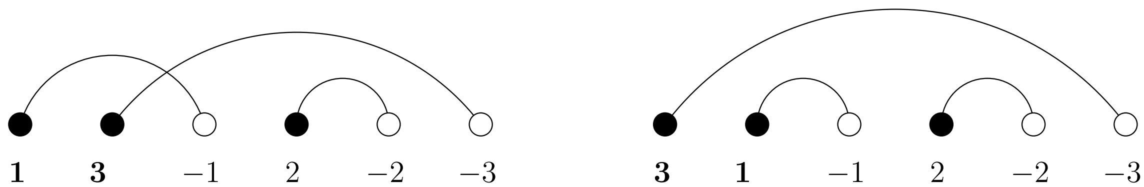
Definition 2.5.13. *A* minimal arc *will be defined to be a component of an arc diagram such that the component consists of exactly one arc.*

For example, in the arc diagram of Example 2.5.11 (see Fig. 22), there is one minimal arc located at the rightmost end of the diagram. However, in the arc diagram of Example 2.5.7 (see Fig. 20), there is no minimal arc since the third arc stretches over the first two; i.e., $a_3 = 1$.

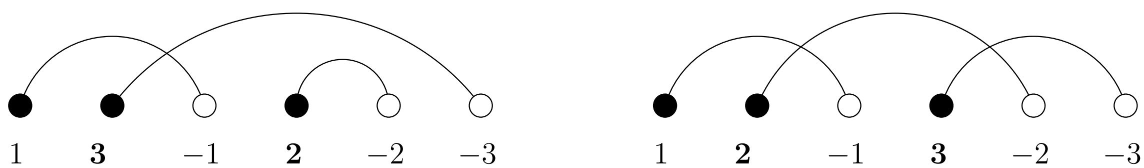
If two standard permutations differ by an adjacent transposition, the two associated arc diagrams will differ by exactly one swap of adjacent ends of two distinct arcs. There may be an interchange of one left and one right end (see Fig. 23(a)) or an interchange of two left ends (see Fig. 23(b)); however, there will not be a swap of two right ends. Due to property (*ii*) of the definition of standard permutations, a swap of two right ends causes the arcs to be relabeled, yielding a nonadjacent transposition whenever the two corresponding left ends are not adjacent. See Fig. 23(c).



(a) A left and right end: $\sigma(132)$ and $\sigma(122)$



(b) Two left ends: $\sigma(132)$ and $\sigma(131)$



(c) Two right ends: $\sigma(132)$ and $\sigma(124)$

Figure 23: Swapping two adjacent ends in the arc diagram

If $\sigma(a_1 \cdots a_n)$ and $\sigma(b_1 \cdots b_n)$ differ by an adjacent transposition, then there exists a unique i such that $b_i = a_i \pm 1$ and $b_j = a_j$ for all $j \neq i$. The converse is not true. For example, the word 122 is obtained from the word 112 by changing the second letter by 1, but the standard permutations $\sigma(112)$ and $\sigma(122)$ differ by a nonadjacent transposition (see Fig. 24).

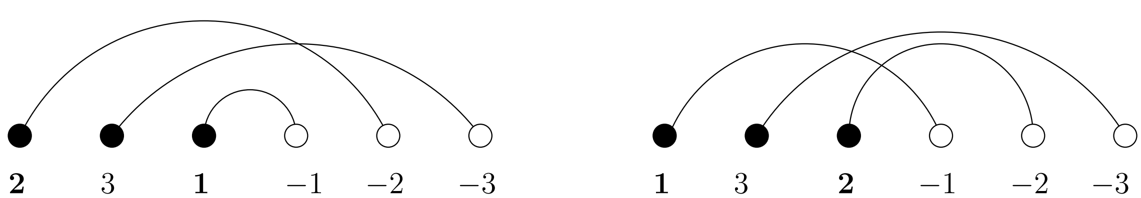


Figure 24: The arc diagrams associated to $\sigma(112)$ and $\sigma(122)$

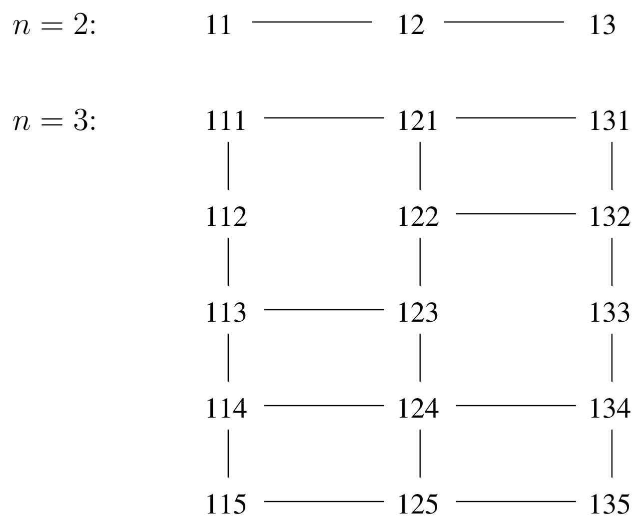


Figure 25: The graph for the standard permutations of $n = 2$ and $n = 3$ where the vertices are the encoded standard permutations, and edges occur where the two standard permutations differ by an adjacent transposition

In fact, the situation in Fig. 24 is a specific instance of a general case where two standard permutations will not differ by an adjacent transposition.

Lemma 2.5.14. *Suppose $a_1 \ldots a_n$ and $b_1 \ldots b_n$ encode two standard permutations where there exists a unique i such that $b_i = a_i \pm 1$ and $b_j = a_j$ for all $j \neq i$. If $a_{i+1} \leq 2i - 1$, $a_i = a_{i+1}$, and $b_i = a_i - 1$, then the two standard permutations $\sigma(a_1 \ldots a_n)$ and $\sigma(b_1 \ldots b_n)$ will not differ by an adjacent transposition.*

Proof. Without loss of generality, let $i = n - 1$. Set $a_{i+1} = k$ where $k \leq 2n - 3$. Then $a_i = k$ and $b_i = k - 1$. In the arc diagram of $a_1 \ldots a_n$, $i + 1$ is located at position k, and i is at position $k + 1$ since in the construction of the diagram the $(i + 1)^{st}$ arc moved the i^{th} arc out of its original position k. However, in the arc diagram of $b_1 \ldots b_n$, $i + 1$ is still located at position k, but i is now located at position $k - 1$. This change is not an interchange of two adjacent left ends of arcs. Hence, the standard permutations do not differ by an adjacent transposition. $\square$

However, it is obvious that the following, more general, statement is true.

Lemma 2.5.15. *If $a_1 \cdots a_n$ and $b_1 \cdots b_n$ satisfy $\sum_{i=1}^{n} |a_i - b_i| = 1$, then $\sigma(a_1 \cdots a_n)$ and $\sigma(b_1 \cdots b_n)$ differ by a single (not necessarily adjacent) transposition.*

2.6 Gray Codes

Gray codes are widely used in computer science to *enumerate* all words, n-tuples, or permutations of a given type. In particular, there is much research on finding Gray codes for classes of permutations where two permutations are considered adjacent if they differ by an involution of some special kind. Gray codes are used in such applications as error correction for digital signals, signal encoding, or image processing [3, 29, 57].

2.6.1 History and General Definitions

Suppose one wants to generate all n-tuples of a certain number k. Each n-tuple or string has the form $a_1, a_2, \ldots, a_n$ where a_i may be any number from 0 to $k-1$ for each $1 \leq i \leq n$. Consider now the specific situation where one would like to generate all binary strings of length n.

These strings may be listed in lexiographical order, starting with $(0, \ldots, 0)_2$ and ending with $(1, \ldots, 1)_2$ after adding 1 repeatedly. While this method will reach all $2^n - 1$ strings, it is not the most efficient way to reach all of the strings. For example, in lexiographic order 100 follows 011. This one step requires all positions to change. If the goal is to reach all $2^n - 1$ strings in an efficient way, where efficiency is determined by the computation cost to implement, then a new method for listing all n-tuples is required.

The Gray binary code, denoted Γ_n, lists all binary strings of length n such that subsequent strings differ by exactly one bit change; i.e., exactly one position changes from 1 to 0 or vice versa. See Table 9 for the Gray binary code for strings of length 2 and 3.

In the 1950's, Frank Gray, a physicist known for his work with color television, used this method to list binary strings in the analogue transmission of digital signals for television. While the Gray binary code was named after Frank Gray since he patented it [30], it was actually used many years prior to the 1950's. In fact, in the 1870's, Émile Bandot used Γ_5 in his telegraph machine. The transactions from the 1878 Paris Exhibition [71] give a detailed description of Bandot's machine. See Fig. 26 for a reproduction of plate 2 from the transactions, which shows Bandot's machine. Those notes also discuss Otto Schäffler's telegraph machine, which also used used this encoding although it was developed independently of Baudot and was not as efficiently implemented. Other types of listings were also used at this time; however, Bandot's was by far the most efficient.

To generate the Gray binary code, also called the binary reflected Gray code, for binary strings of length n, start with $0 \dots 0$. Add 1 to produce the next string the enumeration. At this point, replace $a_1 \dots a_{n-1}$ with the subsequent code from the Gray binary code of length $n-1$. Set $a_n = 0$ to get the next code. Repeat this method until the last code $10 \dots 0$ is reached. Boothroyd [12] published a computer programmable algorithm for this code, which was later modified by Misra [52].

Notice, the binary reflected Gray code is recursively generated. Let $0\Gamma_n$ be the Gray binary code for strings of length n with 0 affixed to the beginning of each string

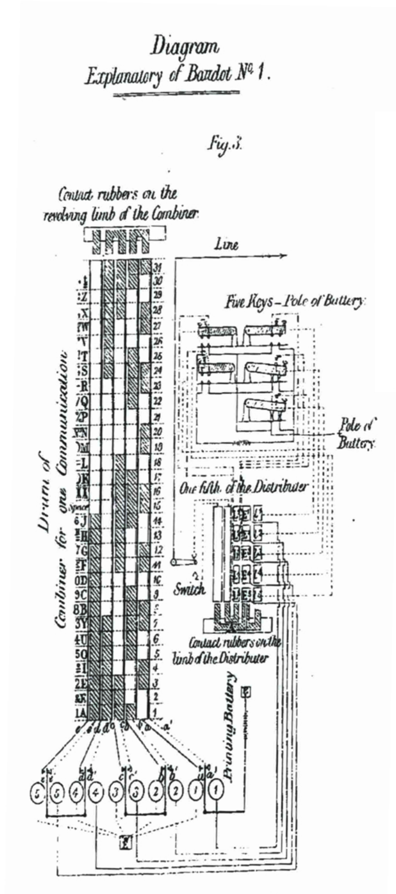


Figure 26: Bandot's telegraph machine

and $1\Gamma_n^R$ be the Gray binary code for strings of length n in reverse order with 1 then affixed to the beginning of each string. Using this notation, an alternate recursive definition for Gray binary code is $\Gamma_{n+1} = 0\Gamma_n, 1\Gamma_n^R$ where Γ_0 is the empty string [10].

In both the method given above and in Boothroyd's algorithm, the binary reflected Gray code is recursively defined. Because of the use of recursive calls, the latter [12, 52] requires $\mathcal{O}(n \cdot 2^n)$ operations to reach all 2^n binary n-tuples [10]. Bitner, Ehrlich, and Reingold [10] were able to produce a streamlined algorithm which reduced the number of operations down to $\mathcal{O}(2^n)$. Later, Ali, Islam, and Foysal were able to improve the algorithm further; however, obviously, not better than $\mathcal{O}(2^n)$. These

length	code			
$n = 2$	00	01	11	10
$n = 3$	000	001	011	010
	110	111	101	100

Table 9: The Gray binary code for $n = 2$ and $n = 3$

two improved algorithms mentioned made use of loop-free algorithms by inducing pointers to indicate when certain changes occur. Since a purpose of this Gray code is to reduce signal transmission error or to list all n-tuples of binary strings, having an efficient algorithm is desirable.

The binary reflected Gray code is only one specific Gray code used to generate all binary n-tuples [27]. In general, such codes are called *binary Gray codes.*

Definition 2.6.1. *Given a finite set X, let $\mathcal{S}_X$ be the set of all strings comprised of elements of X, where the strings adhere to some guideline such as being an n-tuple or a permutation. A Gray code is a sequence such that each element of $\mathcal{S}_X$ is reached exactly once, and consecutive strings in the sequence differ by a specified closeness condition.*

For Gray codes of the binary system, two strings are close if they differ by 1 bit, similarly for a list of n-tuples. When dealing with a sequence of permutations, two permutations are close if they differ by a transposition. Obviously, this closeness condition changes as the set of objects changes. If the first and last object in the Gray code also satisfy the closeness condition, then the Gray code is a *Gray cycle.*

Consider the unit n-cube, whose vertices are given by $(v_1, \ldots, v_n)$ where $v_i \in \{0, 1\}$

for each i. It is clear that each of the 2^n vertices corresponds to a string in a binary Gray code, and the edges of the n-cube represent a 1-bit change between the two strings corresponding to the associated vertices. Hence, a binary Gray code is also an oriented Hamiltonian path along the edges of the n-cube, and a binary Gray cycle is an oriented Hamiltonian cycle on the n-cube. Recall, a Hamiltonian path of a graph is a path such that each vertex is reached exactly once [29].

For non-binary Gray codes, form a graph from the objects of the underlaying set. Let the objects be the vertices of the graph, and any two objects which are close, as defined by the closeness condition, are connected by an edge. The Gray code is an oriented Hamiltonian path on the created graph. For example, Conway, Sloane, and Wilks [17] gave Gray codes for finite reflection groups using Hamiltonian cycles. Gray codes are not necessarily unique so long as multiple Hamiltonian paths exist; however, depending on the underlying set of objects and the closeness condition, a Gray code may not always be possible. See Fig. 28 from Example 2.6.4.

Definition 2.6.2. *Let A be a set of objects, and $f : A \to A$ be a closeness condition. The* Gray graph *has vertices corresponding to the objects from set A, and any two vertices are connected by an edge if and only if they satisfy the closeness condition.*

A *transposition Gray code* is a Gray code where two objects are considered close if they differ by a transposition of two elements. For example, the permutations 1**3**5**2**4 and 1**2**5**3**4 differ by a transposition. Gray codes for permutations are transposition Gray codes. In an *adjacent transposition Gray code*, subsequent codes differ by an adjacent transposition. For example, the permutations 13**52**4 and 13**25**4 differ by an adjacent transposition.

Consider permutations of $\{1, \ldots, n\}$ which adhere to certain topological conditions. A transposition Gray code may also be found in these situations. However, it might not be possible to find an adjacent transposition Gray code. Knuth [45] gave the following example in his book The Art of Computer Programming, vol 4.

Example 2.6.3. *Suppose a Gray code is needed for all permutations of* $\{1, 2, 3, 4\}$ *such that 1 precedes 3, 2 precedes 3, and 2 precedes 4. Then there are four adjacent transposition Gray codes for this set. Read the following two lists forward and backward to get these Gray codes: 1243, 1234, 2134, 2143, 2413 and the sequence 2134, 1234, 1243, 2143, 2413.*

Suppose one needs to generate all permutations of a set or a multiset. Just as with n-tuples, these permutations may be listed in lexicographical order; however, just as with the n-tuples, this is not the most cost efficient way to list all the permutations.

Example 2.6.4. *Consider the multiset:* $\{1, 1, 2, 2\}$*. The permutations of this set listed in lexicographical order are* $1122, 1212, 1221, 2112, 2121, 2211$*. Fig. 27 shows the graph formed by all arrangements of the multiset. Permutations are joined by an edge in the graph if they differ by a single transposition. A Hamiltonian path for this graph exists. Thus, there exists a transposition Gray code on the multiset. One such Gray code is* $1122, 1212, 2112, 2121, 1221, 2211$.

