THE CENTER PROBLEM ON A CENTER MANIFOLD IN $\mathbb{R}^3$

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Abstract. Fix a collection of polynomial vector fields on $\mathbb{R}^3$ with a singularity at the origin, for every one of which the linear part at the origin has two pure imaginary and one non-zero eigenvalue. We show that the set of such systems having a center on the local center manifold at the origin corresponds to a variety in the space of admissible coefficients. We explicitly compute it for several families of systems with quadratic higher order terms.

1. Introduction

Suppose an analytic system of differential equations $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$ on $\mathbb{R}^3$ has an isolated fixed point at the origin, $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, and that the linear part $\mathrm{d}\mathbf{f}(\mathbf{0})$ has one non-zero and two pure imaginary eigenvalues. We investigate the nature of the local flow on a neighborhood of $\mathbf{0}$ on the local center manifold at the origin. In particular, we wish to distinguish whether the origin is a center or a focus.

The problem of the center has a solution dating back to the work of Lyapunov, which overcomes the difficulty that the center manifold at an isolated singularity of a real analytic vector field need not be analytic. Briefly put, there is a center on the center manifold if and only if there exists an analytic first integral on a neighborhood of the origin, in which case the local center manifold is analytic. See Theorem 3.

In this paper we investigate the special case in which the components of $\mathbf{f}$ in $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$ are polynomial functions. We will show that, as in the two-dimensional case, the set of systems with a center corresponds to a variety in the set of admissible coefficients, determined by an explicitly computable collection of polynomials in the coefficients, called the focus quantities. In actual practice the computation of the focus quantities themselves and the characterization of the variety that they determine are challenging problems, in terms of both the computational and the theoretical work that must be done. Because of the complex eigenvalues and the relative advantages of working with varieties over $\mathbb{C}$ rather than over $\mathbb{R}$, complexification of the real system is an integral part of our approach.

We carry out the method for two sub-families of family (18) below, the most general real system whose complexification is (17), the nonlinearities of which are the complex analogue in this context of the Hoyer system on $\mathbb{R}^3$ ([14]).

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2. Background

Suppose U is an open neighborhood of the origin in $\mathbb{R}^3$, $\mathbf{f} : U \to \mathbb{R}^3$ is a real analytic mapping, and that $\mathrm{d}\mathbf{f}(\mathbf{0})$ has one non-zero and two pure imaginary eigenvalues. By an invertible linear change of coordinates and a possibly negative rescaling of time the system of differential equations $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$ can be written in the form

$$\begin{aligned} \dot{u} &= -v + P(u,v,w) = \widetilde{P}(u,v,w) \\ \dot{v} &= u + Q(u,v,w) = \widetilde{Q}(u,v,s) \\ \dot{w} &= -\lambda w + R(u,v,w) = \widetilde{R}(u,v,w) \end{aligned} \tag{1}$$

where λ is a positive real number. We will let $\mathfrak{X}$ denote the corresponding vector field

$$\mathfrak{X} = \widetilde{P}\tfrac{\partial}{\partial u} + \widetilde{Q}\tfrac{\partial}{\partial v} + \widetilde{R}\tfrac{\partial}{\partial w} \tag{2}$$

on a neighborhood of the origin.

A local first integral of system (1) is a nonconstant differentiable function Ψ from a neighborhood of the origin in $\mathbb{R}^3$ into $\mathbb{R}$ that is constant on trajectories of (1), equivalently, such that

$$\mathfrak{X}\Psi = \widetilde{P}\Psi_u + \widetilde{Q}\Psi_v + \widetilde{R}\Psi_w \equiv 0. \tag{3}$$

A formal first integral for system (1) is a formal power series Ψ in u, v, and w that is not merely a constant and is such that when $\widetilde{P}$, $\widetilde{Q}$, and $\widetilde{R}$ are expanded in power series about the origin, every coefficient in the formal power series in (3) is zero.

