ON REDUCED UNICELLULAR HYPERMONOPOLES

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Abstract. The problem of counting unicellular hypermonopoles by the number of their hyperedges is equivalent to describing the cycle length distribution of a product of two circular permutations, first solved by Zagier. The solution of this problem has also been used in the study of the cycle graph model of Bafna and Pevzner and of related models in mathematical biology. In this paper we develop a method to compute the finite number of reduced unicellular hypermonopoles of a given genus. The problem of representing any hypermap as a drawing is known to be simplifiable to solving the same problem for reduced unicellular hypermonopoles. We also outline a correspondence between our hypermap model, the cycle graph model of Bafna and Pevzner, and the polygon gluing model of Alexeev and Zograf. Reduced unicellular hypermonopoles correspond to reduced objects in the other models as well, and the notion of genus is the same.

Introduction

In the study of the combinatorics of the symmetric group many authors have been interested in the statistics of cycle lengths of products of pairs of permutations. In this note we revisit the particular case of counting the cycles of the product of two circular permutations, or dually finding the number of decompositions of a given permutation as a product of two circular permutations.

Our renewed interest in this topic comes from the study of *hypermaps*. In a recent paper [6] we showed that the problem of drawing a hypermap may be reduced to considering the same problem for a *unicellular hypermonopole* of the same genus. A hypermap $H = (\sigma, \alpha)$ is a pair of permutations generating a transitive permutation group. A hypermonopole is a hypermap with a single vertex, that is, σ is a circular permutation, it is called *unicellular* if it has only one face, meaning that $\alpha^{-1}\sigma$ is also a circular permutation. The main result of this short paper concerns the enumeration of unicellular hypermonopoles without *buds*, meaning that α has no fixed point, we call such a unicellular hypermonopole *reduced* .

Our main tool is the enumeration formula obtained by Zagier [17] for the number of unicellular hypermonopoles having a given number of cycles. Notice that the number of cycles k and the genus g of a unicellular hypermonopole of S_n satisfy $k = n - 2g$. A combinatorial bijective proof of Zagier's formula was given by Cori, Marcus and Schaeffer [7].

In the field of genome rearranging, Bafna and Pevzner [2] introduced a cycle graph model that is cryptomorphic to studying the product of a pair of circular permutations.

Date: March 28, 2024.
2020 *Mathematics Subject Classification.* Primary 05C30; Secondary 05C10, 05C15.
Key words and phrases. set partitions, noncrossing partitions, genus of a hypermap.

Their aim was to determine the minimum number of "transpositions"[1] needed to reduce a permutation (considered as a word) to the identity permutation. This allows to model the evolution of the DNA of some viruses. These questions have motivated several researchers to focus on the products of circular permutations. This is the case of the work of A. Hultman in his thesis [10], and more recently that of Alexeev and Zograf [1] who introduced gluings of polygons to represent these products.

Our paper is organized as follows. In the Preliminaries we recall the notions of the theory of hypermaps and state Zagier's formula and some of its reformulations. Section 2 is devoted to the description of the relationship between Zagier's factorization problem, counting unicellular hypermonopoles, and the models of Bafna and Pevzner [2] and of Alexeev and Zograf [1]. In Section 3 we give our main result expressing the number of reduced unicellular hypermonopoles on n points with a given number of hyperedges. We already observed in [6] that the number of all reduced unicellular hypermonopoles of a given genus is finite. Our main result allows to count these finite numbers explicitly, and we give the first values of these numbers.

It is worth noting that reduced unicellular hypermonopoles correspond to cycle graphs having breakpoints everywhere in the model of Bafna and Pevzner [2] and that the genus of the corresponding polygon gluing diagram in the work of Alexeev and Zograf [1] is the same as the genus of the corresponding unicellular hypermonopole. The three models are intimately related, hence the counting problem we solved has also some significance in the related models in mathematical biology.