Now, consider the graph formed by all permutations of the multiset where an edge occurs if the joined permutations differ by a single adjacent transposition. See Fig. 28. Obviously, there is no Hamiltonian path for this graph. Hence, no adjacent transpo-

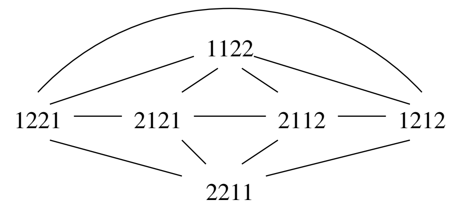


Figure 27: The graph associated to the multiset: $\{1, 1, 2, 2\}$

sition Gray code exists.

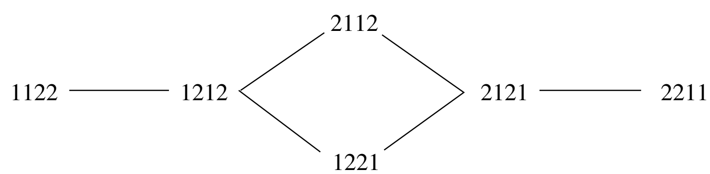


Figure 28: The graph associated to $\{1, 1, 2, 2\}$

There is a simple way to generate the permutations on $[1, n]$ in $n!$ steps by making $n! - 1$ adjacent interchanges. Although this algorithm was originally defined for the permutations on $[1, n]$, the same method can easily be extended to any set of distinct objects which have an ordering. Thus, while not all multisets have an adjacent transposition Gray code, permutations of distinct objects will always have an adjacent transposition Gray code.

The *Johnson-Trotter algorithm* generates an adjacent transposition Gray code on the set $\mathcal{S}_X$ of all permutations of the set X. When $X = \{1, \ldots, n\}$, this algorithm recursively defines the Gray code as follows: set the initial permutation to $12 \ldots n$. Let the element n move from one end of the permutation to the other. Whenever, n reaches the end of a permutation, the next string in the Gray code on $\mathcal{S}_{[1,n-1]}$ replaces the permutation consisting of the first $n - 1$ elements [41, 70]. See Table 10 for the

Gray code when $n = 2$ and $n = 3$.

length	code					
$n = 2$	1**2**	**2**1				
$n = 3$	12**3**	1**3**2	**3**12	**3**21	2**3**1	21**3**

Table 10: The Gray code produced by the Johnson-Trotter algorithm

While the Gray code produced by the Johnson-Trotter algorithm was known before Trotter [70] published its first computer implementation in the 1960's, Trotter's algorithm was efficient even though it utilized recursive calls. In the 1970's, Ehrlich [26] and later Dershowitz [20] developed a simplified loop-free version of the Johnson-Trotter algorithm.

The *inversion table* of a permutation is an n-tuple $c_1 \ldots c_n$ where c_i is the number of elements in the permutation to the right of i that are also less than i. Hence, $0 \leq c_i \leq i - 1$.

Dijkstra [22] showed that the inversion tables associated to an adjacent transposition Gray code is itself a Gray code when traversed in the same order as the original Gray code. Algorithm 2.6.5 rewrites Dijkstra's improvement of the Johnson-Trotter algorithm using inversion tables and Knuth's [45] notation.

For example, the left side of Table 11 gives the adjacent transposition Gray code on the permutations of $[1, 4]$. Read down the first column, then up the second column. Continue reading the table up and down the columns. Notice, if the right side of Table 11 is read up and down the columns in the same way, it also produces a Gray code where exactly one position changes by 1.

Johnson-Trotter algorithm						the inversion tables					
1234	132**4**	312**4**	321**4**	231**4**	213**4**	000**0**	001**0**	002**0**	012**0**	011**0**	010**0**
12**4**3	13**4**2	31**4**2	32**4**1	23**4**1	21**4**3	000**1**	001**1**	002**1**	012**1**	011**1**	010**1**
1**4**23	1**4**32	3**4**12	3**4**21	2**4**31	2**4**13	000**2**	001**2**	002**2**	012**2**	011**2**	010**2**
4123	**4**132	**4**312	**4**321	**4**231	**4**213	000**3**	001**3**	002**3**	012**3**	011**3**	010**3**

Table 11: The Gray code for $n = 4$ and its associated inversion table

Given a sequence $a_1 \ldots a_n$ of n distinct elements, where $a_1 < \ldots < a_n$, the Johnson-Trotter algorithm generates all of its permutations by interchanging adjacent elements. The algorithm below uses the corresponding sequence of inversions $c_1 \ldots c_n$ to determine which adjacent elements are interchanged where $0 \leq c_i < j$ is defined as above for $1 \leq j \leq n$. An array $\sigma_1 \ldots \sigma_n$ will be used to store when c_j changes.

Algorithm 2.6.5 (Dijkstra)**.** *The Improved Johnson-Trotter Algorithm*

initialize: set $c_j = 0$ and $\sigma_j = 1$ for all $1 \leq j \leq n$
step 2: display $a_1 \ldots a_n$
step 3: set $j = n$ and $s = 0$
where s is the number of k such that $k > j$ and $c_k = k - 1$
step 4: set $q = c_j + \sigma_j$
if $q < 0$, then go to step 7
if $q = j$, then go to step 6
step 5: interchange a_{j-c_j+s} and a_{j-q+s}
(since $\sigma_j = \pm 1$, these indices are adjacent)
set $c_j = q$
go to step 2
step 6: if $j = 1$, end algorithm
if $j \neq 1$, let $s = s + 1$
step 7: set $\sigma_j = -\sigma_j$ and $j = j - 1$
go to step 4

Example 2.6.6. *Use the Johnson-Trotter algorithm to generate the Gray code for permutations of length $n = 3$. See Table 12 for the resulting Gray code.*

Gray code	inversions	who's moving?
$a_1a_2a_3$	$c_1c_2c_3$	$\sigma_1\sigma_2\sigma_3$
123	000	1,1,1
132	001	1,1,1
312	002	1,1,1
321	012	1,1,-1
231	011	1,1,-1
213	010	1,1,-1
end		1,-1,1

Table 12: Using the Johnson-Trotter algorithm for $n = 3$

Since there is exactly one transposition per step in the Johnson-Trotter algorithm, odd and even permutations will be listed in alternating order [70]. Thus, all even permutations can be generated by just skipping the odd permutations in the full Gray code.

Depending on the specifics of the objects in the Gray code and the closeness condition, removing an unwanted block from the Gray code may still yield a Gray code. However, any special requirements of the Gray code will still need to be checked. If the original Gray code was an adjacent transposition Gray code, then after removing any block, the resulting list is nearly always a Gray code on the subset of objects, but it often be just a transposition Gray code [14].

Example 2.6.7. *If the second block in the adjacent transposition Gray code of length* 4 *is removed (see Table 11), the resulting list is still a Gray code; however the codes* **4**12**3** *and* **3**12**4** *differ by a non-adjacent transposition.*

2.6.2 Indecomposable Permutations

The results in Chapter 4 are motivated by King's recent paper [42], providing a transposition Gray code for the set of all indecomposable permutations of a finite set.

The combinatorial interest in these permutations is long-standing; see Comtet [16, p. 261] and Stanley [68, Ch. 1, Exercise 32]. As shown by Ossona de Mendez and Rosenstiehl [54], indecomposable permutations are bijectively equivalent to rooted hypermaps. A recent generalization of Dixon's famous result [23], stating that a random pair of permutations almost always generates a transitive group, is also related to enumerating indecomposable permutations; see Cori [18].

A permutation π of $\mathcal{S}_n$ is *indecomposable*, also called irreducible or connected, if there is no $m < n$ such that π sends $\{1, \ldots, m\}$ into $\{1, \ldots, m\}$. King found a *transposition Gray code* for such indecomposable permutations [42]; i.e., a Gray code for the graph whose vertices are indecomposable permutations and whose edges connect permutations that differ by a (not necessarily adjacent) transposition. It is still open whether there is an adjacent transposition Gray code for indecomposable permutations.

Let π be a permutation of length n. Define π_i to be the permutation on $[1, i]$ contained in π. For example, if $\pi = 642153$, then $\pi_3 = 213$. Define two related vectors p and spp, where $p[i]$ is the location of i in π_i and $spp[i]$ is the length of the smallest prefix of π_i which is also a permutation.

Example 2.6.8. *Consider* $\pi = 642153$ *as before. Calculate* π_i *for each* i*. Use* π_i *to*

$$\begin{aligned}
\pi_1 &= \mathbf{1} \\
\pi_2 &= \mathbf{2}1 \\
\pi_3 &= 21\mathbf{3} \\
\pi_4 &= \mathbf{4}213 \\
\pi_5 &= 421\mathbf{5}3 \\
\pi_6 &= \mathbf{6}42153
\end{aligned}$$

compute $p = [1, 1, 3, 1, 4, 1]$ *and* $spp = [1, 2, 2, 4, 5, 6]$.

Since $\pi = \pi_n$, π is indecomposable if and only if $spp[n] = n$. Thus, as can be seen in Example 2.6.8, $\pi = 642153$ is indecomposable. When determining whether a permutation is indecomposable by hand, it is often quicker to look at the given permutation. Using King's [42] lemma, given below, a computer can quickly calculate the *spp* of each permutation of length n, thereby determining whether the permutation is indecomposable or not.

Lemma 2.6.9 (King). *Let π be a permutation on $[1, d]$. For $1 < i \leq n$,*

$$spp[i] = \begin{cases} i & , \text{ if } p[i] \leq spp[i-1] \\ spp[i-1] & , \text{ otherwise.} \end{cases}$$

By definition, $spp[1] = 1$ for any permutation. Consider what happens when inserting i into a permutation of $[1, i-1]$. If i comes before the end of the smallest prefix which is also a permutation; i.e., $p[i] \leq spp[i]$, the resulting permutation on $[1, i]$ will be indecomposable. However, if i is inserted after the end of this smallest prefix, then the original prefix will still be the smallest prefix which is itself a permutation.

Using this information, walk through the Johnson-Trotter Gray code for permutations of length n. Update p and *spp* for each code. Keep the permutation if and only if $spp[n] = n$. The resulting is a list of all indecomposable permutations of length n. See Table 13.

For $n \leq 5$, the list of indecomposable permutations resulting from just removing all the decomposable permutations from the Johnson-Trotter Gray code is itself a Gray code. However, this method does not work starting with $n = 6$. See Table 13. The permutations **61523**4 and **51236**4 do not differ by a single transposition even

$n=3$	$n=6$
312	612345
321	612354
231	612534
	615234
	512364
	...

Table 13: Indecomposable permutations in Johnson-Trotter order

though they are consecutive codes using the above method. Thus, King developed the following method to compute a transposition Gray code for indecomposable permutations.

Define I_n to be the set of indecomposable permutations on $[1, n]$. Then $I_1 = 1$, $I_2 = 1$, and

$$|I_n| = \sum_{j=0}^{n-2} (n-j-1)j!|I_{n-j-1}|.$$

King's Gray code for indecomposable permutations will be constructed in blocks. To do this, let $\pi = p_1 \ldots p_n$. Set $r = p_1$, the first element of the permutation, and let j be the smallest number such that $p_2 \ldots p_{j+1}$ is a permutation on $[1, j]$. Set $I_{n,r,j}$ to be the set of all permutations on $[1, n]$ with fixed r and j and $I_{n,r}$ to be the set of all permutations on $[1, n]$ with fixed r. Then $I_{n,r} = \bigcup_{0 \leq j \leq r-2} I_{n,r,j}$.

Example 2.6.10. *Let $\pi = 52137486$. Then $r = 5$ and $j = 3$ since $p_2p_3p_4 = 213$.*

Using the notation r and j, $|I_n|$ may be rewritten as

$$|I_n| = \sum_{r=2}^{n} \sum_{j=0}^{r-2} |I_{n-j-1}|j!$$

To construct the Gray graph for I_n, let the set of objects used as the vertices be permutations in I_n, and two permutations are connected by an edge if and only if they differ by a transposition. Call this Gray graph G_n. Then the subgraph induced by $I_{n,r}$ is $G_{n,r}$, and the subgraph induced by $I_{n,r,j}$ is $G_{n,r,j}$.

Let P_j be the transposition Gray graph of S_n and has the property that $G_{n,r,j} \cong P_j \times G_{n-j-1}$. King also defines $top_{n,r,j}$ and $bot_{n,r,j}$ which allow each $G_{n,r,j}$ to be ordered.

Lemma 2.6.11 (King)**.** *If $G_{n,n,j}$ has a Hamiltonian path from $top_{n,n,j}$ to $bot_{n,n,j}$ for all j, then so does G_n.*

Lemma 2.6.12 (King)**.** *Given any permutation on $[1, n]$, then is a Hamiltonian path for P_n beginning at π and ending at a permutation whose final vertex differs from π by a transposition of the last two positions.*

Putting all of this together, the transposition Gray code for indecomposable permutations is as follows. Let G_n be determined by $G_{n,2} \to G_{n,3} \to \ldots \to G_{n,n}$ where $G_{n,2} = G_{n,2,0}$, and for $r > 2$, $G_{n,r}$ is determined by $G_{n,r,r-3} \to G_{n,r,r-5} \to \ldots \to G_{n,r,j}$. If $j = 1$, then the path finishes with $G_{n,r,0} \to G_{n,r,2} \to \ldots \to G_{n,r,r-2}$, and if $j = 0$, the path ends with $G_{n,r,1} \to G_{n,r,3} \to \ldots \to G_{n,r,r-2}$.

When traveling from $G_{n,r}$ to $G_{n,r+1}$, one travels from $bot_{n,r,j}$, for an appropriate j, to $bot_{n,r+1,j}$. However, when traveling within a single $G_{n,r}$, one switches between the $G_{n,r,j}$ at either $top_{n,r,j}$ or $bot_{n,r,j}$. This is similar to the Johnson-Trotter algorithm where one travels up and down the columns; see Table 11. As a side note, when $j = 0$ or 1, $G_{n,r,j}$ is as expected; however, when $j \geq 2$, the definition of $top_{n,r,j}$ and $bot_{n,r,j}$ can cause $G_{n,r,j}$ to be quirky. Since this phenomenon does not have any bearing on the results discussed in the rest of the dissertation, see [42] for further explanation.