When w and $\dot{w}$ are absent from (1) so that the system is on $\mathbb{R}^2$, the singularity at the origin is either a center (a punctured neighborhood is composed entirely of periodic orbits) or a focus (every trajectory near the origin spirals towards the origin, or every trajectory does so in reverse time). The center problem is the problem of distinguishing between the two cases. It was solved by Poincaré and Lyapunov in terms of the existence or non-existence of a local first integral. A proof appears in [17].

Theorem 1 (Poincaré-Lyapunov Center Theorem)**.** *The analytic system of differential equations*

$$\begin{aligned} \dot{u} &= -v + P(u,v) \\ \dot{v} &= u + Q(u,v) \end{aligned} \tag{4}$$

has a center at the origin if and only if it admits a local analytic first integral of the form $\Psi(u,v) = u^2 + v^2 + \cdots$. Moreover existence of a formal first integral Ψ implies existence of a local analytic first integral of the same form.

In the special case that P and Q in (4) are homogeneous quadratic polynomials the center problem has been solved in explicit terms through the work of many individuals [1, 2, 8, 9, 11, 12, 18, 19, 20]. We give the version of the solution presented in [6].

Theorem 2 (Quadratic Center Theorem)**.** *The system*

$$\begin{aligned} \dot{u} &= -v - bu^2 - (B+2c)uv - dv^2 \\ \dot{v} &= u + au^2 + (A+2b)uv + cv^2 \end{aligned}$$

has a center at the origin if and only if at least one of the following three conditions is satisfied:

I. $a + c = b + d$;
II. $A(a + c) = B(b + d)$ *and* $aA^3 - (3b + A)A^2B + (3c + B)AB^2 - dB^3 = 0$;
III. $A + 5b + 5d = B + 5a + 5c = ac + bd + 2a^2 + 2d^2 = 0$.

It is worth noting that the conditions are polynomial, hence the set of quadratic systems with a center corresponds to a variety in the six-dimensional space of coefficients. Each condition yields a component of that variety. The set of solutions of the system of polynomial equations in Condition III, for example, defines the variety that corresponds to the set of common zeros of elements of the ideal

$$\langle A + 5b + 5d, B + 5a + 5c, ac + bd + 2a^2 + 2d^2\rangle$$

in the polynomial ring $\mathbb{R}[a, b, c, d, A, B]$. The center variety likewise exists for all systems (4) for which P and Q lie in a fixed family of polynomials without constant or linear terms.

Returning to the three-dimensional analytic system (1), for every $r \in \mathbb{N}$ there exists in a sufficiently small neighborhood of the origin a C^r invariant manifold W^c, the local center manifold, that is tangent to the (u, v)-plane at the origin and which contains all the recurrent behavior of system (1) in a neighborhood of the origin in $\mathbb{R}^3$ ([5, §4.1], [15, §2], [21]). It is not necessarily unique, but the local flows near the origin on any two C^{r+1} center manifolds are C^r conjugate in a neighborhood of the origin ([4]). This fact justifies our abuse of language in speaking below of a center on "the" center manifold. The following theorem of Lyapunov is proved in [3, §13]. Analyticity of W^c is a consequence of the analyticity of the distinguished normalizing transformation that brings the system to its quasi-normal form. Uniqueness follows from the same fact (as well as from a general theorem, Theorem 3.2 in [21]).

Theorem 3 (Lyapunov Center Theorem)**.** *For system* (1) *with corresponding vector field* (2)*, the origin is a center for* $\mathfrak{X}|W^c$ *if and only if* $\mathfrak{X}$ *admits a real analytic local first integral of the form* $\Phi(u, v, w) = u^2 + v^2 + \cdots$ *in a neighborhood of the origin in* $\mathbb{R}^3$*. Moreover when there exists a center the local center manifold* W^c *is unique and is analytic.*

In the next section we show that, as in the two-dimensional case, when the functions P, Q, and R in (1) are polynomials in some specified family, the set of systems for which the origin is a center for $\mathfrak{X}|W^c$ are precisely those whose coefficients lie in a variety V_C in the space of admissible coefficients. We would like to be able to state and prove a direct analogue of Theorem 2 in the case that P, Q, and R are homogeneous quadratic polynomials, but at the moment to do so is computationally out of reach. In Section 4 we particularize to the family (18) and find the center variety for two sub-families of that family.