1. Preliminaries

1.1. **Hypermaps and their two disk diagrams.** A *hypermap* (σ, α) is a pair of permutations of a set $\{1, 2, \ldots, n\}$ generating a transitive permutation group. It is used to represent a (connected) hypergraph on an oriented surface. The *points* $1, 2, \ldots, n$ are the points of incidence between vertices and hyperedges. The cycles of σ list these points around the vertices in counterclockwise order, whereas the cycles of α list these points hyperedges in clockwise order. The cycles of $\alpha^{-1}\sigma$ represent then the faces of the hypermap (σ, α). The *genus* $g(\sigma, \alpha)$ of the surface on which such a hypermap may be drawn is given by the following formula due to Jacques [11]:

$$n + 2 - 2g(\sigma, \alpha) = z(\sigma) + z(\alpha) + z(\alpha^{-1}\sigma), \tag{1.1}$$

where $z(\pi)$ denotes the number of cycles of the permutation π. This paper is motivated by the following observation in [6]: using a sequence of *topological hyperdeletions* $(\sigma, \alpha) \mapsto (\sigma, \alpha\delta)$ and of *topological hypercontractions* $(\sigma, \alpha) \mapsto (\gamma\sigma, \gamma\alpha)$, each hypermap may be reduced to a hypermap (σ', α') of the same genus, such that $z(\sigma') = 1$, that is, (σ', α') is a *hypermonopole*, and $z(\alpha^{-1}\sigma') = 1$, that is, (σ', α') *unicellular*. We refer the interested reader for the detailed description of the process to [6]. The final conclusion is that if we are able to draw the unicellular hypermonopole (σ', α') by some means in the plane then we may easily extend such a figure to a drawing of (σ, α) by replacing the vertex σ' with a noncrossing partition having $z(\sigma)$ parts and the face $\alpha^{-1}(\sigma)$ with a noncrossing partition having $z(\alpha^{-1}\sigma)$ parts. Noncrossing partitions, introduced by

[1]The transpositions in biology are different from those considered in algebra, they may be obtained by a composition of two usual transpositions

Kreweras [12], are partitions of the set $\{1, 2, \ldots, n\}$ such that the polygons representing the parts do not cross if we place the points in the cyclic order of $(1, 2, \ldots, n)$ on a circle.

There are infinitely many unicellular hypermonopoles of a fixed genus: trivially, for each n the hypermap (σ, α) with $\sigma = (1, 2, \ldots, n)$ and $\alpha = (1)(2)\cdots(n)$ is a unicellular hypermonopole of genus zero. Fortunately, in the case of unicellular hypermonopoles it is easy to remove or reinsert *buds*: a bud is a fixed point i of α. For $n \geq 2$, the removal of i from the cycles representing σ and α results in a hypermap (σ', α'), which is still a hypermonopole (as $z(\sigma') = z(\sigma) = 1$) and it is also still unicellular: the action of $\alpha'^{-1}\sigma'$ on a $j \notin \{i, \sigma^{-1}(i)\}$ is the same as that of $\alpha^{-1}\sigma$, whereas $\alpha'^{-1}\sigma'$ takes $\sigma^{-1}(i)$ into $\alpha'^{-1}(\sigma(i))$. (Note that neither $\sigma^{-1}(i)$ nor $\sigma(i)$ is equal to i, as σ is circular and has at least 2 points.) Hence the cycle representing $\alpha'^{-1}\sigma'$ is obtained from the cycle representing $\alpha^{-1}\sigma$ by simply removing the point i from the cyclic list. Clearly if we are able to draw (σ', α') in the plane, adding a bud to he figure amounts to adding a single point.

Definition 1.1. *We call a unicellular hypermonopole* reduced *if it contains no bud.*

It has been first observed in [6] that there are only finitely many reduced unicellular hypermonopoles of a fixed genus.

Lemma 1.2. *If (σ, α) is genus g a reduced unicellular hypermonopole on n points then $2g + 1 \leq n \leq 4g$ holds.*

Proof. Substituting $z(\sigma) = 1$ and $z(\alpha^{-1}\sigma) = 1$ into (1.1) we obtain

$$n = z(\alpha) + 2g(\sigma, \alpha). \tag{1.2}$$

Since each cycle of α has at least 2 elements we get $z(\alpha) \leq n/2$, yielding the upper bound for n. The lower bound is a direct consequence of $z(\alpha) \geq 1$. □

1.2. **Products of two circular permutations.** A permutation is *circular* if it has exactly one cycle. In his paper we will need the number of pairs of circular permutations of $\{1, 2, \ldots, n\}$ whose product has exactly k cycles. The answer to this question was first given by Zagier [17, application 3 of Theorem 1].