Example 2.6.13. *Suppose one wants to find G_4. According to the above method, the path will be determined by $G_{4,2} \to G_{4,3} \to G_{4,4}$ where $G_{4,2} = G_{4,2,0}$, $G_{4,3}$ is determined by $G_{4,3,0} \to G_{4,3,1}$, and $G_{4,4}$ is determined by $G_{4,4,1} \to G_{4,4,0} \to G_{4,4,2}$. Using the facts that r is the first number of the permutation, j is the length of the smallest permutation on $[1, j]$ that starts at position 2, and $G_{n,r,j}$ is the set of all permutations on $[1, n]$ with these properties, one can determine each $G_{n,r,j}$. Apply the concepts of $top_{n,r,j}$ and $bot_{n,r,j}$ to get the order of each $G_{n,r,j}$. Hence, $G_{4,4,0} = \{4231, 4321, 4312\}$, $G_{4,4,1} = \{4132\}$, $G_{4,4,2} = \{4213, 4123\}$, $G_{4,3,0} = \{3241, 3421, 3412\}$, $G_{4,3,1} = \{3142\}$, and $G_{4,2,0} = \{2341, 2431, 2413\}$. Putting this altogether yields a Gray code for indecomposable permutations on $[1, 4]$:*

$$\begin{array}{l}
\left.\begin{array}{l}2341\\2431\\2413\end{array}\right\} G_{4,2} \\
\left.\begin{array}{l}\left.\begin{array}{l}3412\\3421\\3241\end{array}\right\} G_{4,3,0}\\ \left.3142\right\} G_{4,3,1}\end{array}\right\} G_{4,3} \\
\left.\begin{array}{l}\left.4132\right\} G_{4,4,1}\\ \left.\begin{array}{l}4231\\4321\\4312\end{array}\right\} G_{4,4,0}\\ \left.\begin{array}{l}4213\\4123\end{array}\right\} G_{4,4,2}\end{array}\right\} G_{4,4}
\end{array}$$

CHAPTER 3: NONCROSSING PARTITION STATISTICS

3.1 The Toric Contribution of the Adin h-Vector

Let $\mathcal{P}$ be a d-dimensional cubical complex. In this section, the toric contribution of the normalized Adin h-vector will be determined using the information from Chapter 2.3.4. Start with Definition 2.2.1 for the toric f polynomial of $\mathcal{P}$

$$f(\mathcal{P}, x) = \sum_{F \in \mathcal{P}} g(\partial F, x)(x-1)^{d-dimF} \tag{7}$$

where F is a face of $\mathcal{P}$. Then F is a j-cube for some $0 \leq j \leq d$ or $F = \emptyset$. Hence,

$$f(\mathcal{P}, x) = (x-1)^{d+1} + \sum_{j=0}^{d} f_j \cdot (x-1)^{d-j} g(L_j, x)$$

where $(x-1)^{d+1}$ is the contribution when $F = \emptyset$. Applying Equation (3),

$$f(\mathcal{P}, x) = (x-1)^{d+1} + \sum_{j=0}^{d} 2^{d-j} \sum_{i=0}^{j} \binom{d-i}{d-j} [h_{i+1} + h_i](x-1)^{d-j} g(L_j, x).$$

Since $h_0 = 1$, collecting the contribution of each h_i yields

$$\begin{aligned} f(\mathcal{P}, x) = & \left[(x-1)^{d+1} + \sum_{j=0}^{d} 2^{d-j} \binom{d}{j} (x-1)^{d-j} g(L_j, x)\right] + h_{d+1} \cdot g(L_d, x) \\ & + \sum_{i=1}^{d} h_i \sum_{j=i-1}^{d} 2^{d-j} \left[\binom{d-i}{d-j} + \binom{d+1-i}{d-j}\right] (x-1)^{d-j} g(L_j, x). \end{aligned} \tag{8}$$

Thus, $(x-1)^{d+1}$ will be part of the toric contribution of h_0.

Definition 3.1.1. *Suppose $\mathcal{P}$ is a d-dimensional cubical complex. Let $Q_{d,k}(x)$ be the polynomial which gives the toric contribution of the normalized Adin h-vector of*

$f(\mathcal{P}, x)$, *namely*

$$f(\mathcal{P}, x) = \sum_{k=0}^{d+1} h_k \cdot Q_{d,k}(x). \tag{9}$$

By Equation (8), the polynomials $Q_{d,k}(x)$ have an explicit formula:

$$Q_{d,k}(x) = \begin{cases} (x-1)^{d+1} + \sum_{j=0}^{d} 2^{d-j} \binom{d}{j} (x-1)^{d-j} g(L_j, x) & \text{if } k = 0, \\ \sum_{j=k-1}^{d} 2^{d-j} \left[\binom{d-k}{d-j} + \binom{d+1-k}{d-j} \right] (x-1)^{d-j} g(L_j, x) & \text{if } 1 \le k \le d, \\ g(L_d, x) & \text{if } k = d+1. \end{cases} \tag{10}$$

Table 15 in Chapter 3.3 lists $Q_{d,k}(x)$ for small d.

3.2 Contribution of Shelling Components to the Toric h-Vector

The original plan was to find a relationship between the toric h-polynomial and the Adin h-vector, namely $Q_{d,k}(x)$. However, both Chan [15] and Hetyei [38] looked at the contribution of shelling components to the toric h polynomial. Since the polynomials $Q_{d,k}(x)$ satisfy the relations found in these two papers, it was hypothesized that there may be a relation between the polynomials $Q_{d,k}(x)$ the contribution of the cubical shelling components to the toric h polynomial. The following section explores this relationship.

Consider a shelling $F_1, \ldots, F_m$ of $\mathcal{P}$, and let (i, j) be the type of the t^{th} facet in the shelling. According to Adin, $\sum_k h_k x^k = \sum_t \Delta_t h(x)$, where $\Delta_t h(x)$ is the contribution from F_t. See Chapter 2.3.3 for a definition of a shelling and shelling component types. In Chapter 2.3.6, an explicit formula for the Adin h-vector was found in terms of the number of types of cubical shelling components, namely Equation (4).

Applying Equations (4) and (9) to $\sum_k h_k x^k$ gives the contribution of $c_{i,j}$ in terms of the polynomials $Q_{d,k}(x)$:

$$f(\mathcal{P},x) = \sum_{k=1}^{d} h_k \cdot Q_{d,k}(x) = \sum_{j=0}^{d-1}\sum_{i=1}^{d-j} c_{i,j} \sum_{k=j+1}^{i+j} \left(\frac{1}{2}\right)^i \binom{i-1}{k-1-j} Q_{d,k}(x).$$

Define $C_{d,i,j}(x)$ to be the toric contribution of all shelling components of type (i,j). Hence,

$$C_{d,i,j}(x) = \sum_{k=j+1}^{i+j} \left(\frac{1}{2}\right)^i \binom{i-1}{k-1-j} Q_{d,k}(x) = \sum_{\ell=1}^{i} \left(\frac{1}{2}\right)^i \binom{i-1}{\ell-1} Q_{d,\ell+j}(x). \tag{11}$$

In particular if $i = 1$, $C_{d,1,j}(x) = \frac{1}{2}Q_{d,j+1}(x)$. Additionally, $C_{d,0,0}(x) = 1$ and $C_{d,0,d}(x) = c_{0,d} \cdot x^{d+1}$.

3.3 A Combinatorial Interpretation

In this section, the formulas for both $C_{d,i,j}(x)$ and $Q_{d,k}(x)$ are developed using the weight function defined in Chapter 2.4.9 as applied to noncrossing partitions.

3.3.1 $C_{d,i,j}(x)$ as the Total Weight of Objects

Lemma 3.3.1. *Let $\pi \in NC(d)$, and suppose $\mathcal{S} = \{[n,m]\}$ where $[n,m]$ may or may not be wrapped. Let $d-j$ be the number of elements in $[n,m]$. Hence, $0 \leq j \leq d-1$. Then*

$$C_{d,1,j}(x) = \sum_{\pi \in NC(d)} wt_{\mathcal{S}}(\pi) = \frac{1}{2}Q_{d,j+1}(x). \tag{12}$$

Proof. Definition 2.4.9 may be rephrased as follows. Consider the special objects: singleton and antisingleton elements in $\mathcal{S}$ as well as all nonsingleton blocks in π. Assign the weight x to each special object in π. All others receive a weight of 1.

Next, consider a related weight function $wt'_{\mathcal{S}}$, where any special object is assigned a weight of $x+1$. All others have weight 1. This modified weight function $wt'_{\mathcal{S}}$

differs from $wt_{\mathcal{S}}$ in that there exists the option to choose whether or not to mark each special object. Each marked object has weight x, and an unmarked object has weight 1. Then $wt'_{\mathcal{S}}(\pi)$ is computed by replacing every x in $wt_{\mathcal{S}}(\pi)$ with $x+1$. The $x+1$ counts both options.

By Equation (11), $\frac{1}{2}Q_{d,j+1}(x+1) = C_{d,1,j}(x+1)$, and by Equation (10), $Q_{d,j+1}(x+1) = \sum_{j=k-1}^{d} 2^{d-j}\left[\binom{d-k}{d-j} + \binom{d+1-k}{d-j}\right] x^{d-j} g(L_j, x+1)$. The rest of this proof will show that $C_{d,1,j}(x+1) = \sum_{\pi \in NC(d)} wt'_{\mathcal{S}}(\pi)$, which is equivalent to the desired result.

To compute $\sum_{\pi \in NC(d)} wt'_{\mathcal{S}}(\pi)$, choose π in $NC(d)$, and mark as many nonsingleton blocks, antisingleton elements in $\mathcal{S}$, and last elements where all other elements in the block are marked and contained in one interval of $\mathcal{S}$ as desired.

Define *type a* elements to be the marked antisingletons of π. Define *type b* elements to be marked last elements of a block of π whose other elements are all type a and the entire block is completely contained in one interval of $\mathcal{S}$. Marked singletons are type b elements.

Remove all type a and type b elements from π. Let k be the number of elements remaining in the partition. Call the noncrossing partition formed by these k elements π_k. Any marked object left in π_k must be a nonsingleton block. By Lemma 2.4.12, the weight of all possible π_k in $NC(k)$ is $g(L_k, x+1)$. Since type a and type b elements are only located in $\mathcal{S}$, there are at most $d-j$ of them. Since $d-k$ is the number of removed type a and type b elements, $j \leq k \leq d$.

Next, reinsert $d-k$ type a and type b elements into π_k, and count the number of ways that this can be done. If only the position and order of the marked type a and type b elements is known, the original π can be recovered from π_k. Consider the linear representation of π_k. Insert elements at each position where type a and type b elements are known to be located. For type b elements, do nothing else. Each inserted type a element joins the block to which the element directly following it belongs. Thus, excluding the situation where a type b element is directly preceded by a type a element, all type b elements are singletons. The original partition π can be constructed from this newly created arc diagram. (See Example 3.3.2.)

For fixed k, one can determine where to insert the $d-k$ type a and type b elements. Type b elements may be inserted anywhere in $\mathcal{S}$. Type a, or antisingleton, elements may be inserted anywhere in $\mathcal{S}$ except at the end of an interval of $\mathcal{S}$, namely at position m.

Unless an inserted antisingleton element is placed immediately prior to a singleton element in π_k, the elements of π_k will have the same role in π as they did in π_k. If an antisingleton element is inserted before a singleton element in π_k, the singleton element will change to a last element of a nonsingleton block. Singleton elements cannot be marked in π_k, so the new nonsingleton block is still unmarked. Thus, this change will not affect the overall weight.

One may also insert consecutive type a and type b elements. Suppose one wants to insert two consecutive type a and type b elements. If these two elements are located at positions $m-1$ and m, then there are only two options: ab and bb. Obviously, for the first option, a two element block is inserted into π_k, and for the second option, two singletons are inserted. However, if two consecutive elements are inserted prior to positions $m-1$ and m, there are four options: aa, ab, ba, and bb, where the type b element of ab is a last element, but all other type b elements are singletons.

Suppose one wants to insert an arbitrary number of consecutive type a and type b elements. Each inserted element may be either type a or type b so long as it is not located at position m. This is possible since type b elements may be connected to a type a element. Within the string of consecutive type a and type b elements, the type b elements indicate the end of a block (either nonsingleton or singleton).

If m is a type b element, then the remaining $d-k-1$ type a and type b elements are placed in the other $d-j-1$ positions of $[n, m]$, and the total number of ways to arrange these elements is $\binom{d-j-1}{d-k-1}2^{d-k-1}$. If m is not a type b element, the last element of $[n, m]$ is an unmarked element. The element at position m could be located in a marked nonsingleton block, but it will never be a type a element. In this situation, all type a and type b elements are in the remaining $d-j-1$ positions of $\mathcal{S}$. The total number of ways to arrange these elements is $\binom{d-j-1}{d-k}2^{d-k}$.

Summing over all partitions of $NC(d)$ yields a total weight of

$$\sum_{k=j}^{d}\left[\binom{d-j-1}{d-k-1}2^{d-k-1}+\binom{d-j-1}{d-k}2^{d-k}\right]x^{d-k}g(L_k, x+1),$$

where x^{d-k} is the weight of the type a and type b elements, and $g(L_k, x+1)$ gives the weight of all the other elements, namely the weight of all π_k. Thus,

$$\sum_{\pi\in NC(d)} wt'_{\mathcal{S}}(\pi)=\sum_{k=j}^{d}\left[\binom{d-j-1}{d-k-1}+\binom{d-j-1}{d-k}2\right]2^{d-k-1}x^{d-k}g(L_k, x+1).$$

Apply Pascal's identity as well as Equations (10) and (11) to get

$$=\sum_{k=j}^{d}\left[\binom{d-j-1}{d-k}+\binom{d-j}{d-k}\right]2^{d-k-1}x^{d-k}g(L_k, x+1)$$

$$=\frac{1}{2}Q_{d,j+1}(x+1)=C_{d,1,j}(x+1).$$

□

Note $k = d$ implies that $d - k = 0$, yielding the special case where there are no type a or type b elements. At the other extreme, $k = j$ means that $d - k = d - j$, or every position in $\mathcal{S}$ is a type a or type b element.

Example 3.3.2. *Let* $\pi = (145)(23)(6)$ *and* $\mathcal{S} = \{[4, 6]\}$. *Mark the elements at positions* 4 *and* 6. *Then the element at position* 4 *is a type* a *element and the one at position* 6 *is type* b *(see Fig. 29). Remove them. Call the remaining partition* π_4*; see*

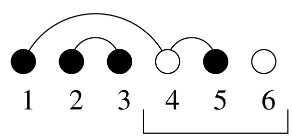


Figure 29: The arc diagram of $\pi = (145)(23)(6)$

Fig. 30. Working left to right, insert a type a *element at position* 4 *and then a type* b

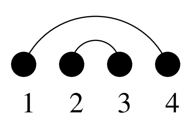


Figure 30: The arc diagram of π_4

element at position 6. *Connect the type* a *element to the element directly following it in the arc diagram. The resulting arc diagram is equivalent to the one in Fig. 29, and the partition is the original* π. *See Fig. 31. Hence, as stated in the above proof, if it*

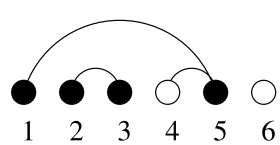


Figure 31: Inserting type a and b elements into π_4 to get π

is known where all type a and b elements are located, the original π can be recovered from π_k.

The following identity will be used in Theorem 3.3.4, the main result of this section.

Lemma 3.3.3. *Let $d > 0$, $i > 0$, $0 \leq j \leq d - i$, and $j \leq k \leq d$. Then*

$$\sum_{\ell=0}^{i} \binom{i}{\ell}\binom{d-i-j}{d-k-\ell} \cdot 2^{i-\ell} = \sum_{\ell=0}^{i} \binom{i}{\ell}\binom{d-j-\ell}{d-k}.$$

Proof.

$$\sum_{\ell=0}^{i} \binom{i}{\ell}\binom{d-i-j}{d-k-\ell} \cdot 2^{i-\ell} = \sum_{\ell=0}^{i} \binom{i}{\ell}\binom{d-i-j}{d-k-\ell} \sum_{m=0}^{i-\ell} \binom{i-\ell}{m}$$

Interchange the summations and apply the equality $\binom{i}{\ell}\binom{i-\ell}{m} = \binom{i}{m}\binom{i-m}{\ell}$. Finally, use the Chu-Vandermonde identity to get the desired result. Thus, the left-hand side equals

$$\sum_{m=0}^{i} \binom{i}{m} \sum_{\ell=0}^{i-m} \binom{i-m}{\ell}\binom{d-i-j}{d-k-\ell} = \sum_{m=0}^{i} \binom{i}{m}\binom{d-m-j}{d-k}.$$

□

Theorem 3.3.4. *Let $\mathcal{S}$ be defined as above, where i is the number of intervals in $\mathcal{S}$ and j is the number of elements in $[1, d]$ but not in any interval of $\mathcal{S}$. For $i > 0$,*

$$C_{d,i,j}(x) = \sum_{\pi \in NC(d)} wt_{\mathcal{S}}(\pi).$$

Proof. Let $\mathcal{S} = \{S_1, \ldots, S_i\}$ where $S_n = [k_n, l_n]$ for $1 \leq n \leq i$. Recall, if $[k_i, l_i]$ is wrapped, then d may be an antisingleton element. If $i = 1$, then the result follows from Lemma 3.3.1. For the rest of the proof, suppose $i > 1$.