3. The Focus Quantities and the Center Variety

By Theorem 3 existence of a center of $\mathfrak{X}|W^c$ is equivalent to existence of a first integral for $\mathfrak{X}$, so we can restrict our efforts to investigation of conditions for existence of an integral Φ, which can be assumed to have no constant term, hence must have the form $\Phi(u, v, w) = u^2 + v^2 + \cdots$. We further suppose that each of P, Q, and R is a sum of homogeneous polynomials of degrees between 2 and some number N, although not all terms need be admissible, as illustrated in the system (18) that we investigate in Section 4.

We begin by introducing the complex variable $x = u + iv$. Then the first two equations in (1) are equivalent to a single equation $\dot{x} = ix + X(x, \bar{x}, w)$, where X is a sum of homogeneous polynomials of degrees between 2 and N. Adjoining to this equation its complex conjugate, replacing $\bar{x}$ everywhere by y, regarding y as an independent complex variable, and replacing w by z simply as a notational convenience we obtain the complexification of family (1),

$$\begin{aligned} \dot{x} &= ix + \sum_{p+q+r=2}^{N} a_{pqr} x^p y^q z^r \\ \dot{y} &= -iy + \sum_{p+q+r=2}^{N} b_{pqr} x^p y^q z^r \\ \dot{z} &= -\lambda z + \sum_{p+q+r=2}^{N} c_{pqr} x^p y^q z^r, \end{aligned} \tag{5}$$

where $b_{qpr} = \bar{a}_{pqr}$ and the c_{pqr} are such that $\sum_{p+q+r=2}^{N} c_{pqr} x^p \bar{x}^q w^r$ is real for all $x \in \mathbb{C}$, for all $w \in \mathbb{R}$. Let $\mathfrak{Z}$ denote the corresponding vector field on $\mathbb{C}^3$. Existence of a first integral $\Phi(u, v, w) = u^2 + v^2 + \cdots$ for a system in family (1) is equivalent to existence of a first integral

$$\Psi(x, y, z) = xy + \sum_{j+k+\ell=3} v_{jk\ell} x^j y^k z^\ell \tag{6}$$

for the corresponding system in family (5).

We first characterize existence of a formal first integral in terms of a normal form of systems in family (5). By normal form we mean the system after a formal change of variables $\mathbf{x} = \mathbf{x}_1 + \mathbf{h}(\mathbf{x}_1)$ that eliminates all nonresonant terms, where the term $x^p y^q z^r$ in the mth equation in (5) is nonresonant if there is no solution $(p, q, r) \in \mathbb{N}_0^3$ ($\mathbb{N}_0 = \{0, 1, 2, \ldots\}$) with $p + q + r \geq 2$ to the equation

$$((p, , q, r), \kappa) - \kappa_m = -\lambda + (p - q)i - \kappa_m = 0,$$

where $(\kappa_1, \kappa_2, \kappa_3) = (i, -i, -\lambda)$. (See for example [3] and [17] for a full discussion of normal forms.) Thus any normal form of a system in family (5) has the form

$$\begin{aligned} \dot{x}_1 &= ix_1 + x_1 X(x_1 y_1) \\ \dot{y}_1 &= -iy_1 + y_1 Y(x_1 y_1) \\ \dot{z}_1 &= -\lambda z_1 + z_1 Z(x_1 y_1). \end{aligned} \tag{7}$$

Except for the last theorem in this section we will not assume that (5) arises as the complexification of a real system, hence the coefficients in (5) and (7) are unrestricted.