Theorem 1.3 (Zagier)**.** *The probability that the product of two cyclic permutations of $\{1, 2, \ldots, n\}$ has k cycles is*

$$P(n, k) = \frac{1 + (-1)^{n-k}}{(n + 1)!} c(n + 1, k).$$

Here $c(n + 1, k) = |s(n + 1, k)|$ is the number of permutations of $\{1, 2, \ldots, n + 1\}$ with k cycles, and $s(n + 1, k)$ is a Stirling number of the first kind.

The first purely combinatorial proof of this result was provided by Cori, Marcus and Schaeffer [7, Corollary 1]. Note that $P(n, k) = 0$ if $n - k$ is odd. This is obvious: all circular permutations have the same parity, hence the product of two circular permutations must be an even permutation. The parity of a permutation is the parity of the number of its cycles of even length in its cycle decomposition. This number can not be even if $n - k$ is odd. After noting that one of the two circular permutations may be fixed to be $(1, 2, \ldots, n)$, Zagier's result may be restated in combinatorial terms as follows.

Theorem 1.4. *The number of circular permutations* ψ *of* $\{1,\ldots,n\}$ *such that the product* $(1,\ldots,n)\psi$ *has exactly* k *cycles is*

$$H(n,k)=\begin{cases} c(n+1,k)/\binom{n+1}{2} & \text{if } n-k \text{ is even,}\\ 0 & \text{if } n-k \text{ is odd.}\end{cases} \tag{1.3}$$

n \ k	1	2	3	4	5	6	7	8	9	10
0	1									
1	0	1								
2	1	0	1							
3	0	5	0	1						
4	8	0	15	0	1					
5	0	84	0	35	0	1				
6	180	0	469	0	70	0				
7	0	3044	0	1869	0	126	0	1		
8	8064	0	26060	0	5985	0	210	0	1	
9	0	193248	0	152900	0	16401	0	330	0	1

TABLE 1. The values of $H(n,k)$ for $n\le 9$.

Table 1 shows the values of $H(n,k)$ for $n\le 9$. It is worth noting that when $n-k$ is even, the sign of $s(n+1,k)$ is negative, hence we may also replace $c(n+1,k)$ with $-s(n+1,k)$ in (1.3) above. The numbers $H(n,k)$ were later rediscovered by A. Hultman in his MS Thesis [10] who defined them in terms of counting alternating cycles in the cycle graph of a permutation. The numbers $H(n+1,k)$ were named *Hultman numbers* in the work of Doignon and Labarre [8]. The equivalence of the two definitions is made apparent in [4, Corollary 1], which is based on a result of Doignon and Labarre [8]. Citing Stanley [15], M. Bóna and R. Flynn [4, p. 931] also published (1.3) for these numbers. A simple proof of (1.3) (relying on Hultman's definition) was also found by S. Grusea and A. Labarre [9, Section 7]. We will use the following lemma of S. Grusea and A. Labarre [9, Lemma 8.1].

Lemma 1.5 (Grusea-Labarre)**.** *The numbers* $H(n,k)$ *satisfy*

$$\sum_{k=0}^{n} H(n,k)x^k=\frac{(x)^{(n+1)}-(x)_{n+1}}{(n+1)n}$$

Here $(x)^{(n+1)}=x\cdot(x+1)\cdots(x+n)$ *and* $(x)_{n+1}=x\cdot(x-1)\cdots(x-n)$ *are falling, respectively rising factorials (Pochhammer symbols).*

2. Basic facts about unicellular hypermonopoles

As a direct consequence of the definitions we may observe the following.

Remark 2.1. For a unicellular hypermonopole (σ,α) the permutations σ and $\pi=\alpha^{-1}\sigma$, representing the unique vertex, respectively, the unique face of the hypermap, are circular permutations. Conversely, given a pair (σ,π) of circular permutations of the same set, there is a unique unicellular hypermonopole (σ,α) satisfying $\pi=\alpha^{-1}\sigma$.