Consider the weight function $wt'_{\mathcal{S}}$ defined as in the proof of Lemma 3.3.1, where it is optional to mark the special objects for some $\pi \in NC(d)$. It will be shown that $C_{d,i,j}(x+1) = \sum_{\pi \in NC(d)} wt'_{\mathcal{S}}(\pi)$.

To compute $\sum_{\pi \in NC(d)} wt'_{\mathcal{S}}(\pi)$, choose π in $NC(d)$, and mark as many nonsingleton blocks, antisingleton elements in $\mathcal{S}$, and last elements where all other elements of the block are marked and contained in a single interval of $\mathcal{S}$ as desired. Recall that for each interval S_n, the last element l_n cannot be a marked antisingleton element. Define *type a* elements and *type b* elements as before.

Remove all type a and type b elements, and let k be the number of elements remaining in the partition. As before, call the noncrossing partition formed by these k elements π_k. Recall, the weight of all possible π_k in $NC(k)$ is $g(L_k, x+1)$, and $j \leq k \leq d$.

Next, count the number of ways the $d-k$ type a and type b elements may be reinserted into π_k at positions in $\mathcal{S}$. Type a elements cannot be inserted at the end of any interval in $\mathcal{S}$. Thus, type a elements can be inserted at $d-j-i$ possible locations. Type b elements may be inserted at any position in $\mathcal{S}$, including the last position of any interval in S. Let ℓ be the number of type b elements inserted at the end of some interval in $\mathcal{S}$. Then there are $\binom{i}{\ell}$ ways to select these intervals. The remaining $d-k-\ell$ type a and b elements are inserted at the positions in $\mathcal{S}$ that are not at the end of any interval. Hence, $\max(0, i+j-k) \leq \ell \leq \min(i, d-k)$.

The total possible ways to reinsert these elements is

$$\sum_{\ell=\max(0,i+j-k)}^{\min(i,d-k)} \binom{i}{\ell}\binom{d-j-i}{d-k-\ell} 2^{d-k-\ell} = \sum_{\ell=0}^{i} \binom{i}{\ell}\binom{d-j-i}{d-k-\ell} 2^{d-k-\ell}.$$

Summing over all partitions π of $NC(d)$ and calculating the weight of the partitions yields

$$\sum_{\pi\in NC(d)} wt'_{\mathcal{S}}(\pi) = \sum_{k=j}^{d}\sum_{\ell=0}^{i}\binom{i}{\ell}\binom{d-j-i}{d-k-\ell}2^{d-k-\ell}x^{d-k}g(L_k,x+1)$$
$$= \sum_{k=j}^{d}\sum_{\ell=0}^{i}\binom{i}{\ell}\binom{d-j-i}{d-k-\ell}2^{i-\ell}2^{d-k-i}x^{d-k}g(L_k,x+1).$$

Apply Lemma 3.3.3 to get

$$\sum_{\pi\in NC(d)} wt'_{\mathcal{S}}(\pi) = \sum_{k=j}^{d}\sum_{\ell=0}^{i}\binom{i}{\ell}\binom{d-j-\ell}{d-k}2^{d-k-i}x^{d-k}g(L_k,x+1).$$

Note, if $\ell > k-j$, $\binom{d-j-\ell}{d-k} = 0$. Similarly, if $\ell > i$, $\binom{i}{\ell} = 0$. Hence,

$$\sum_{\pi\in NC(d)} wt'_{\mathcal{S}}(\pi) = \sum_{k=j}^{d}\sum_{\ell=0}^{k-j}\binom{i}{\ell}\binom{d-j-\ell}{d-k}2^{d-k-i}x^{d-k}g(L_k,x+1)$$
$$= \sum_{k=j}^{d}\left[\binom{i}{0}\binom{d-j}{d-k} + \binom{i}{1}\binom{d-j-1}{d-k} + \right.$$
$$\left. \ldots + \binom{i}{k-j}\binom{d-k}{d-k}\right] 2^{d-k-i}x^{d-k}g(L_k,x+1).$$

Applying Pascal's identity to each of the $\binom{i}{\ell}$ yields

$$\sum_{\pi\in NC(d)} wt'_{\mathcal{S}}(\pi) = \sum_{k=j}^{d}\left(\frac{1}{2}\right)^{i}\left\{\binom{i-1}{0}\binom{d-j}{d-k} + \left[\binom{i-1}{0} + \binom{i-1}{1}\right]\binom{d-j-1}{d-k} + \right.$$
$$\left. \ldots + \left[\binom{i-1}{k-j} + \binom{i-1}{k-j-1}\right]\binom{d-k}{d-k}\right\} 2^{d-k}x^{d-k}g(L_k,x+1).$$

Rearranging the terms gives

$$\sum_{\pi\in NC(d)} wt'_{\mathcal{S}}(\pi) = \sum_{k=j}^{d}\left(\frac{1}{2}\right)^{i}\left\{\binom{i-1}{0}\left[\binom{d-j-1}{d-k} + \binom{d-j}{d-k}\right] + \right.$$
$$\left. \ldots + \binom{i-1}{k-j}\left[\binom{d-k-1}{d-k} + \binom{d-k}{d-k}\right]\right\} 2^{d-k}x^{d-k}g(L_k,x+1)$$

$$= \sum_{k=j}^{d} \sum_{\ell=1}^{k-j+1} \left(\frac{1}{2}\right)^i \binom{i-1}{\ell-1} \left[\binom{d-\ell-j}{d-k} + \binom{d-\ell-j+1}{d-k}\right] 2^{d-k} x^{d-k} g(L_k, x+1).$$

Interchange the summations, rearrange terms, and use Equations (10) and (11) to get

$$\begin{aligned}\sum_{\pi\in NC(d)} wt'_{\mathcal{S}}(\pi) &= \sum_{\ell=1}^{i} \left(\frac{1}{2}\right)^i \binom{i-1}{\ell-1} \sum_{k=\ell+j-1}^{d} \left[\binom{d-\ell-j}{d-k}\right. \\ &\qquad \left. + \binom{d-\ell-j+1}{d-k}\right] 2^{d-k} x^{d-k} g(L_k, x+1) \\ &= \sum_{\ell=1}^{i} \left(\frac{1}{2}\right)^i \binom{i-1}{\ell-1} Q_{d,\ell+j}(x+1) \\ &= C_{d,i,j}(x+1).\end{aligned}$$

□

Thus, it was found that $C_{d,i,j}(x) = \sum_{\pi\in NC(d)} wt_{\mathcal{S}}(\pi)$ for some family $\mathcal{S}$ of intervals of $[1,d]$ where $\mathcal{S}$ consists of i intervals and a total of $d-j$ elements are contained in its intervals.

The proof of Theorem 3.3.4 could be shorten by using the following lemma instead of Lemma 3.3.3. In Lemma 3.3.5, a combinatorial reason is given for why both sides of the equation will count the same number of objects. Using this lemma, allows one to condense several steps at the end of the proof of Theorem 3.3.4.

Lemma 3.3.5. *Let $d > 0$, $i > 0$, $0 \le j \le d-i$, and $j \le k \le d$. Then*

$$\sum_{\ell=0}^{i} \binom{i}{\ell} \binom{d-i-j}{d-k-\ell} \cdot 2^{i-\ell} = \sum_{m=0}^{i-1} \binom{i-1}{m} \left[2\binom{d-m-j-1}{d-k} + \binom{d-m-j-1}{d-k-1}\right].$$

Proof. The following proof will show that both sides of the given equation count all pairs (X, f) such that X is a subset of $\{1, \dots, d-j\}$ and $f : \{1, \dots, i\}\backslash X \to \{1, 2\}$ is a 2-coloring of $\{1, \dots, i\}\backslash X$.

On the left hand side, fix the size ℓ of $X \cap \{1,\ldots,i\}$. The binomial coefficients count the number of ways to select $X \cap \{1,\ldots,i\}$ and $X \cap \{i+1,\ldots,d-j\}$, respectively. Finally, $2^{i-\ell}$ is the number of ways to select f.

On the right hand side, set m as the size of the set $Y := f^{-1}(1) \cap \{1,\ldots,i-1\}$. In other words, Y is the set of elements of color 1 that are different from i. There are $\binom{i-1}{m}$ ways to select Y. The elements of $\{1,\ldots,i-1\} \backslash Y$ either belong to X or have color 2. If i does not belong to X, then there are two ways to select the color of i, and X is a subset of $(\{1,\ldots,i-1\} \backslash Y) \uplus \{i+1,\ldots,d-j\}$, which may be selected $\binom{d-m-j-1}{d-k}$ ways. The elements of the remaining set $\{1,\ldots,d-j\} \backslash (X \uplus Y \uplus \{i\})$ must have color 2. A similar reasoning for the case when i belongs to X shows that the elements of $X \backslash \{i\}$ may be selected in $\binom{d-m-j-1}{d-k-1}$ ways, completing the proof that the right hand side counts the same set of objects as the left hand side. □

The result of Theorem 3.3.4 could also have been proved by induction. This alternative method is outlined in the following remark and provides an intuitive reason for why $C_{d,i,j}(x)$ is a linear combination of the polynomials $Q_{d,k}(x)$.

Remark 3.3.6. *For $i > 1$, $C_{d,i,j}(x) = \frac{1}{2}[C_{d,i-1,j}(x) + C_{d,i-1,j+1}(x)]$. Let $\mathcal{P}$ be a d-dimensional cubical complex, and assume that $F_1,\ldots,F_k$ is a shelling of $\mathcal{P}$. For $1 \leq m \leq k$, suppose $F_m \cap (F_1 \cup \cdots \cup F_{m-1})$ is a shelling component of type (i_m, j_m) where $i_m > 0$. Adding F_m to $F_1 \cup \cdots \cup F_{m-1}$ results either in introducing a new antipodally unpaired facet to the component or F_m is the second facet of an antipodal pair of a previously listed facet. In the first case, $i_m = i_{m-1} + 1$ and $j_m = j_{m-1}$. In the second case, $i_m = i_{m-1} - 1$ and $j_m = j_{m-1} + 1$. Hence, for $i > 1$, $C_{d,i+1,j}(x) =$*

$\frac{1}{2}[C_{d,i,j}(x) + C_{d,i,j+1}(x)]$.

Applying this relation repeatedly, gives the result that for $i > 1$,

$$\begin{aligned}
C_{d,i,j}(x) &= \frac{1}{2}[C_{d,i-1,j}(x) + C_{d,i-1,j+1}(x)] \\
&= \frac{1}{2}\left[\frac{1}{2}\left(C_{d,i-2,j}(x) + C_{d,i-2,j+1}(x)\right) + \frac{1}{2}\left(C_{d,i-2,j+1}(x) + C_{d,i-2,j+2}(x)\right)\right] \\
&= \left(\frac{1}{2}\right)^2 [C_{d,i-2,j}(x) + 2C_{d,i-2,j+1}(x) + C_{d,i-2,j+2}(x)] \\
&= \dots \\
&= \left(\frac{1}{2}\right)^{i-1} \left[\binom{i-1}{0} C_{d,1,j}(x) + \dots + \binom{i-1}{i-1} C_{d,1,i+j-1}(x)\right].
\end{aligned}$$

Hence, $C_{d,1,j}(x), \dots, C_{d,1,i+j-1}(x)$ *acts as a basis for* $C_{d,i,j}(x)$. *Substituting* $Q_{d,j+1}(x) = 2C_{d,1,j}(x)$ *yields Equation (11):*

$$C_{d,i,j}(x) = \left(\frac{1}{2}\right)^{i} \left[\binom{i-1}{0} Q_{d,j+1}(x) + \dots + \binom{i-1}{i-1} Q_{d,i+j}(x)\right].$$

After it has been shown that $C_{d,i,j}(x) = \sum_{\pi \in NC(d)} wt_{\mathcal{S}}(\pi)$ *for some* $\mathcal{S}$ *consisting of* i *intervals and* j *elements in* $[1, d]$ *but not in any interval of* $\mathcal{S}$, *consider the following interpretation of* $C_{d,i,j}(x) = \frac{1}{2}[C_{d,i-1,j}(x) + C_{d,i-1,j+1}(x)]$.

Pick an $\mathcal{S}$ *which has* i *intervals and* $d - j$ *elements contained in those intervals such that one of those intervals consists of a single element. Note, since* $\mathcal{S}$ *is not specified other than number of intervals and elements for* $C_{d,i,j}(x) = \sum_{\pi \in NC(d)} wt_{\mathcal{S}}(\pi)$, *specifying additional conditions here will not change the result. In this terminology, if an extra unpaired facet is added to the shelling, one of the intervals of* $\mathcal{S}$ *is divided; whereas, if a facet which completes a pair of antipodal facets is added, it corresponds to an element being moved out of* $\mathcal{S}$. *Hence, a chosen* $\mathcal{S}$ *can be built in one of two moves: start with some other* $\mathcal{S}$ *which has* $i - 1$ *intervals. Divide one of the intervals* *in two; i.e., start with* $C_{d,i-1,j}(x)$ *to get* $C_{d,i,j}(x)$*. Alternatively, one element may be taken from the complement of* $\mathcal{S}$ *and moved into* $\mathcal{S}$*; i.e., start from* $C_{d,i-1,j+1}(x)$*. The element moved into* $\mathcal{S}$ *is not joined to any existing interval in* $\mathcal{S}$ *since that would constitute a second move; hence, the number of intervals of* $\mathcal{S}$ *also increases.*

Recall that type b elements are defined as last elements of a block where all other elements in that block of the partition are type a, and the entire block is contained in the same interval of $\mathcal{S}$. However, in Definition 2.4.9, the weight function only gives weight to antisingleton and singleton elements. Marked singleton elements are a specific case of type b elements. The question that naturally arises is how often are type b elements singletons?

Consider strings of n consecutive type a and type b elements. There are 2^n such strings with a total of $n \cdot 2^{n-1}$ type b elements contained in all 2^n strings. Let N_n be the number of nonsingleton type b elements contained in all 2^n strings. See Table 14.

n	total strings	number of b	N_n
1	2	1	0
2	4	4	1
3	8	12	4
4	16	32	12
...	...	...	...
n	2^n	$n \cdot 2^{n-1}$	$(n-1) \cdot 2^{n-2}$

Table 14: Statistics on strings of type a and type b elements

N_n counts the number of pairs ab in strings of length n. Consider now strings of length $n+1$, which are formed by inserting either a type a or type b element at the beginning of a length n string. If a type b element is at position 1, then N_n counts the number of nonsingleton type b elements over all these length $n+1$ strings. However, if a type a element is at position 1, then N_n+2^{n-1} counts the number of nonsingleton type b elements over all length $n+1$ strings. The extra 2^{n-1} counts all the situations where the second element of the string is type b, transforming that type b element from being counted as a singleton element in the length n string to a nonsingleton element in the length $n+1$ string. Hence, $N_{n+1} = 2N_n + 2^{n-1}$. Using the initial values (see Table 14) yields a closed formula: $N_n = (n-1)\cdot 2^{n-2}$.

Then the percentage of nonsingleton type b elements among length n strings is

$$\frac{\text{number of nonsingleton type } b\text{'s}}{\text{total number of type } b\text{'s}} = \frac{N_n}{n\cdot 2^{n-1}} = \frac{(n-1)\cdot 2^{n-2}}{n\cdot 2^{n-1}} = \frac{n-1}{2n}.$$

Notice the number of nonsingleton type b elements is strictly less than half of the total number of type b elements. However, as the length of the string approaches infinity, the number of nonsingleton type b elements in the string approaches 50%. On the other hand, when n is small, type b elements are predominately singleton elements.