The only if part of the following theorem, when existence of an analytic integral is hypothesized, appears with practically the same proof in §6 of [3]. The remainder can also be drawn from various parts of [3]. For convenience of the reader we present the proof.

Theorem 4. *A system of the form* (5) *admits a formal first integral of the form* (6) *if and only if the functions X and Y in any normal form* (7) *satisfy $X + Y \equiv 0$.*

Proof. Suppose system (5) has a formal first integral of the form $\Psi(x,y) = xy + \cdots$. If $\mathbf{x} = \mathbf{H}(\mathbf{x_1})$ is the normalizing transformation that converts (5) into a normal form (7), then $F = \Psi \circ \mathbf{H}$ is a formal first integral for the normal form, hence

$$\begin{aligned} ix_1\frac{\partial F}{\partial x_1}(x_1,y_1,z_1) - iy_1\frac{\partial F}{\partial y_1}(x_1,y_1,z_1) - \lambda z_1\frac{\partial F}{\partial z_1}(x_1,y_1,z_1) \\ = -x_1\frac{\partial F}{\partial x_1}(x_1,y_1,z_1)X(x_1y_1) - y_1\frac{\partial F}{\partial y_1}(x_1,y_1,z_1)Y(x_1y_1) \\ - z_1\frac{\partial F}{\partial z_1}(x_1,y_1,z_1)Z(x_1y_1)\,. \end{aligned} \tag{8}$$

Since $\mathbf{H}$ has the form $\mathbf{x} = \mathbf{x}_1 + \mathbf{h}(\mathbf{x}_1)$, writing $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ and $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $F(x_1,y_1,z_1)$ has the form

$$F(x_1,y_1,z_1) = \sum_{\substack{\alpha \\ |\alpha|\geq 2}} F^{(\alpha)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3} = x_1y_1 + \cdots\,. \tag{9}$$

Equation (8) then reads

$$\begin{aligned} &\sum_{\substack{\alpha \\ |\alpha|\geq 2}} (\alpha_1 i - \alpha_2 i - \lambda\alpha_3) F^{(\alpha)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3} \\ &\quad = -\left[\sum_{\alpha,|\alpha|\geq 2} \alpha_1 F^{(\alpha)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3}\right]\left[\sum_{j=1}^{\infty} X^{(j+1,j,0)}(x_1y_1)^j\right] \\ &\quad\quad -\left[\sum_{\alpha,|\alpha|\geq 2} \alpha_2 F^{(\alpha)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3}\right]\left[\sum_{j=1}^{\infty} Y^{(j,j+1,0)}(x_1y_1)^j\right] \\ &\quad\quad -\left[\sum_{\alpha,|\alpha|\geq 2} \alpha_3 F^{(\alpha)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3}\right]\left[\sum_{j=1}^{\infty} Z^{(j,j,1)}(x_1y_1)^j\right]\,. \end{aligned} \tag{10}$$

Using (9) for the basis step and using (10) and the fact that $-\lambda\alpha_3 + (\alpha_1 - \alpha_2)i$ is nonzero if $|\alpha|$ is odd and holds only for $\alpha = (\alpha_1,\alpha_1,0)$ if $|\alpha|$ is even for the inductive step, by mathematical induction we have that $F(x_1,y_1,z_1)$ is a function of x_1y_1 alone. Writing $F(x_1,y_1,z_1) = f(x_1y_1)$,

$$x_1\frac{\partial F}{\partial x_1}(x_1,y_1,z_1) = x_1y_1f'(x_1y_1) \quad \text{and} \quad y_1\frac{\partial F}{\partial y_1}(x_1,y_1,z_1) = x_1y_1f'(x_1y_1)$$

so that, letting $\zeta = x_1y_1$, (8) becomes

$$0 \equiv \zeta f'(\zeta)(X(\zeta) + Y(\zeta))\,.$$

But because F is a formal first integral it is not a constant, so we immediately obtain $X(\zeta) + Y(\zeta) \equiv 0$.