Indeed, the unique α satisfying $\pi = \alpha^{-1}\sigma$ is

$$\alpha = \sigma\pi^{-1}. \tag{2.1}$$

Regardless of α, the pair (σ, α) is always a hypermap because σ is a circular permutation, and any permutation group containing is transitive. Theorem 1.4 and Remark 2.1 have the following consequence.

Corollary 2.2. *The number of unicellular hypermonopoles (σ, α) satisfying $\sigma = (1, \ldots, n)$ and $z(\alpha) = k$ is the number $H(n, k)$ given in (1.3).*

In the rest of this section we show that unicellular hypermonopoles are bijectively equivalent to two other combinatorial models for the Hultman numbers $H(n + 1, k)$.

The first model we consider is the *cycle graph model* introduced by Bafna and Pevzner [2]. We present it using the same simplification that was introduced in [8] where the vertex $n + 1$ is identified with 0, and we adjust that model even further. Let us fix the circular permutation $\sigma = (0, 1, \ldots, n)$ and let π be any circular permutation of the set $\{0, 1, \ldots, n\}$. The *cycle graph* $G(\pi)$ of the permutation π is a digraph on the vertex set $\{0, 1, \ldots, n\}$ whose edges are colored with two colors:

(1) the black edges go from i to $\pi^{-1}(i)$ (modulo $n + 1$) for $0 \leq i \leq n$;
(2) the grey edges go from i to $i + 1$ (modulo $n + 1$) for $0 \leq i \leq n$.

Each vertex is the head, respectively tail of one edge of each color, hence the cycle graph may be uniquely decomposed into disjoint color-alternating cycles. Note that even though edges do not repeat in such cycles, vertices may occur twice. To remedy this slight confusion, we introduce two copy of each vertex i: a negative copy i^- and a positive copy i^+. Each negative vertex i^- will be the head of a black edge whose tail is $\pi(i)^+$ and it will be the tail of a grey edge whose head is $(i + 1)^+$. Equivalently, each positive vertex i^+ will be the head of a grey edge whose tail is $(i - 1)^-$ and it will be the tail of a black edge whose head is $\pi^{-1}(i)^+$. Instead of using colors we will label the black edges with π^{-1} and the grey edges with σ. For example, for $n = 7$ and $\pi = (0, 4, 1, 6, 2, 5, 7, 3)$ we obtain the following two cycles

$$0^- \xrightarrow{\sigma} 1^+ \xrightarrow{\pi^{-1}} 4^- \xrightarrow{\sigma} 5^+ \xrightarrow{\pi^{-1}} 2^- \xrightarrow{\sigma} 3^+ \xrightarrow{\pi^{-1}} 7^- \xrightarrow{\sigma} 0^+ \xrightarrow{\pi^{-1}} 3^- \xrightarrow{\sigma} 4^+ \xrightarrow{\pi^{-1}} 0^- \quad \text{and}$$

$$6^- \xrightarrow{\sigma} 7^+ \xrightarrow{\pi^{-1}} 5^- \xrightarrow{\sigma} 6^+ \xrightarrow{\pi^{-1}} 1^- \xrightarrow{\sigma} 2^+ \xrightarrow{\pi^{-1}} 6^-$$

Using this notation one may notice immediately that these cycles may be uniquely reconstructed from the positive vertices only: we may identify the first cycle with $(1, 5, 3, 0, 4)$ and the second cycle with $(7, 6, 2)$. Observe next that

$$\alpha = (1, 5, 3, 0, 4)(7, 6, 2) = \sigma\pi^{-1},$$

that is, $\pi = \alpha^{-1}\sigma$. The cycle graph of π may be identified with the unicellular hypermonopole (σ, α) whose only vertex is σ and only face is $\pi = \alpha^{-1}\sigma$. The number of alternating cycles in $G(\pi)$ is $z(\alpha)$, the number of hyperedges.