In the situations where type a and type b elements are inserted into a partition at positions in $\mathcal{S}$, as in the previous proof, the only restriction on the length of these inserted strings is that the string must stay within an interval of $\mathcal{S}$. As such, many of the inserted strings of type a and type b elements may be short, which implies that, overall, the majority of type b elements will be singleton elements.

3.3.2 $Q_{d,k}(x)$ as the Total Weight of Objects

Theorem 3.3.7. *For* $0 \leq k \leq d+1$,

$$Q_{d,k}(x) = \begin{cases} 2\sum_{\pi \in NC(d)} wt_k(\pi) & \text{if } 1 \leq k \leq d, \\ \sum_{\pi \in NC(d)} wt_k(\pi) & \text{if } k = 0 \text{ or } d+1. \end{cases}$$

Proof. Case 1: Suppose $1 \leq k \leq d$. Definition 2.4.11 defines the weight function $wt_k(\pi)$ for any $\pi \in NC(d)$. Apply this definition to Lemma 3.3.1 to get the desired result.

Case 2: Let $k = d+1$. By Lemma 2.4.12, $Q_{d,d+1}(x) = g(L_d, x) = \sum_{\pi \in NC(d)} x^{\text{block}(\pi)}$, which gives the desired result.

Case 3: Suppose $k = 0$. Then $\mathcal{S} = \{[1,d]^*\}$. Let $wt'_0(\pi)$ be the weight of π in $NC(d)$ where marked special objects have a weight of x and all others a weight of 1 as defined in the proof of Lemma 3.3.1. This case be will proved by showing that $Q_{d,0}(x+1) = \sum_{\pi \in NC(d)} wt'_0(\pi)$.

To compute $\sum_{\pi \in NC(d)} wt'_0(\pi)$, choose π in $NC(d)$, and mark as many nonsingleton blocks, antisingleton elements of π, and last elements of π where all other elements of the block have been marked as desired. Since $\mathcal{S} = \{[1,d]^*\}$, d is an antisingleton of π if and only if d is a singleton of $\alpha(\pi)$. Define *type a* and *type b* elements as in the proof of the Lemma 3.3.1.

Let $d - \ell$ be the number of type a and type b elements in π. Then $0 \leq \ell \leq d$, and $\pi_\ell \in NC(\ell)$ as defined in Lemma 3.3.1. Suppose $\ell \neq 0, 1$. Then $0 \leq d - \ell < d - 1$. Summing over all partitions and ways to mark the elements yields $\sum_{\ell=2}^{d} 2^{d-\ell}\binom{d}{d-\ell} x^{d-\ell} g(L_\ell, x+1)$.

When $\ell = 0$, all elements are type a or type b elements. The total weight for each of these partitions is x^d. Let n be the number of blocks of the partition, and suppose $n > 1$. The type b elements will determine the position of the blocks of the partition since each block ends with a type b element. Consider the partition in its circular representation, and pick the location of the type b elements. There are $\sum_{n=2}^{d} \binom{d}{n} = 2^d - d - 1$ ways to choose these positions. Thus, the total weight of all such partitions is $(2^d - d - 1)x^d$. On the other hand, suppose the partition has only one block. Then, obviously, $\pi = (1 \ldots d)$ whose weight is $x^d(x+1)$, where $(x+1)$ is the weight of the block, which can either be marked or unmarked.

When $\ell = 1$, all elements except for one are type a or type b elements. Suppose the partition has at least two blocks. Once the unmarked position is chosen, there are 2^{d-1} ways to arrange the type a and type b elements. However, if all the $d-1$ marked elements are type a, the partition has exactly one block. Hence, there are really $2^{d-1} - 1$ ways to arrange the type a and type b elements, giving a total weight of $d(2^{d-1} - 1)x^{d-1}$. If the partition has one block, then $\pi = (1 \ldots d)$, and its weight is $dx^{d-1}(x+1)$.

Summing over all partitions of $NC(d)$ yields a total weight of

$$\begin{aligned}
\sum_{\pi \in NC(d)} wt'_0(\pi) &= \left[x^d(x+1) + (2^d - d - 1)x^d\right] + \left[dx^{d-1}(x+1) + d(2^{d-1} - 1)x^{d-1}\right] \\
&\qquad + \sum_{\ell=2}^{d} 2^{d-\ell} \binom{d}{d-\ell} x^{d-\ell} g(L_\ell, x+1) \\
&= x^{d+1} + \sum_{\ell=0}^{d} 2^{d-\ell} \binom{d}{\ell} x^{d-\ell} g(L_\ell, x+1) \\
&= Q_{d,0}(x+1).
\end{aligned}$$

Thus, $Q_{d,0}(x) = \sum_{\pi \in NC(d)} wt_0(\pi)$. $\square$

The next two corollaries follow directly from the result of Theorem 3.3.7; these two properties are illustrated in Table 15 for small d.

Corollary 3.3.8. *The coefficients of $Q_{d,k}(x)$ are nonnegative integers for $0 \leq k \leq d+1$.*

Corollary 3.3.9. *The coefficients of $Q_{d,k}(x)$ are even for $1 \leq k \leq d$.*

$d \backslash k$	0	1	2	3	4	5
0	x	1				
1	x^2	$2x$	1			
2	x^2+x^3	$4x^2$	$4x$	$1+x$		
3	$4x^3+x^4$	$2x^2+8x^3$	$10x^2$	$8x+2x^2$	$1+4x$	
4	$2x^3+11x^4+x^5$	$12x^3+16x^4$	$4x^2+24x^3$	$24x^2+4x^3$	$16x+12x^2$	$1+11x+2x^2$

Table 15: The polynomials $Q_{d,k}(x)$ for small d

3.4 The Duality of the Polynomials $C_{d,i,j}(x)$

Suppose $\mathcal{S} = \{[k_1, l_1], \ldots, [k_i, l_i]\}$ as defined in Section 3.3 where $[k_n, l_n] \subseteq [1, d]$ for $1 \leq n \leq i$. Consider also $\mathcal{S}'$, $[1, d] - \mathcal{S}$, and $[1, d] - \mathcal{S}'$ as defined in Section 3.3.

Lemma 3.4.1. *Let $\pi \in NC(d)$ and $\mathcal{S}$ as defined above. Then $wt_{\mathcal{S}}(\pi) \cdot wt_{\mathcal{S}'}(\alpha(\pi)) = x^{d+1}$.*

Proof. By Equation (5), $wt_{\mathcal{S}}(\pi) = x^{\text{block}(\pi)} \cdot x^{\text{sing}_{\mathcal{S}}(\pi)} \cdot x^{\text{sing}_{[1,d]-\mathcal{S}'}(\alpha(\pi))}$, and $wt_{\mathcal{S}'}(\alpha(\pi)) = x^{\text{block}(\alpha(\pi))} \cdot x^{\text{sing}_{\mathcal{S}'}(\alpha(\pi))} \cdot x^{\text{sing}_{[1,d]-\mathcal{S}}(\pi)}$. By definition, $\text{sing}(\pi) = \text{sing}_{\mathcal{S}}(\pi) + \text{sing}_{[1,d]-\mathcal{S}}(\pi)$.

Using these definitions and applying Lemma 2.4.5 yields

$$\begin{aligned} wt_{\mathcal{S}}(\pi)\cdot wt_{\mathcal{S}'}(\alpha(\pi)) &= x^{\text{block}(\pi)}\cdot x^{\text{sing}_{\mathcal{S}}(\pi)}\cdot x^{\text{sing}_{[1,d]-\mathcal{S}'}(\alpha(\pi))} \\ &\quad \cdot x^{\text{block}(\alpha(\pi))}\cdot x^{\text{sing}_{\mathcal{S}'}(\alpha(\pi))}\cdot x^{\text{sing}_{[1,d]-\mathcal{S}}(\pi)} \\ &= x^{\text{block}(\pi)+\text{sing}(\pi)+\text{sing}(\alpha(\pi))+\text{block}(\alpha(\pi))} \\ &= x^{|\pi|+|\alpha(\pi)|} = x^{d+1}. \end{aligned}$$

□

An immediate consequence of Lemma 3.4.1 is that if $wt_{\mathcal{S}}(\pi) = x^k$ for some k, then $wt_{\mathcal{S}'}(\alpha(\pi)) = x^{d+1-k}$. Note, Lemma 3.4.1 also holds for $\mathcal{S} = \emptyset$ (with $\mathcal{S}' = \{[1,d]^*\}$) and for $\mathcal{S} = \{[1,d]^*\}$ (where $\mathcal{S}' = \emptyset$) since $\text{sing}_{\emptyset}(\pi) = 0$ and $\text{sing}_{\{[1,d]^*\}}(\pi) = \text{sing}(\pi)$.

Theorem 3.4.2. *Let $i > 0$ and $0 \le j \le d-i$. Then for $0 \le k \le d+1$, $[x^k]C_{d,i,j}(x) = [x^{d+1-k}]C_{d,i,d-i-j}(x)$.*

Proof. By Theorem 3.3.4, $C_{d,i,j}(x) = \sum_{\pi\in NC(d)} wt_{\mathcal{S}}(\pi)$ for some set $\mathcal{S}$ of pairwise disjoint intervals of $[1,d]$ defined as in Section 3.3. Recall $\mathcal{S}$ consists of i intervals and contains a total of $d-j$ elements in its intervals. Define $\mathcal{S}'$ as before. Then $\mathcal{S}'$ has i intervals and contains a total of $i+j$ elements in its intervals. Pick any $\pi \in NC(d)$, and let $k \ge 0$ be such that $wt_{\mathcal{S}}(\pi) = x^k$. Define $\alpha(\pi)$ as before. By Lemma 3.4.1, $wt_{\mathcal{S}'}(\alpha(\pi)) = x^{d+1-k}$, which means that $C_{d,i,d-i-j}(x) = \sum_{\pi\in NC(d)} wt_{\mathcal{S}'}(\alpha(\pi))$. Thus, $[x^k]\sum_{\pi\in NC(d)} wt_{\mathcal{S}}(\pi) = [x^{d+1-k}]\sum_{\pi\in NC(d)} wt_{\mathcal{S}'}(\alpha(\pi))$, implying that $[x^k]C_{d,i,j}(x) = [x^{d+1-k}]C_{d,i,d-i-j}(x)$. □

Using these results, the duality of the polynomials $Q_{d,k}(x)$ is easily shown as seen below.

Lemma 3.4.3. *For* $0 \leq \ell \leq \lfloor d/2 \rfloor$*,* $[x^\ell]Q_{d,d+1}(x) = [x^{d+1-\ell}]Q_{d,0}(x)$*.*

In the proof of Proposition 2.6 in [64], Stanley showed that

$$x^{d+1}g(L_d, 1/x) = (x-1)^{d+1} + \sum_{\ell=0}^{d} 2^{d-\ell}\binom{d}{\ell}(x-1)^{d-\ell}g(L_\ell, x).$$

Reinterpreting this in terms of the polynomials $Q_{d,k}(x)$ yields $x^{d+1}Q_{d,d+1}(1/x) = Q_{d,0}(x)$, which gives the desired result. Additionally, if $\ell > \lfloor d/2 \rfloor$, then $[x^\ell]Q_{d,d+1}(x) = 0 = [x^{d+1-\ell}]Q_{d,0}(x)$.

Lemma 3.4.4. *Let* $\pi \in NC(d)$ *and* $0 \leq k \leq d+1$*, then* $wt_k(\pi) \cdot wt_{d+1-k}(\alpha(\pi)) = x^{d+1}$*.*

Proof. If $k = 0$ or $k = d+1$, the result follows directly from the proof of Lemma 3.4.3 and the definition of $wt_k(\pi)$. Suppose $1 \leq k \leq d$ and $\pi \in NC(d)$. Let $\mathcal{S} = \{[k, d]\}$. Then $\mathcal{S}' = \{[d+1-k, d]\}$. By Definition 2.4.11 and Lemma 3.4.1, $wt_k(\pi) \cdot wt_{d+1-k}(\alpha(\pi)) = wt_{\mathcal{S}}(\pi) \cdot wt_{\mathcal{S}'}(\alpha(\pi)) = x^{d+1}$. □

A direct consequence of Lemma 3.4.4 is that if $\pi \in NC(d)$ where $wt_k(\pi) = x^\ell$ for some ℓ then $wt_{d+1-k}(\alpha(\pi)) = x^{d+1-\ell}$.

Theorem 3.4.5. *Let* $0 \leq k \leq d+1$*. Then* $[x^\ell]Q_{d,k}(x) = [x^{d+1-\ell}]Q_{d,d+1-k}(x)$ *for* $0 \leq \ell \leq d+1$*.*

Proof. When k equals 0 or $d+1$, apply Lemma 3.4.3. If $1 \leq k \leq d$, let $\mathcal{S} = \{[k, d]\}$; then $\mathcal{S}' = \{[d+1-k, d]\}$. Apply Theorem 3.4.2 to get $[x^\ell]C_{d,1,k-1}(x) = [x^{d+1-\ell}]C_{d,1,d-k}(x)$. Thus, $[x^\ell]Q_{d,k}(x) = [x^{d+1-\ell}]Q_{d,d+1-k}(x)$ since, by Equation (11), $Q_{d,d+1-k}(x) = 2 \cdot C_{d,1,d-k}(x)$. □

Example 3.4.6. *Let P be a d-cube. The normalized Adin h-vector for P is $h_k = 1$ for all $0 \leq k \leq d+1$. Hence, $f(P,x) = \sum_{k=0}^{d+1} Q_{d,k}(x)$. Note, $h(P,x) = x^{d+1} f(P,1/x) = f(P,x)$. Thus, the h polynomials will be symmetric (see Table 16) because of the duality of the polynomials $Q_{d,k}(x)$.*

d	$h(P,x)$
0	$x+1$
1	x^2+2x+1
2	x^3+5x^2+5x+1
3	$x^4+12x^3+14x^2+12x+1$
4	$x^5+27x^4+42x^3+42x^2+27x+1$

Table 16: The h polynomials for the d-cube for small d

Remark 3.4.7. *For any d-dimensional cubical sphere $\mathcal{P}$, the face poset is Eulerian. Thus, the toric h polynomial satisfies the generalized Dehn-Sommerville equations $h(\mathcal{P},x) = x^{d+1} h(\mathcal{P},1/x)$ [64]. In terms of the polynomials $Q_{d,k}(x)$ this statement is the same as*

$$\sum_{k=0}^{d+1} h_k Q_{d,k}(x) = x^{d+1} \sum_{k=0}^{d+1} h_k Q_{d,k}\left(\frac{1}{x}\right).$$

Adin [2] showed that the Dehn-Sommerville equations holding for $\mathcal{P}$ may be restated as $h_k = h_{d+1-k}$ holding for the Adin h-vector. Combining these produces

$$\sum_{k=0}^{d+1} h_k Q_{d,k}(x) = \sum_{k=0}^{d+1} h_k x^{d+1} Q_{d,d+1-k}\left(\frac{1}{x}\right).$$

Since the cubical Dehn-Sommerville equations are a complete set of linear relations even for cubical polytopes [32], we know that $h_0, \ldots, h_{\lfloor \frac{d+1}{2} \rfloor}$ are linearly independent.