Conversely, if $X + Y \equiv 0$ then $\widehat{\Psi}(x_1,y_1,z_1) = x_1y_1$ is a first integral of (7). Since the coordinate transformation $\mathbf{x} = \mathbf{x}_1 + \mathbf{h}(\mathbf{x}_1)$ has an inverse of the form $\mathbf{x}_1 = \mathbf{x} + \widehat{\mathbf{h}}(\mathbf{x})$, system (5) therefore admits a formal first integral of the form $\Psi(x,y,z) = xy + \cdots$. □

Theorem 5. *Fix a system* (1) *in which the functions P, Q, and R are real analytic on a neighborhood of the origin. The following statements are equivalent.*
1. The origin is a center for $\mathfrak{X}|W^c$, W^c the local center manifold at the origin.

2. System (1) *admits a formal first integral.*
3. System (1) *admits a local analytic first integral.*

Proof. The equivalence of the first and third statements is Theorem 3. The third statement implies the second. If the second statement holds, then by Theorem 4 the functions X and Y in any normal form (7) of the complexification (5) of (1) satisfy $X + Y \equiv 0$. In such a case $\Psi(x_1, y_1, z_1) = x_1 y_1$ is an analytic first integral of (7). But in §5 of [3] it is shown that for family (1) the condition $X + Y \equiv 0$ implies that the distinguished normalizing transformation $\mathbf{x} = \mathbf{x}_1 + \mathbf{h}(\mathbf{x}_1)$ that transforms (5) into (7) is real analytic, since it doubles as a normal form on an invariant surface. Since the normalizing transformation has an analytic local inverse, the analytic integral Ψ yields an analytic integral $\Phi(u, v, w) = u^2 + v^2 + \cdots$ of (1). □

We now investigate the existence of a first integral Ψ for a system in family (5) by computing the coefficients of $\mathfrak{Z}\Psi$ and equating them to zero. When Ψ has the form (6) the coefficient $g_{k_1k_2k_3}$ of $x^{k_1}y^{k_2}z^{k_3}$ in $\mathfrak{Z}\Psi$ is

$$
\begin{aligned}
&(-\lambda k_3 + (k_1 - k_2)i)v_{k_1k_2k_3} \\
&\qquad + a_{k_1,k_2-1,k_3} + b_{k_1-1,k_2,k_3} \\
&\qquad + \sum_{r=0}^{\min\{k_3,N\}} \left[\sum_{\substack{j+k=3+r-k_3 \\ j\geq 1, k\geq 0}}^{k_1+k_2+r-1} j\, a_{k_1-j+1,k_2-k,r}\, v_{j,k,k_3-r} \right] \\
&\qquad + \sum_{r=0}^{\min\{k_3,N\}} \left[\sum_{\substack{j+k=3+r-k_3 \\ j\geq 0, k\geq 1}}^{k_1+k_2+r-1} k\, b_{k_1-j,k_2-k+1,r}\, v_{j,k,k_3-r} \right] \\
&\qquad + \sum_{r=0}^{\min\{k_3,N\}} \left[\sum_{\substack{j+k=2+r-k_3 \\ j\geq 0, k\geq 0}}^{k_1+k_2+r-2} (k_3 - r + 1)\, c_{k_1-j,k_2-k,r}\, v_{j,k,k_3-r+1} \right].
\end{aligned}
\tag{11}
$$