Bafna and Pevzner [2] call a pair $(i, \pi(i)$ a *breakpoint* if $\pi(i) \neq \sigma(i)$. In our setting $(i, \pi(i))$ is a breakpoint if and only if $\alpha^{-1}\sigma(i) \neq \sigma(i)$, equivalently $\sigma(i)$ is not a bud of (σ, α). Reduced unicellular hypermonopoles bijectively correspond to circular permutations for which every pair $(i, \pi(i))$ is a breakpoint, see Lemma 3.1 in the next section.

The other model is the one introduced by Alexeev and Zograf [1]. Consider a $2n$ sided polygon whose boundary consists of n black sides followed by n grey sides, the black sides are oriented in the the counterclockwise direction and the grey sides are oriented in the clockwise direction, as shown in Fig. 1. Pairwise gluing of black sides

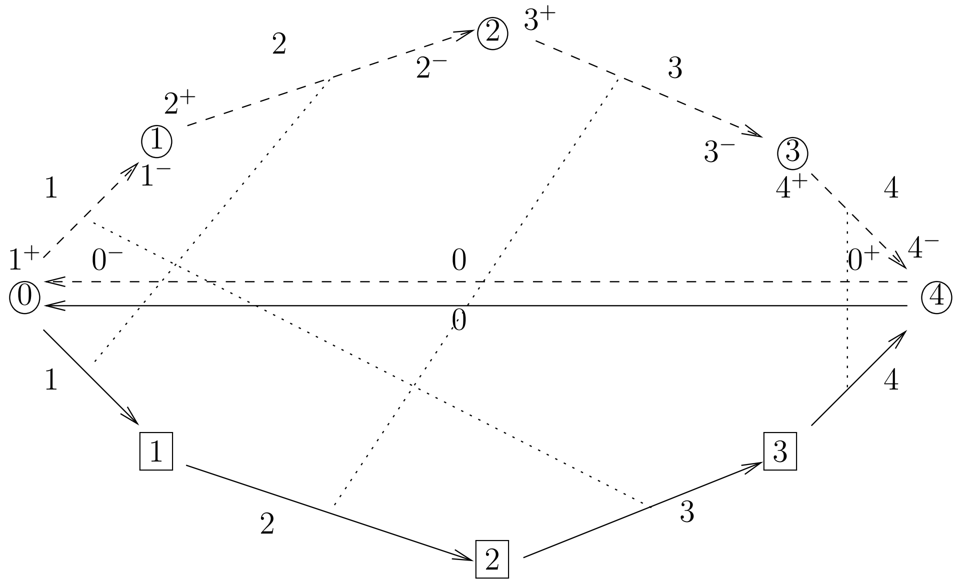

Figure 1. A polygon gluing diagram for $\pi = (0, 2, 3, 1, 4)$

with gray sides (respecting orientation) gives an orientable topological surface without boundary of topological genus $g \geq 0$ (the genus g depends on the gluing). We cut the polygon along the diagonal connecting the vertex n and the vertex 0, and we add a directed edge of each color from n to 0. By assuming that these added edges will be glued together we don't change the genus. We number all edges by their tail end, and we use the gluing pattern to define the circular permutation $\pi = (\pi_0, \pi_1, \dots, \pi_n)$: we define π_i as the grey edge that is glued with the black edge i. As in the previous model, we define σ as the circular permutation $(0, 1, \dots, n)$. For the gluing pattern shown in Fig. 1 we obtain $\pi = (0, 2, 3, 1, 4)$. As before, let us define $\alpha = \sigma\pi^{-1}$, in our example we obtain $\alpha = (0)(1, 4, 2)(3)$. According to Alexeev and Zograf [1] it is easy to see that the alternating cycles of $G(\pi)$ are in bijection with the vertices of the glued polygon. We can make it easier to see this by adding the signed labels i^- and i^+ along each grey edge labeled i as shown in Fig. 1. For example, the alternating cycle

$$0^- \xrightarrow{\sigma} 1^+ \xrightarrow{\pi^{-1}} 3^- \xrightarrow{\sigma} 4^+ \xrightarrow{\pi^{-1}} 1^- \xrightarrow{\sigma} 2^+ \xrightarrow{\pi^{-1}} 0^-$$

corresponds to the identification $\textcircled{0} = \boxed{2} = \textcircled{3} = \boxed{3} = \textcircled{1} = \textcircled{0}$. The verification of the details is left to the reader.