Comparing the contributions of h_k *and* h_{d+1-k} *on both sides of the last equation yields*

$$Q_{d,k}(x) + Q_{d,d+1-k}(x) = x^{d+1}\left(Q_{d,d+1-k}\left(\frac{1}{x}\right) + Q_{d,k}\left(\frac{1}{x}\right)\right). \tag{13}$$

Conversely, it is not difficult to show that the Dehn-Sommerville equations, stated for the Adin h-vector, and Equation (13) imply $f(\mathcal{P},x) = x^{d+1}f(\mathcal{P},1/x)$. *Equation (13) is a direct consequence of Theorem 3.4.5, but Theorem 3.4.5 can not be derived from it.*

Remark 3.4.8. *Theorem 3.4.5 could also be shown directly by specializing the proof of Theorem 3.4.2. After that Theorem 3.4.2 can be proved by combining Theorem 3.4.5 and Equation (11) as follows:*

$$\begin{aligned} x^{d+1}C_{d,i,d-i-j}(1/x) &= \sum_{k=d+1-i-j}^{d-j}\left(\frac{1}{2}\right)^i\binom{i-1}{k-1-d+i+j}x^{d+1}Q_{d,k}(1/x) \\ &= \sum_{k=d+1-i-j}^{d-j}\left(\frac{1}{2}\right)^i\binom{i-1}{k-1-d+i+j}Q_{d,d+1-k}(x). \end{aligned}$$

Set $\ell = d+1-k$ *to get*

$$x^{d+1}C_{d,i,d-i-j}(1/x) = \sum_{\ell=j+1}^{i+j}\left(\frac{1}{2}\right)^i\binom{i-1}{i+j-\ell}Q_{d,\ell}(x) = C_{d,i,j}(x).$$

CHAPTER 4: A GRAY CODE FOR SHELLINGS OF THE HYPERCUBE

As a consequence of Lemma 2.3.3 and how the hypercube, or n-cube, was defined in Chapter 2.3, shellings of the boundary complex of an n-cube may be described in the following way. Recall, using this notation means that each enumeration of the facets of the boundary of the n-cube will be identified with a signed permutation of the set $\{\pm 1, \ldots, \pm n\}$.

Lemma 4.0.9. *An enumeration $(F_1, \ldots, F_{2n})$ of the facets of the boundary complex of the n-cube is a shelling if and only if for each $m < 2n$, the set $\{F_1, \ldots, F_m\}$ contains at least one antipodally unpaired facet.*

Proof. The cubical complex $F_1 \cup \cdots \cup F_m$ is shellable and $(n-1)$-dimensional, and as such, it is homeomorphic to a $(n-1)$-ball or an $(n-1)$-sphere. Since the boundary complex $F_1 \cup \cdots \cup F_{2n}$ is an $(n-1)$-sphere, the proper subcomplex $F_1 \cup \cdots \cup F_m$ can only be an $(n-1)$-ball. By [25, Lemma 3.3] (see part (i) of Lemma 2.3.3 above), $F_1 \cup \cdots \cup F_m$ must contain at least one antipodally unpaired facet.

Conversely, assume that $F_1 \cup \cdots \cup F_m$ contains at least one antipodally unpaired facet for each $m < 2n$. In other words, each $F_m \cap (F_1 \cup \cdots \cup F_{m-1})$ is a cubical shelling component in the shelling of an n-dimensional cubical complex, and its type (i_m, j_m) satisfies $i_m > 0$. Adding F_m to $F_1 \cup \cdots \cup F_{m-1}$ results either in introducing a new antipodally unpaired facet or F_m is the antipodal pair of a previously listed facet. In the first case, $i_m = i_{m-1} + 1$ and $j_m = j_{m-1}$. In the second case, $i_m = i_{m-1} - 1$ and $j_m = j_{m-1} + 1$. Finally, for $m = 2n$, $F_{2n} \cap (F_1 \cup \cdots \cup F_{2n-1})$ is a shelling component of type $(0, n-1)$. □

For each initial segment of an enumeration of the facets of a shelling of the boundary of the n-cube, there is at least one antipodal pair of facets such that exactly one of the two facets belongs to the shelling component; i.e., the facet k is listed among the first i facets of the shelling but facet $-k$ facet is not [25].

Corollary 4.0.10. *A permutation of $\{\pm 1, \ldots, \pm n\}$ is sign-connected if and only if it represents a shelling of the facets of the hypercube. Similarly, a sign-disconnected permutation represents an enumeration of the facets that is not a shelling.*

As a result of Corollary 4.0.10, the number of sign-connected permutations equals the number of shellings of the boundary of the n-cube, which differ by an isometry. Given any n, this number can be found by the recursive formula $a_n = (2n-1)!! - \sum_{k=1}^{n-1}(2k-1)!! \cdot a_{n-k}$. The sequence $\{a_n\}$ is sequence A000698 in the On-Line Encyclopedia of Integer Sequences [1].

4.1 Equivalence Classes of the Enumerations of the Facets of the n-Cube

The isometries of the n-cube permute its facets, inducing a B_n action on the enumerations of all facets of the boundary of the n-cube. This action is free; i.e., any nontrivial isometry takes each enumeration into a different enumeration.

Definition 4.1.1. *Two enumerations of the facets of the n-cube will be considered equivalent if they can be transformed into each other by an isometry of the n-cube.*

As noted in Chapter 2.3, every enumeration of the facets of the boundary of the n-cube can be identified with a signed permutation. The induced action of B_n on these signed permutations is generated by the following operations:

1. For each $k \in \{1, \ldots, n\}$ there is a reflection ε_k interchanging k with $-k$ and leaving all other entries unchanged;

2. For each $\{i, j\} \subset \{1, \ldots, n\}$ there is a reflection $\rho_{i,j}$ interchanging i with j and $-i$ with $-j$, leaving all other entries unchanged.

It is worth noting that the elements of B_n may be identified with signed permutations in the usual way; the action of B_n considered is the action of B_n on itself, via conjugation. There exist $2^n \cdot n!$ symmetries of the n-cube [40]. Since the B_n action is free, all equivalence classes have the same cardinality, giving a total of $\dfrac{(2n)!}{2^n \cdot n!}$ equivalence classes.

From Chapter 2.5.1, there are a total of $\dfrac{(2n)!}{2^n \cdot n!}$ standard permutations.

Lemma 4.1.2. *Each equivalence class of the enumerations of the facets of an n-cube corresponds to exactly one standard permutation.*

Proof. Since the number of equivalence classes and the number of standard permutations are the same, it only remains to be shown that every $\pi \in \mathcal{S}_{\{\pm 1, \ldots, \pm n\}}$ is equivalent to a standard permutation.

Pick any $\pi \in \mathcal{S}_{\{\pm 1, \ldots, \pm n\}}$, and suppose Π is the equivalence class that contains π. Applying only reflections $\varepsilon_k \in B_n$, π may be replaced with a $\pi' \in \Pi$ that satisfies condition (1) in Definition 2.5.1. Applying only reflections $\rho_{i,j} \in B_n$, π' may be replaced with a $\pi'' \in \Pi$ that also satisfies condition (2) in Definition 2.5.1. Note that the application of an operator $\rho_{i,j}$ leaves the validity of condition (1) unchanged. □

The set of sign-connected permutations is closed under the action of B_n; thus, the equivalence classes of shellings may be thought of as types of shellings.

4.2 A Gray Code for all Standard Permutations

In this section, a Gray code will be defined for the standard permutations of the set $\{\pm 1, \ldots, \pm n\}$ when $n \geq 1$. Because standard permutations can be written in the form $\sigma(a_1 \cdots a_n)$, first define a code on the words $a_1 \cdots a_n$ (see Definition 2.5.8). Call this listing the *Gray code for words.* To get the Gray code for the standard permutations, simply convert each word into its associated standard permutation.

Recursively define the Gray code for words as follows. For $n = 1$, there is only one word to list, namely 1. To write the Gray code for words of length n, start with $11 \ldots 1$. Increase a_n by 1 to get the next word: $11 \ldots 12$. Continue to increase a_n by 1 to generate a list of codes. Stop increasing a_n once $a_n = 2n - 1$ (recall $1 \leq a_i \leq 2i - 1$ for any i). Now replace $a_1 \cdots a_{n-1}$ with the next word in the Gray code for words of length $n - 1$ to get $11 \ldots 12(2n - 1)$. Decrease a_n by 1 to get the next string of codes. When $a_n = 1$, replace $a_1 \cdots a_{n-1}$ with the next word in the Gray code of length $n - 1$. Continue in this fashion until the Gray code terminates with $135 \ldots (2n - 1)$. See Table 17 for the Gray code when $n = 2$ and $n = 3$.

Theorem 4.2.1. *The enumeration defined above is an adjacent transposition Gray code for the standard permutations of the set* $\{\pm 1, \ldots, \pm n\}$.

Proof. Proceed by induction on n. Compare two consecutive words $a_1 \cdots a_n$ and

length	code				
$n=2$	11	12	13		
$n=3$	111	112	113	114	115
	125	124	123	122	121
	131	132	133	134	135

Table 17: The Gray Code for $n = 2$ and $n = 3$

$b_1 \cdots b_n$ in the list. They will differ at exactly one letter. There are two cases, depending on whether this letter is the last letter of the word or some other letter.

Case 1: $a_n \neq b_n$. In this case, the first $n-1$ arcs in the two associated arc diagrams remain stationary, and the n^{th} arc moves to the right or to the left by one vertex position. This move switches the left end of the n^{th} arc with an adjacent end of a different arc. As mentioned at the end of Chapter 2.5, switching adjacent ends of two arcs corresponds to an adjacent transposition on the associated signed permutation.

Case 2: $a_n = b_n$. This case occurs only when $a_n = 1$, corresponding to the n^{th} arc stretching over the first $n-1$ arcs, or when $a_n = 2n-1$, meaning the n^{th} arc is a minimal arc. In either situation, the n^{th} arc will not affect whether or not the move produces an adjacent transposition in the associated standard permutations. By the recursive definition of the Gray code, replace $a_1 \cdots a_{n-1}$ with the next word in the Gray code for words of length $n-1$. By the induction hypothesis, consecutive words in the Gray code for words of length $n-1$ correspond to standard permutations on $\{\pm 1, \ldots, \pm(n-1)\}$ that differ by an adjacent transposition. □

Fig. 32 illustrates the relationship between the Gray code for signed permutations and its associated arc diagrams and words.

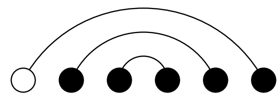
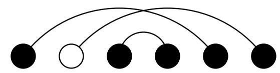
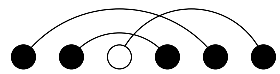
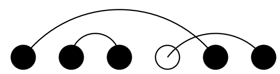
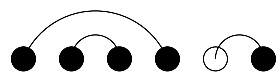
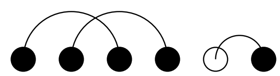
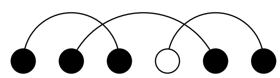
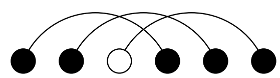
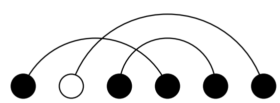
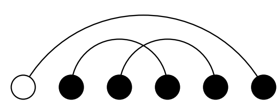
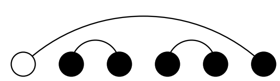
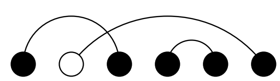
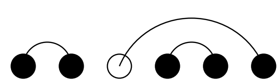
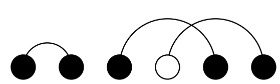
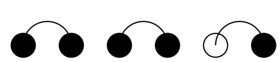
Signed Permutations	Arc Diagrams	Words
3,2,1,−1,−2,−3		111
2,3,1,−1,−2,−3		112
2,1,3,−1,−2,−3		113
2,1,−1,3,−2,−3		114
2,1,−1,−2,3,−3		115
1,2,−1,−2,3,−3		125
1,2,−1,3,−2,−3		124
1,2,3,−1,−2,−3		123
1,3,2,−1,−2,−3		122
3,1,2,−1,−2,−3		121
3,1,−1,2,−2,−3		131
1,3,−1,2,−2,−3		132
1,−1,3,2,−2,−3		133
1,−1,2,3,−2,−3		134
1,−1,2,−2,3,−3		135

Figure 32: The Gray code for $n = 3$ in terms of signed permutations, arc diagrams, and words

4.3 Properties of the full Gray Code

The Gray code for the equivalence classes of the enumerations of the facets of the boundary of the n-cube, denoted by their standard permutations, will be referred to as the *full Gray code*. As seen in Chapter 4.2, this Gray code may be defined in terms of a list of standard permutations in the form $\sigma(a_1 \cdots a_n)$. Let $\underline{a} = a_1 \cdots a_n$. The notation $\tau(\underline{a})$ will represent the truncated word $a_1 \cdots a_{n-1}$, obtained by removing the last letter of $\underline{a}$.

Definition 4.3.1. *A* run *in the Gray code is defined to be a maximal sublist of consecutive words such that $\tau(\underline{a})$ is identical for all $\underline{a}$ in the sublist.*

All words in a run form a sublist of codes corresponding to standard permutations differing by an adjacent transposition. In the run, only a_n changes, either increasing from 1 to $2n-1$ or decreasing from $2n-1$ to 1. A run is increasing if a_n increases to get each subsequent word in the run. If an increasing run is the k^{th} run in the Gray code, then $k \equiv 1 \bmod 2$. Hence, increasing runs will also be referred to as odd runs. A run is decreasing if a_n decreases to get each subsequent code in the run. If a decreasing run is the k^{th} run in the Gray code, then $k \equiv 0 \bmod 2$. Decreasing runs will also be referred to as even runs. Odd and even refer to the count of the run and not whether a_{n-1} is odd or even. For example, 1111 through 1117 is the first run in the Gray code for words of length 4, and $a_3 = 1$. However, 12561 through 12569 is the 37^{th} run in the Gray code for words of length 5, and $a_4 = 6$.

Several properties of the Gray code follow directly from its definition:

1. Suppose $\sigma(\underline{b})$ immediately follows $\sigma(\underline{a})$ in the Gray code, and $\underline{a}$ and $\underline{b}$ are in

different runs. If $\underline{a}$ is in an odd (increasing) run, then $\underline{b}$ is in an even (decreasing) run and $a_n = b_n = 2n-1$. If $\underline{a}$ is in an even (decreasing) run, then $\underline{b}$ is in an odd (increasing) run and $a_n = b_n = 1$.

2. In every run, there is at least one arc-disconnected word (when $a_n = 2n-1$) and there are at least two arc-connected words (when $a_n = 1$ or 2).

3. There are $(2n-3)!!$ runs in the Gray code where each word that encodes a standard permutation has length n.

Lemma 4.3.2. *If $\sigma(\underline{a})$ is the m^{th} standard permutation in the Gray code, then $(n-1) + \sum_{i=1}^{n} a_i \equiv m$ mod 2.*

Proof. Use induction on m. If $m = 1$, then $a_1 = a_2 = \cdots = a_n = 1$ and $(n-1) + \sum_{i=1}^{n} a_i = 2n-1 \equiv m \bmod 2$. Assume the statement is true for the m^{th} standard permutation $\sigma(\underline{a})$, and let $\underline{b}$ be the word which encodes the $(m+1)^{st}$ standard permutation in the Gray code. As noted above, there exists a unique i such that $b_i = a_i \pm 1$ and $b_j = a_j$ for all $j \neq i$. Hence the parity of $(n-1) + \sum_{i=1}^{n} b_i$ is the opposite of the parity of $(n-1) + \sum_{i=1}^{n} a_i$; that is, $(n-1) + \sum_{i=1}^{n} b_i \equiv m+1 \bmod 2$. □

Lemma 4.3.3. *Suppose $\sigma(\underline{a})$ is the m^{th} code in the Gray code. If $\underline{a}$ is in an increasing run, $m \equiv a_n$ mod 2, and if $\underline{a}$ is in a decreasing run, $m \equiv (a_n - 1)$ mod 2.*

Proof. Let k be the number of increasing runs which occur before the run in which $\underline{a}$ is located. Recall, there are $2n-1$ codes in each run.

Case 1: $\underline{a}$ is in an increasing run.

Then there are k decreasing runs which occur before the run in which $\underline{a}$ is located. Thus, $m = (2k)(2n-1) + a_n \equiv a_n \bmod 2$.