The maximum of the sum of the subscripts on $v_{\alpha\beta\gamma}$ in the sums is $k_1 + k_2 + k_3 - 1$. Thus except when $(k_1, k_2, k_3) = (K, K, 0)$ for $K \in \mathbb{N}$, the equation $g_{k_1k_2k_3} = 0$ can be solved uniquely for $v_{k_1k_2k_3}$ in terms of the known quantities $v_{\alpha\beta\gamma}$ with $\alpha + \beta + \gamma < k_1 + k_2 + k_3$. A formal first integral Ψ thus exists if $g_{kk0} = 0$ for all $k \in \mathbb{N}$. An obstruction to the existence of the formal series Ψ occurs when the coefficient g_{KK0} is nonzero. This coefficient is the Kth *focus quantity*,

$$
\begin{aligned}
g_{KK0} = &\sum_{\substack{j+k=2 \\ j\geq 0, k\geq 0}}^{2K-1} [j\, a_{K-j+1,K-k,0} + k\, b_{K-j,K-k+1,0}]\, v_{j,k,0} \\
&+ \sum_{\substack{j+k=2 \\ j\geq 0, k\geq 0}}^{2K-2} c_{K-j,K-k,0}\, v_{j,k,1},
\end{aligned}
\tag{12}
$$

where we have incorporated the summands in the second line in (11) into the sums by making the natural assignments $v_{110} = 1$ and $v_{\alpha\beta\gamma} = 0$ for $\alpha + \beta + \gamma = 2$ but $(\alpha, \beta, \gamma) \neq (1, 1, 0)$.

Remark 6. The focus quantity g_{KK0} is obviously a polynomial in the coefficients $a_{\alpha\beta\gamma}$, $b_{\alpha\beta\gamma}$, $c_{\alpha\beta\gamma}$ of (5), but contains λ in the denominator of its coefficients. A similar sequence of such quantities could be found based on Theorem 4 by zeroing coefficients of $X + Y$. Indeed, computations presented in Section 4 were double-checked in this fashion.

The focus quantities g_{110} and g_{220} are uniquely determined, but the remaining ones depend on the choices made for v_{KK0}, $K \in \mathbb{N}$, $K \geq 2$. Once such an assignment is made Ψ is determined and satisfies

$$\mathfrak{Z}\Psi(x, y, z) = g_{110}\, xy + g_{220}\, (xy)^2 + g_{330}\, (xy)^3 + \cdots . \tag{13}$$

Vanishing of all the focus quantities is sufficient for existence of the formal first integral. We show that it is necessary by proving that if for one choice of the v_{KK0} at least one focus quantity is non-zero, then the same is true for every other choice of the v_{KK0}. To shorten the notation we let (a, b, c) stand for the coefficient string

$$(a_{200}, \ldots, a_{00N}, b_{200}, \ldots, b_{00N}, c_{200}, \ldots, c_{00N}).$$

Theorem 7. *Let Ψ be a formal series of the form* (6) *and let $g_{110}, g_{220}, \ldots$ be polynomials in (a, b, c) that satisfy* (13) *with respect to the system* (5)*. Then system* (5) *with $(a, b, c) = (a^*, b^*, c^*)$ admits a formal first integral of the form* (6) *if and only if $g_{kk0}(a^*, b^*, c^*) = 0$ for all $k \in \mathbb{N}$.*

Proof. If for admissible (a^*, b^*, c^*), $g_{kk0}(a^*, b^*, c^*) = 0$ for all $k \in \mathbb{N}$ then Ψ is a formal first integral for the corresponding family in (5).

For the converse, suppose that, contrary to what we wish to prove, there exists a formal first integral of (5) when $(a, b, c) = (a^*, b^*, c^*)$ but that for some $K \in \mathbb{N}$, $K \geq 2$, $g_{kk0}(a^*, b^*, c^*) = 0$ for $1 \leq k \leq K - 1$ but $g_{KK0}(a^*, b^*, c^*) \neq 0$. Because there exists a formal first integral of the form (6) by Theorem 4 the functions X and Y in (7) satisfy

$$X + Y \equiv 0. \tag{14}$$

On the other hand, if $\mathbf{x} = \mathbf{H}(\mathbf{x}_1)$ is the normalizing transformation that produced (7), then in the new variables (13) becomes, writing $F = \Psi \circ \mathbf{H}$,