3. Counting reduced unicellular hypermonopoles

In this section we express the number of reduced unicellular hypermonopoles in terms of the number $H(n, k)$. In doing so, the following lemma will be useful.

Lemma 3.1. *A unicellular hypermonopole (σ,α) satisfying $\sigma=(1,\ldots,n)$ is reduced if and only if there is no $i\in\{1,\ldots,n\}$ that the circular permutation $\pi=\alpha^{-1}\sigma$ takes into $i+1$. Here addition is performed modulo n.*

Indeed, $\alpha^{-1}\sigma(i)=i+1$ is equivalent to $\alpha(i+1)=i+1$. Let us also note the following consequence of the proof of Lemma 1.2.

Corollary 3.2. *If (σ,α) is a reduced unicellular hypermonopole on n points then $1\leq z(\alpha)\leq n/2$ holds.*

Proposition 3.3. *Given $n\geq 2$ and $1\leq k\leq n/2$, the number of reduced unicellular hypermonopoles (σ,α) satisfying $\sigma=(1,\ldots,n)$ and $z(\alpha)=k$ is given by*

$$r(n,k)=\sum_{i=0}^{k-1}(-1)^i\binom{n}{i}H(n-i,k-i). \tag{3.1}$$

Proof. We compute $r(n,k)$ using inclusion-exclusion. Let $\mathcal{H}_{n,k}$ be the set of all unicellular hypermonopoles (σ,α) satisfying $\sigma=(1,\ldots,n)$ and $z(\alpha)=k$. For each $j\in\{1,\ldots,n\}$, let $\mathcal{H}_{n,k,j}$ be the subset of $\mathcal{H}_{n,k}$ also satisfying $\alpha^{-1}\sigma(j)=j+1$. Clearly we have

$$r(n,k)=\left|\mathcal{H}_{n,k}-\bigcup_{j=1}^{n}\mathcal{H}_{n,k,j}\right|.$$

Using the inclusion-exclusion formula we obtain

$$r(n,k)=\sum_{i=0}^{n}(-1)^i\sum_{\{j_1,\ldots,j_i\}\subseteq\{1,\ldots,n\}}\left|\mathcal{H}_{n,k,j_1}\cap\ldots\cap\mathcal{H}_{n,k,j_i}\right|. \tag{3.2}$$

First we show that it suffices to perform the summation on the right hand side only up to $i=k-1$. All hypermaps $(\sigma,\alpha)\in\mathcal{H}_{n,k,j_1}\cap\ldots\cap\mathcal{H}_{n,k,j_i}$ have the property that $j_1,\ldots,j_i$ are fixed points of α. Since $z(\alpha)=k$, we may restrict the summation in (3.2) to $i\leq k$. Furthermore the case $i=k$ is possible only if the cycles $(j_1),\ldots,(j_k)$ are all the cycles of α in which case $k=n$ in contradiction with $k\leq n/2$.

From now on let us fix a subset $\{j_1,\ldots,j_i\}$ of $\{1,\ldots,n\}$ for some $i\leq k-1$. Let σ' be the circular permutation of $\{1,\ldots,n\}-\{j_1,\ldots,j_i\}$ obtained from $(1,\ldots,n)$ by removing the elements $j_1,\ldots,j_i$. Given any unicellular hypermonopole $(\sigma,\alpha)\in\mathcal{H}_{n,k,j_1}\cap\ldots\cap\mathcal{H}_{n,k,j_i}$, let us define the unicellular hypermonopole (σ',α') on the set of points $\{1,\ldots,n\}-\{j_1,\ldots,j_i\}$ by the following procedure:

1. We define $\pi=\alpha^{-1}\sigma$ as the unique face of (σ,α).
2. We define the unique face π' of (σ',α') as the circular permutation obtained by removing the elements $j_1,\ldots,j_i$ from from π .
3. The permutation α' is given by $\alpha'=\sigma'\pi'^{-1}$.