Case 2: $\underline{a}$ is in a decreasing run.

Then there are $k-1$ decreasing runs which occur before the run in which $\underline{a}$ occurs, and since $\underline{a}$ is in a decreasing run, $\underline{a}$ is the $(2n-1) - a_n + 1$ word in the run. Thus, $m = (2k)(2n-1) - a_n + 1 \equiv (a_n - 1) \bmod 2$. □

Corollary 4.3.4. *$\underline{a}$ is in an increasing run if and only if $a_1 + a_2 + \ldots + a_{n-1} + (n-2)$ is odd, otherwise $\underline{a}$ is in a decreasing run.*

Lemma 4.3.5. *Suppose that $\sigma(\underline{a})$ is immediately followed by $\sigma(\underline{b})$ in the full Gray code. Let i be the unique index such that $b_i = a_i \pm 1$ and $b_j = a_j$ for all $j \neq i$. If $i < n$, then either $a_k = 1$ for all $k > i$ or $a_k = 2k - 1$ for all $k > i$.*

Proof. By the recursive nature of the Gray code, for each $k > i$ either $a_k = 1$ or $a_k = 2k - 1$. The goal is to show that in a single word it is not possible to have both $a_k = 1$ and $a_j = 2j - 1$ for some k and j both exceeding i. If there is such a change, then there is a least k exceeding i such that exactly one of a_k and a_{k+1} is equal to one. Since $a_j = b_j$ for $j \geq k+2$, the word $b_1 \cdots b_{k+1}$ immediately follows $a_1 \ldots a_{k+1}$ in the full Gray code for words of length $k+1$. By the recursive nature of the Gray code, assume $k = n-1$. It will be shown by way of contradiction that the case when $a_n = 2n-1$ and $a_{n-1} = 1$ is impossible. The case when $a_{n-1} = 2n-3$ and $a_n = 1$ is analogous.

Since $a_n = b_n = 2n - 1$, $\underline{a}$ is in an increasing run. By Corollary 4.3.4, $a_1 + \ldots + a_{n-1} + (n-2) \equiv 1 \bmod 2$. Thus, $a_1 + \ldots + a_{n-2} + (n-3) \equiv 1 \bmod 2$, which means $\tau(\underline{a})$

is in an increasing run in the Gray code for words of length $n-1$. Since $a_{n-1}=1$, $b_{n-1}=2$, which contradicts $i<n-1$. □

Corollary 4.3.6. *If* $\underline{a}=a_1\cdots a_k 1\cdots 1$ *and the next word in the Gray code for words is* $a_1\cdots a'_k 1\cdots 1$*, then* $a'_k=a_k+1$ *exactly when* a_k *is even, and* $a'_k=a_k-1$ *exactly when* a_k *is odd. Similarly, if* $\underline{a}=a_1\cdots a_k(2k+1)\cdots(2n-1)$ *and the next word in the Gray code for words is* $a_1\cdots a'_k(2k+1)\cdots(2n-1)$*, then* $a'_k=a_k+1$ *exactly when* a_k *is odd, and* $a'_k=a_k-1$ *exactly when* a_k *is even.*

Lemma 4.3.2 has an additional consequence.

Corollary 4.3.7. *Under the conditions of Lemma 4.3.5, if* $\underline{a}=a_1\cdots a_k 1\cdots 1$*, then* $a_1+\ldots+a_k+(k-1)=0$ *mod 2. If* $\underline{a}=a_1\cdots a_k(2k+1)\cdots(2n-1)$*, then* $a_1+\ldots+a_k+(k-1)=1$ *mod 2.*

Lemma 4.3.8. *In each run in the Gray code for words of length* n*, there exists* k *such that* $a_n\leq 2k-2$ *if and only if* $\underline{a}$ *is arc-connected; and if* $a_n\geq 2k-1$*,* $\underline{a}$ *is arc-disconnected. This* k *is the least index* $k'\leq n-1$ *such that* $a_j\geq 2k'-1$ *holds for all* $j\in\{k',k'+1,\ldots,n-1\}$ *if such an index exists; otherwise* $k=n$*.*

Proof. Consider a run in the Gray code for words of length n. All words in the run have the same truncated word $\tau(\underline{a})=a_1\cdots a_{n-1}$.

Case 1: $\tau(\underline{a})$ is arc-connected. Applying Corollary 2.5.12 to $\tau(\underline{a})$, here there is no $k'\leq n-1$ such that $a_j\geq 2k'-1$ holds for all $j\in\{k',k'+1,\ldots,n-1\}$. By the same Corollary 2.5.12, $\underline{a}$ is arc-disconnected if and only if there exists a k such that $a_k=2k-1$ and $a_j\geq a_k=2k-1$ for all $j>k$. Since $a_1\cdots a_{n-1}$ is arc-connected, the only such possibility for k is n. Thus, $a_1 \cdots a_n$ is arc-disconnected if and only if $a_k = a_n = 2n - 1$.

Case 2: $\tau(\underline{a})$ is arc-disconnected. Applying Corollary 2.5.12 to $\tau(\underline{a})$ yields a smallest $k' \leq n - 1$ such that $a_{k'} = 2k' - 1$ and $a_j \geq 2k' - 1$ for $k' < j < n$. Clearly, if $a_n \geq 2k' - 1$, then $\underline{a}$ is arc-disconnected. It is only left to show that $\underline{a}$ is arc-connected whenever $a_n \leq 2k' - 2$. If $\underline{a}$ is arc-disconnected for some $a_n \leq 2k' - 2$, then by Corollary 2.5.12, there is a k'' such that $a_j \geq 2k'' - 1$ holds for all $j \geq k''$. By the minimality of k', $k'' \geq k'$. On the other hand, $a_n \geq 2k'' - 1$ and $a_n \leq 2k' - 2$ imply $k'' < k'$, a contradiction. □

Lemma 4.3.9. *If k is defined as in Lemma 4.3.8, then $\sigma(a_1 \cdots a_{k-1})$ is the first sign-connected component of the standard permutation $\sigma(a_1 \cdots a_{n-1})$.*

Proof. Assume there is a least index $k' \leq n - 1$ such that $a_j \geq 2k' - 1$ holds for all $j \in \{k', k' + 1, \ldots, n - 1\}$. In this case, $k = k'$ and, by Corollary 2.5.12, the standard permutation $\sigma(a_1 \cdots a_{k-1})$ is sign-connected. Since $a_j \geq 2k - 1$ holds for $j \in k, \ldots, n - 1$, the left endpoints of the corresponding arcs are all to the right of the arcs associated to $\sigma(a_1 \cdots a_{k-1})$. Therefore, $\sigma(a_1 \cdots a_{k-1})$ is the first sign-connected component of $\sigma(\tau(\underline{a}))$.

The remaining case is when there is no $k' \leq n - 1$ satisfying $a_j \geq 2k' - 1$ for all $j \in \{k', k' + 1, \ldots, n - 1\}$. In this case, $k = n$, and Corollary 2.5.12 implies $\sigma(\tau(\underline{a}))$ is sign-connected. □

Lemma 4.3.10. *Suppose $\underline{a}$ is immediately followed by $\underline{b}$ in the full Gray code for words, and let k be the unique index such that $b_k \neq a_k$. If $b_k = a_k + 1$ and $\underline{a}$ is arc-*

disconnected, then $\underline{b}$ is arc-disconnected; and if $b_k = a_k - 1$ and $\underline{a}$ is arc-connected, then $\underline{b}$ is arc-connected.

Proof. Consider first the case when $k = n$. In this case $\underline{a}$ and $\underline{b}$ belong to the same run and the statement follows from Lemma 4.3.8. Finally, consider the case when $k \neq n$. In this case Lemma 4.3.5 implies that either $a_i = 1$ holds for all $i > k$ or $a_i = 2i - 1$ holds for all $i > k$. By Lemma 4.3.8, $\underline{a}$ is arc-connected exactly when $a_i = 1$ holds for all $i > k$ and $\underline{a}$ is arc-disconnected exactly when $a_i = 2i - 1$ holds for all $i > k$. The same characterization also applies to $\underline{b}$ since $a_i = b_i$ for all $i > k$. □

4.4 Restricting the Gray Code to the Shelling Types of the n-Cube

The goal of this chapter is to define a Gray code for the facet enumerations of the boundary of the n-cube, restricted to shelling types. This is equivalent to finding a Gray code for the arc-connected words $a_1 \cdots a_n$ since arc-connected words encode sign-connected standard permutations which in turn represent shellings of the boundary of the n-cube.

In this section, it will be shown that the sublist obtained by removing all arc-disconnected words from the full Gray code of words of Chapter 4.2 yields a Gray code for the sign-connected standard permutations. This sublist of the standard permutations will be referred to as the *connected Gray code*. This will be shown in two stages. First the following, weaker statement is proven.

Theorem 4.4.1. *If $\underline{a}$ and $\underline{b}$ are arc-connected, but every code listed between these two words in the full Gray code for words is arc-disconnected, then $\sigma(\underline{a})$ and $\sigma(\underline{b})$ differ by a single transposition.*

Theorem 4.4.1 is an immediate consequence of Lemma 2.5.15 and of the following statement.

Proposition 4.4.2. *If $\underline{a}$ and $\underline{b}$ are as in Theorem 4.4.1, then there is a unique $k < n$ such that $a_k \neq b_k$. Furthermore, $a_n = b_n = 2i - 2$ holds for some $i \in \{2, \ldots, n\}$, satisfying $a_i = b_i = 2i - 1$ and $a_j, b_j \geq 2i - 1$ for all $j \in \{i, \ldots, n-1\}$.*

Proof. Without loss of generality one may assume that $\underline{b}$ follows $\underline{a}$ in the full Gray code for words and that there is at least one arc-disconnected word between them. By Lemma 4.3.8, $\underline{a}$ cannot be at the end of a run; by Lemma 4.3.10, $\underline{a}$ must be in an increasing run. Thus, $a_1 \cdots a_{n-1}a_n$ is arc-connected, and the next consecutive word in the Gray code of words is $a_1 \cdots a_{n-1}a'_n$, where $a'_n = a_n + 1$. This word is arc-disconnected as are all the remaining codes in the run up to and including $a_1 \cdots a_{n-1}(2n-1)$.

The next run in the Gray code for words starts with $c_1 \cdots c_{n-1}(2n-1)$, which is arc-disconnected. This run decreases down to $c_1 \cdots c_{n-1}1$, an arc-connected code. Since $b_1 \cdots b_n$ is the first arc-connected code following $\underline{a}$, $c_1 \cdots c_{n-1} = b_1 \cdots b_{n-1}$. Thus, the next run in the Gray code of words actually starts with $b_1 \cdots b_{n-1}(2n-1)$, and the subsequent codes in the run, down to $b_1 \cdots b_{n-1}(b_n + 1)$, are arc-disconnected.

By the construction of the Gray code, $\tau(\underline{a})$ and $\tau(\underline{b})$ are consecutive words in the full Gray code for words of length $n-1$; thus, there is exactly one $k < n$ such that $b_k = a_k \pm 1$. By Lemma 4.3.5, $\tau(\underline{a})$ has either the form $a_1 \cdots a_k 1 \cdots 1$ or the form $a_1 \cdots a_k(2k+1) \cdots (2n-3)$.

Case 1: $k < n-1$, and $\tau(\underline{a}) = a_1 \cdots a_k 1 \cdots 1$ where $a_k \neq 1$. By assumption,

$b_k = a_k \pm 1$. Thus, both $\tau(\underline{a})$ and $\tau(\underline{b})$ are arc-connected, and Lemma 4.3.9 implies that $a_n = 2n - 2 = b_n$.

Case 2: $k < n - 1$, and $\tau(\underline{a}) = a_1 \cdots a_k(2k+1)\cdots(2n-3)$ where $a_k \neq 2k-1$. By Corollary 2.5.12, $\tau(\underline{a})$ is arc-disconnected, so there must exist an $i \leq k+1$ such that $a_i = 2i - 1$ and $a_j \geq 2i - 1$ for $i < j < n$. Choose i to be the smallest index with this property. By Lemma 4.3.8, the word $a_1 \cdots a_k(2k+1)\cdots(2n-3)(2i-2)$ is arc-connected, but the rest of the codes in the run, $a_1 \cdots a_k(2k+1)\cdots(2n-3)(2i-1)$ through $a_1 \cdots a_k(2k+1)\cdots(2n-3)(2n-1)$, are arc-disconnected. Thus, $a_n = 2i-2$.

Since $\underline{b}$ is in a decreasing run immediately following the run in which $\underline{a}$ is contained, the run of $\underline{b}$ starts with $b_1 \cdots b_k(2k+1)\cdots(2n-3)(2n-1)$ where $b_k = a_k \pm 1$. By Corollary 4.3.6, if $b_k = a_k + 1$, then a_k is odd, and if $b_k = a_k - 1$, then a_k is even. Since $i \neq k$ and $i \leq k+1$, there are two subcases:

Case 2a: $i = k+1$. In this case, $a_1 \cdots a_k$ is arc-connected and $a_n = 2i - 2 = 2k$. We claim that $b_1 \cdots b_k$ is also arc-connected. This is an immediate consequence of Lemma 4.3.10 when $b_k = a_k - 1$. If $b_k = a_k + 1$, then b_k is even and $b_1 \cdots b_k$ is arc-connected because, by Lemma 4.3.8, the first arc-disconnected code in an increasing run ends with an odd letter. Thus, both $\underline{a} = a_1 \cdots a_k(2k+1)\cdots(2n-3)(2k)$ and $b_1 \cdots b_k(2k+1)\cdots(2n-3)(2k)$ are arc-connected codes, and there are only arc-disconnected codes between these two words in the full Gray code for words. Therefore $b_n = 2k$.

Case 2b: $i < k$. In this case, $a_1 \cdots a_{i-1}$ is arc-connected and $2i - 1 \leq a_k < 2k - 1$. By Corollary 2.5.12, $a_1 \ldots a_k$ is arc-disconnected and, by Lemma 4.3.9, the first sign-connected component of $\sigma(a_1 \cdots a_k)$ is $\sigma(a_1 \cdots a_{i-1})$. When $b_k = a_k + 1$, an immediate consequence of Lemma 4.3.10 is that $b_1 \cdots b_k$ is arc-disconnected. If $b_k = a_k - 1$, then a_k is even and $b_1 \cdots b_k$ is arc-disconnected. Since a_k is strictly greater than $2i-1$, $b_k \geq 2i-1$.

Since $b_1 \cdots b_m = a_1 \cdots a_m$ for any m strictly between i and k and $\sigma(a_1 \cdots a_{i-1})$ is the first sign-connected component of $\sigma(a_1 \cdots a_k)$, then $\sigma(b_1 \cdots b_{i-1})$ is the first sign-connected component of $\sigma(b_1 \cdots b_k)$. Hence, both $a_1 \cdots a_i \cdots a_k(2k+1) \cdots (2n-3)(2i-2)$ and $b_1 \cdots b_i \cdots b_k(2k+1) \cdots (2n-3)(2i-2)$ are arc-connected codes such that there are only arc-disconnected words between them in the full Gray code for words. Thus, $b_n = 2i-2$.

Case 3: $k = n-1$ (namely, $b_{n-1} = a_{n-1} \pm 1$). By Lemma 4.3.8, there exists i such that $a_1 \cdots a_{n-1}(2i-2)$ is arc-connected but $a_1 \cdots a_{n-1}(2i-1)$ is arc-disconnected. If $\tau(\underline{a})$ and $\tau(\underline{b})$ are both arc-connected, this is a degenerate case of case 2a with $i = k+1 = n$ and $a_n = b_n = 2n-2$. If $\tau(\underline{a})$ and $\tau(\underline{b})$ are both arc-disconnected, this is a degenerate case of case 2b with $k = n-1$, $i < k$ and $a_n = b_n = 2i-2$. The remaining case is when $\tau(\underline{a})$ is arc-connected but $\tau(\underline{b})$ is arc-disconnected and the case when $\tau(\underline{a})$ is arc-disconnected but $\tau(\underline{b})$ is arc-connected. The following will show, by way of contradiction, that neither of these cases can occur.