$$\begin{aligned} &\Big[ix_1 + x_1 X(x_1 y_1)\Big] \frac{\partial F}{\partial x_1}(x_1, y_1, z_1) \\ &+ \Big[-iy_1 + y_1 Y(x_1 y_1)\Big] \frac{\partial F}{\partial y_1}(x_1, y_1, z_1) + \Big[-\lambda z_1 + z_1 Z(x_1 y_1)\Big] \frac{\partial F}{\partial z_1}(x_1, y_1, z_1) \\ &= g_{KK0}(a^*, b^*, c^*)[x_1 + h_1(x_1, y_1, z_1)]^K [y_1 + h_2(x_1, y_1, z_1)]^K + \cdots \\ &= g_{KK0}(a^*, b^*, c^*) x_1^K y_1^K + \cdots . \end{aligned} \tag{15}$$

If we subtract $x_1 X F_{x_1} + y_1 Y F_{y_1} + z_1 Z F_{z_1}$ from each side of (15) then we obtain (10) with the right hand side of (15) added to its right hand side. The induction argument following (10) obviously applies out to order $2K - 1$, and applies to order $2K$ as well because the terms of order $2K$ on the right hand side of (15) are $g_{KK0}(a^*, b^*, c^*) x_1^K y_1^K$. Therefore

$$\begin{aligned} F(x_1, y_1, z_1) &= x_1 y_1 + f_2\, (x_1 y_1)^2 + \cdots + f_K\, (x_1 y_1)^K + U(x_1, y_1, z_1) \\ &= f(x_1 y_1) + U(x_1, y_1, z_1) \end{aligned}$$

where $U(x_1, y_1, z_1)$ begins with terms of order at least $2K+1$. Thus

$$x_1\frac{\partial F}{\partial x_1} = x_1y_1f'(x_1y_1) + \alpha(x_1, y_1, z_1) \quad \text{and} \quad y_1\frac{\partial F}{\partial y_1} = x_1y_1f'(x_1y_1) + \beta(x_1, y_1, z_1)$$

where $\alpha(x_1, y_1, z_1)$ and $\beta(x_1, y_1, z_1)$ begin with terms of order at least $2K+1$. The left-hand side of (15) is thus

$$\begin{aligned} i[&\alpha(x_1,y_1, z_1) - \beta(x_1, y_1, z_1)] \\ &+ (X(x_1y_1) + Y(x_1y_1))\, x_1y_1\, f'(x_1y_1) \\ &+ X(x_1y_1)\,\alpha(x_1, y_1, z_1) + Y(x_1y_1)\,\beta(x_1, y_1, z_1) + [-\lambda z_1 + z_1]\frac{\partial F}{\partial z_1}(x_1, y_1, z_1)\,. \end{aligned}$$

Hence if we subtract

$$\begin{aligned} i[&\alpha(x_1,y_1, z_1) - \beta(x_1, y_1, z_1)] \\ &+ X(x_1y_1)\,\alpha(x_1, y_1, z_1) + Y(x_1y_1)\,\beta(x_1, y_1, z_1) + [-\lambda z_1 + z_1]\frac{\partial F}{\partial z_1}(x_1, y_1, z_1)\,, \end{aligned}$$

which begins with terms of order at least $2K+1$, from each side of (15) we obtain

$$(X(x_1y_1) + Y(x_1y_1))\, x_1y_1\, f'(x_1y_1) = g_{KK0}(a^*, b^*, c^*)(x_1y_1)^K + \cdots\,. \tag{16}$$

By (14) the left hand side of (16) is identically zero, whereas the right hand side is not, a contradiction. □

Theorem 7 clearly implies the following result.