The operation $(\sigma,\alpha)\mapsto(\sigma',\alpha')$ associates to each element of $\mathcal{H}_{n,k,j_1}\cap\ldots\cap\mathcal{H}_{n,k,j_i}$ a unicellular hypermonopole (σ',α'). The operation is invertible: to obtain π from π' we must insert each $j\in\{j_1,\ldots,j_i\}$ right before $j+1$. Hence we obtain a bijection between the hypermaps in $\mathcal{H}_{n,k,j_1}\cap\ldots\cap\mathcal{H}_{n,k,j_i}$ and the set of all unicellular hypermonopoles with vertex σ'. Therefore we have

$$\left|\mathcal{H}_{n,k,j_1}\cap\ldots\cap\mathcal{H}_{n,k,j_i}\right|=H(n-i,k-i),$$

and the statement is a direct consequence of (3.2). □

Corollary 3.4. *If $n-k$ is odd then $r(n,k)=0$. As a consequence the least value of n for which $r(n,k)>0$ holds for some $k \leq \frac{n}{2}$ is $n=3$.*

Indeed, if $n-k$ is odd then all terms $H(n-i,k-i)=0$ appearing on the right hand side of (3.1) are zero.

The values of $r(n,k)$ for $3 \leq n \leq 12$ are shown in Table 2.

$n \backslash k$	1	2	3	4	5	6
3	1					
4	0	1				
5	8	0				
6	0	36	0			
7	180	0	49			
8	0	1604	0	21		
9	8064	0	5144	0		
10	0	112608	0	7680	0	
11	604800	0	604428	0	5445	
12	0	11799360	0	1669052	0	1485

TABLE 2. The values of $r(n,k)$ for $3 \leq n \leq 12$ and $1 \leq k \leq \lfloor n/2 \rfloor$.

Combining Lemma 1.5 and Equation (3.1) we obtain the following formula.

Theorem 3.5. *The numbers $r(n,k)$ satisfy*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} r(n,k) \cdot x^k = \sum_{i=0}^{n-1} \binom{n}{i} (-x)^i \cdot \frac{(x)^{n-i} - (x)_{n-i}}{(n-i)(n-i+1)}.$$

Proof. By Lemma 1.5 the number $H(n-i,k-i)$ is given by

$$H(n-i,k-i) = [x^{k-i}] \frac{(x)^{n-i} - (x)_{n-i}}{(n-i)(n-i+1)} = [x^k] x^i \frac{(x)^{n-i} - (x)_{n-i}}{(n-i)(n-i+1)}.$$

The statement now follows from Equation (3.1) after noticing that we may extend the upper limit of the summation to n:

$$r(n,k) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} H(n-i,k-i)$$

also holds if we set $H(n,k)=0$ for $k \leq 0$. Lemma 1.5 is still applicable: the expression $((x)^{(n+1)} - (x)_{n+1})/((n+1)n)$ is a polynomial of x with zero constant term, containing no negative powers of x. □

Lemma 1.2 and Proposition 3.3 allow us to compute the number of all reduced unicellular hypermonopoles of a fixed genus, using the following result.

Proposition 3.6. *The number $u(g)$ of all reduced unicellular hypermonopoles of genus g is given by*

$$u(g) = \sum_{n=2g+1}^{4g} r(n, n-2g).$$

Proof. By (1.2) a unicellular hypermonopole with k cycles has genus g if and only if $k = n - 2g$ holds. As seen in Lemma 1.2, n must be at least $2g + 1$ and at most $4g$. $\square$

The first 10 entries of the sequence $\{u(g)\}_{g=1}^{\infty}$ are the following:

$$\begin{aligned}&2, 114, 21538, 8698450, 6113735682, 6641411533106,\\ &10323616703610338, 21755183272319116818,\\ &59718914489141881419202, 207083242485963591169089778.\end{aligned}$$

Acknowledgments

The second author wishes to express his heartfelt thanks to Labri, Université Bordeaux I, for hosting him as a visiting researcher in Summer 2023 and in Spring 2024, where a great part of this research was performed. This work was partially supported by a grant from the Simons Foundation (#514648 to Gábor Hetyei).

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