Assume first that $\tau(\underline{a})$ is arc-connected but is immediately followed by the arc-disconnected word $\tau(\underline{b})$ in the Gray code for words of length $n-1$. By Lemma 4.3.8, $\tau(\underline{a}) = a_1 \cdots a_{n-2}(2i-2)$ for some i. Since $\underline{a}$ is in an increasing run, Corollary 4.3.4 gives $a_1 + \ldots + a_{n-2} + (2i-2) + (n-2) \equiv 1 \bmod 2$. Thus, $a_1 + \ldots + a_{n-2} + (n-3) \equiv 0 \bmod 2$, so $\tau(\underline{a})$ is in a decreasing run in the Gray code of length $n-1$. Hence, $b_{n-1} = a_{n-1} - 1 = 2i-3$. By Lemma 4.3.10, $\tau(\underline{b})$ is arc-connected, which is a contradiction.

Assume finally that $\tau(\underline{a})$ is arc-disconnected but is immediately followed by the arc-connected word in $\tau(\underline{b})$ the Gray code for words of length $n-1$. By Lemma 4.3.8, $\tau(\underline{a}) = a_1 \cdots a_{n-2}(2i-1)$ for some i. Since $\underline{a}$ is in an increasing run, Corollary 4.3.4 gives $a_1+\ldots+a_{n-2}+(2i-1)+(n-2) \equiv 1 \bmod 2$. Thus, $a_1+\ldots+a_{n-2}+(n-3) \equiv 1 \bmod 2$, so $\tau(\underline{a})$ is in an increasing run in the Gray code for words of length $n-1$. Hence, $b_{n-1} = a_{n-1}+1 = 2i$. By Lemma 4.3.10, $\tau(\underline{b})$ is arc-disconnected, a contradiction. □

A second look at the proof of Proposition 4.4.2 allows for the proof of the main result.

Theorem 4.4.3. *If $\underline{a}$ and $\underline{b}$ are arc-connected, but every code listed between these two words in the full Gray code for words is arc-disconnected, then $\sigma(\underline{a})$ and $\sigma(\underline{b})$ differ by an adjacent transposition.*

Proof. This proof will make the same assumptions as in the proof of Proposition 4.4.2, review the same cases, and consider the associated arc diagrams.

In case 1, $a_j = b_j = 1$ for $k < j < n$ implies that the first k arcs are under $n-k-1$ nested arcs. An arc stretching over all previous arcs will not change how any two ends of the previous arcs are interchanged since that move occurs completely under the arcs. The n^{th} arc ($a_n = 2n-2$) will only overlap the $(n-1)^{st}$ arc, meaning that this last arc will not affect how the first k arcs change in the arc diagrams when changing from $\underline{a}$ to $\underline{b}$. By the recursive definition of the full Gray code, $\sigma(a_1 \cdots a_k)$ and $\sigma(b_1 \cdots b_k)$ differ by an adjacent transposition. Hence the arc diagrams encoded by $a_1 \cdots a_k$ and $b_1 \cdots b_k$ differ by exactly two adjacent ends of two distinct arcs swapping positions; and since it is already known that the $(k+1)^{st}$ through n^{th} arcs will not affect that swap, then $\underline{a} = a_1 \cdots a_k 1 \cdots 1(2n-2)$ and $\underline{b} = b_1 \cdots b_k 1 \cdots 1(2n-2)$ encode two standard permutations that differ by an adjacent transposition.

In case 2a, in the arc diagram of $\sigma(\underline{a}) = \sigma(a_1 \cdots a_k(2k+1)\cdots(2n-3)(2k))$, the arcs of $\sigma(a_1 \cdots a_k)$ form the first component of $\sigma(\tau(\underline{a}))$, which is followed by $n-k-1$ minimal arcs (see Chapter 2.5 for a definition). Then the n^{th} arc stretches over the minimal arcs to intersect only the k^{th} arc of the first component of $\sigma(\tau(\underline{a}))$. Thus, the $(k+1)^{st}$ through n^{th} arcs will not affect any moves among the first k arcs. Since $\sigma(a_1 \cdots a_k)$ and $\sigma(b_1 \cdots b_k)$ must differ by an adjacent transposition, the recursive construction of the full Gray code guarantees that $\sigma(\underline{a})$ and $\sigma(\underline{b})$ will also differ by an adjacent transposition.

In case 2b, the arcs of $\sigma(a_1 \cdots a_{j-1})$ form the first connected component of the arc diagram of $\sigma(\tau(\underline{a}))$; the $(k+1)^{st}$ through $(n-1)^{st}$ arcs are minimal arcs located at the right end of the diagram; and the n^{th} arc stretches over the second through last component of the arc diagram, intersecting only the $(j-1)^{st}$ arc of the first connected component. Thus, the $(k+1)^{st}$ through n^{th} arcs will not affect any changes occurring in the first k arcs. The recursive construction of the full Gray code ensures that $\sigma(\underline{a})$ and $\sigma(\underline{b})$ will differ by an adjacent transposition since $\sigma(a_1 \cdots a_k)$ and $\sigma(b_1 \cdots b_k)$ differ by an adjacent transposition.

In case 3, there will only be a degenerate case of case 2a or 2b. In the degenerate case of 2a, the number of minimal arcs to the right of the first connected component of the arc diagram of $\sigma(\tau(\underline{a}))$ is zero (since $\tau(\underline{a})$ is arc-connected). This does not change the conclusion of the argument. The analysis of the degenerate case of 2b is similarly easy. □

Looking at the list of words encoding the standard permutations of the connected Gray code, each run starts with $a_n = 1$ and ends with $a_n = 2i - 2$ for some i or vice versa. The proof of Proposition 4.4.2 describes the relationship between any arc-connected word $\underline{a}$ and the word $\underline{b}$ immediately following it in the connected Gray code. When $\underline{a}$ and $\underline{b}$ are in different runs of the full Gray code for words, $\underline{a}$ and $\underline{b}$ have the following properties:

1. $\sigma(\underline{a})$ and $\sigma(\underline{b})$ differ by an adjacent transposition;

2. $a_n = b_n$;

3. if $a_n = b_n = 2i - 1$ for some $i > 1$, then $\tau(\underline{a})$ and $\tau(\underline{b})$ are either both arc-connected or both arc-disconnected;

4. if $a_n = b_n = 2i - 1$ for some $i > 1$, both $\tau(\underline{a})$ and $\tau(\underline{b})$ are arc-disconnected, and $\sigma(a_1 \cdots a_k)$ is the first sign-connected component of $\sigma(\tau(\underline{a}))$; then the first sign-connected component of $\sigma(\tau(\underline{b}))$ is $\sigma(b_1 \cdots b_k)$.

As an immediate consequence of Theorem 4.4.3, the sublist of the full Gray code obtained by simply removing all of the sign-disconnected standard permutations is also a Gray code. Hence, by working with the words that encode standard permutations, a Gray code has been found, namely the connected Gray code, for the standard permutations that represent the shelling types of the facets of the boundary of the n-cube.

4.5 The Cost to Implement to the Full and Connected Gray Codes

Recall that the number of connected Gray codes of length n is given by $a_n = (2n-1)!! - \sum_{k=1}^{n-1} a_{n-k}(2k-1)!!$. Let $b_n = \frac{a_n}{(2n-1)!!}$. Hence, b_n is the proportion of connected words in in the full Gray code. Using the definition of a_n, b_n may be rewritten as

$$\begin{aligned} b_n &= 1 - \sum_{k=1}^{n-1} \frac{a_{n-k}}{(2(n-k)-1)!!} \cdot \frac{(2(n-k)-1)!!(2k-1)!!}{(2n-1)!!} \\ &= 1 - \sum_{k=1}^{n-1} b_{n-k} \cdot \frac{(2(n-k)-1)!!(2k-1)!!}{(2n-1)!!}. \end{aligned}$$

In order to determine whether the connected Gray code can be generated in constant amortized time, b_n needs to approach a constant as n approaches infinity.

Lemma 4.5.1. *The proportion b_n approaches 1 as n approaches infinity.*

Proof. Obviously, for each n, $0 < b_n < 1$. Thus, $b_n > 1 - \sum_{k=1}^{n-1} \frac{(2(n-k)-1)!!(2k-1)!!}{(2n-1)!!}$. Hence, it is only needed to show that $\lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{(2(n-k)-1)!!(2k-1)!!}{(2n-1)!!} = 0$. Set $\gamma_{n,k} = \frac{(2(n-k)-1)!!(2k-1)!!}{(2n-1)!!}$. Then $\gamma_{n,k} = \gamma_{n,n-k}$ and $\gamma_{n,k} \geq 0$ for all k. Consider:

$$\frac{\gamma_{n,k+1}}{\gamma_{n,k}} = \frac{\frac{(2(n-k-1)-1)!!(2k+1)!!}{(2n-1)!!}}{\frac{(2(n-k)-1)!!(2k-1)!!}{(2n-1)!!}} = \frac{(2(n-k-1)-1)!!(2k+1)!!}{(2(n-k)-1)!!(2k-1)!!} = \frac{2k+1}{2n-2k-1}$$

Hence, $\frac{\gamma_{n,k+1}}{\gamma_{n,k}} \leq 1$ if and only if $\frac{2k+1}{2n-2k-1} \leq 1$, which occurs exactly when $2k+1 \leq 2n-2k-1$. Thus, $k \leq \frac{n-1}{2}$ yielding $\gamma_{n,1} \geq \gamma_{n,2} \geq \ldots \geq \gamma_{\lfloor \frac{n}{2} \rfloor} = \gamma_{\lceil \frac{n}{2} \rceil} \leq \ldots \leq \gamma_{n,n-2} \leq \gamma_{n,n-1}$ where $2\gamma_{n,1} = \gamma_{n,1} + \gamma_{n,n-1}$ and $(n-3)\gamma_{n,2} \geq \gamma_{n,2} + \gamma_{n,3} + \ldots + \gamma_{n,n-2}$. Therefore,

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{(2(n-k)-1)!!(2k-1)!!}{(2n-1)!!} &\leq 2\gamma_{n,1} + (n-3)\gamma_{n,2} \\ &= 2 \cdot \frac{(2(n-1)-1)!!(1)!!}{(2n-1)!!} + (n-3) \cdot \frac{(2(n-2)-1)!!(3)!!}{(2n-1)!!} \end{aligned}$$

$$= 2 \cdot \frac{(2n-3)!!}{(2n-1)!!} + (n-3) \cdot \frac{(2n-5)!! \cdot 3}{(2n-1)!!}$$
$$= \frac{2}{2n-1} + \frac{3(n-3)}{(2n-1)(2n-3)}.$$

Hence, $\sum_{k=1}^{n-1} \frac{(2(n-k)-1)!!(2k-1)!!}{(2n-1)!!} \to 0$ as n approaches infinity, which immediately implies that b_n approaches 1 as n approaches infinity. □

Hence, it is possible that the connected Gray code could be generated in constant amortized time.

Theorem 4.5.2. *The full Gray code can be generated in constant amortized time.*

Proof. Suppose each of the words in the full Gray code have length n, and a code $\underline{a}$ has the form: $a_1 \dots a_i \dots a_n$, where $1 \leq a_i \leq 2i - 1$ for each i. Recall, there are a total of $(2n-1)!!$ codes and $(2n-3)!!$ runs with $2n-1$ codes per run. The claim is that the cost to implement the full Gray code is $\mathcal{O}((2n-1)!!)$. Let i be the index such that a_i changes to get to the next word in the Gray code.

Case 1: $i = n$. In this case, a_n changes $2n-2$ times per run. The last word of the run will not change at position n. Each time a_n changes, it must be verified that $\underline{a}$ is not at the end of a run; i.e., $a_n \neq 1$ in a decreasing run or $a_n \neq 2n-1$ in an increasing run. Thus, the total cost to change a_n throughout the entire Gray code is $(1)(2n-2)(2n-3)!! = 2(n-1)\frac{(2n-1)!!}{2n-1} = \mathcal{O} \cdot (2n-1)!!$.

Case 2: $i < n$. In this case, a_i will only change at the end of a run. However, in the full Gray code of length i, a_i changes $(2i-2)(2i-3)!!$ times. By the recursive nature of the Gray code, this is exactly the number of times a_i changes in the full Gray code of length n. When a_i changes, there are $(n-i+2)$ comparisons to make:

is $a_j = 1$ for all $j > i$ or $a_j = 2j - 1$ for all $j > i$, is $a_i \neq a_n$, and lastly, is a_i odd or even. By Corollary 4.3.6, this last comparison will determine whether a_i will increase or decrease to produce the next word in the Gray code. Since $2 \leq i \leq n-1$, then $3 \leq n-i+2 \leq n$. Thus, the total cost to change a_i is $(n-i+2)(2i-2)(2i-3)!! = (n-i+2)(2i-2) \cdot \frac{(2n-1)!!}{(2n-1)...(2i-1)} = \mathcal{O}(n) \cdots \mathcal{O}(\frac{1}{n^{n-i}}) \cdot (2n-1)!!$. Combining these relations yields a total cost of $\mathcal{O}(1) \cdot (2n-1)!! + \sum_{i=2}^{n-1} \mathcal{O}(n) \cdots \mathcal{O}(\frac{1}{n^{n-i}}) \cdot (2n-1)!! = \mathcal{O}(1) \cdot (2n-1)!! + \mathcal{O}(n\left[\frac{1}{n} + \ldots + \frac{1}{n^{n-2}}\right]) \cdot (2n-1)!! = \mathcal{O}(1 + n \cdot \frac{n^{n-3} + \ldots + 1}{n^{n-2}}) \cdot (2n-1)!! = \mathcal{O}(1) \cdot (2n-1)!! = \mathcal{O}((2n-1)!!)$. □

Theorem 4.5.3. *The connected Gray code can be generated in constant amortized time.*

Proof. Since the cost to generate the full Gray code is $\mathcal{O}((2n-1)!!)$ and the number of words in the code is $(2n-1)!!$, it is not difficult to show that the number of arc-connected words is also $\mathcal{O}((2n-1)!!)$. Thus, it is only needed to show that the cost to skip the disconnected codes is at most $\mathcal{O}((2n-1)!!)$.

To know when to skip words in the Gray code for words, keep track of the k from Lemma 4.3.8. This k can only change at the end of a decreasing run. When k is constant, only a_n changes. However, when k does change, it takes at most $\mathcal{O}(n)$ steps, reading left to right, to find the new value of k. Since there are $(2n-3)!!$ runs in the Gray code, the cost to skip the disconnected codes is $\mathcal{O}(n(2n-3)!!) = \mathcal{O}((2n-1)!!)$. □

CHAPTER 5: CONCLUSION AND FUTURE WORK

The results from Chapter 4 highlight a connection between the study of hypermaps and the theory of shellings, which may be worth exploring further in the future. It also raises the hope that, by using a similar encoding to the one introduced in [37], one may be able to find an adjacent transposition Gray code for indecomposable permutations. Although a large amount of literature exists on shelling and shellability, very little has been done to explore the set of all shellings of the same object. It may be worthwhile to look for a Gray code on all the shellings of other objects related to the hypercube such as orthotopes or cross-polytopes.

Another possible direction to take is to look at what other objects that signed permutations can represent. In particular, signed permutations can represent rooted hypermaps (see [19], [54], and [55]) and Feynman diagrams [47]. However, in both of these two cases, the closeness condition used in this dissertation must be adjusted to account for the situation which arises when a single component rooted hypermap or Feynman diagram is followed by one containing multiple components.

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