Theorem 8. *Let Ψ and g_{kk0} be as in Theorem 7 and suppose there exists another function Ψ' of the form* (6) *and polynomials $g'_{110}(a, b, c), g'_{220}(a, b, c), \dots$ that satisfy* (13) *with respect to the family* (5)*. Then $V_C = V'_C$, where V_C is the variety determined by the ideal $\langle g_{110}, g_{220}, \dots\rangle$ and V'_C is the variety determined by the ideal $\langle g'_{110}, g'_{220}, \dots\rangle$.*

The previous four theorems were derived without any restrictions on the coefficients in (5). When (5) is the complexification of a real system (1) then a real variety is obtained by applying the restrictions on (a, b, c) listed after (5). Since by Theorem 5 existence of a formal integral is equivalent to existence of a local analytic integral, the following theorem is an immediate corollary of Theorem 8.

Theorem 9. *Let* (1) *be a family of polynomial differential equations on $\mathbb{R}^3$. For any system in the family let $\mathfrak{X}$ be the corresponding vector field* (2) *and let W^c be a local center manifold through the origin. Then there exists a variety V_C in the space of admissible coefficients such that the origin is a center for $\mathfrak{X}|W^c$ if and only if the coefficients of the components of $\mathfrak{X}$ lie in V_C.*

4. Center Conditions for Quadratic Families

As stated earlier, we would like to be able to reproduce Theorem 2 in the context of family (1), finding necessary and sufficient conditions for existence of a center at the origin on any center manifold, but insurmountable computational difficulties quickly arise when trying to work in such generality.

For a fixed choice of λ in (1) the center variety V_C is determined by the ideal $\langle g_{kk0} : k \in \mathbb{N}\rangle$ of focus quantities, which by the Hilbert Basis Theorem is finitely generated. The first difficulty in using the focus quantities to find V_C is that they can quickly become enormous, containing hundreds and hundreds of terms. In

the case at hand just a few computations show that λ must be assigned a specific numerical value in order for progress to be possible. For simplicity we fix $\lambda = 1$. Even then computing more than five or six focus quantities becomes infeasible without further restriction.

The next difficulty is that once focus quantities are known, their vanishing gives only necessary conditions for existence of a center, since we do not know how many focus quantities suffice to generate the full ideal $\langle g_{kk0} : k \in \mathbb{N}\rangle$. In practical terms we use the special purpose computational commutative algebra program SINGULAR ([7, 10]) to decompose the radical of $\langle g_{kk0} : k \in \mathbb{N}\rangle$, which actually determines V_C (over $\mathbb{C}$), into an intersection of prime ideals, to which correspond the irreducible components of V_C. Generators of these prime ideals give finite sets of conditions which, if they can be proved to be sufficient for existence a center on W^c, either directly or by proving existence of a local first integral of (1) of the form $u^2 + v^2 + \cdots$ and appealing to Theorem 5, are then a collection of conditions that together characterize existence of a center on W^c. This stage of the computations, using SINGULAR to perform the decomposition, also dictates restrictions on the coefficients in (1).

Among all systems of the form (1) with quadratic nonlinearities, one somewhat natural initial restriction is to look for systems which when complexified yield a system of the form

$$\begin{aligned} \dot{x} &= ix + a_{12}xy + a_{13}xz + a_{23}yz \\ \dot{y} &= -iy + b_{12}xy + b_{13}xz + b_{23}yz \\ \dot{z} &= -z + c_{12}xy + c_{13}xz + c_{23}yz\,. \end{aligned} \tag{17}$$

The requirement that (17) arise as the complexification of a real system then imposes the following constraints:

$$b_{12} = \bar{a}_{12}, \quad b_{13} = \bar{a}_{23}, \quad b_{23} = \bar{a}_{13}, \quad c_{23} = \bar{c}_{13}, \quad c_{12} \in \mathbb{R}.$$

We write

$$a_{12} = u_1 + iv_1, \quad a_{23} = u_2 + iv_2, \quad a_{13} = u_3 + iv_3, \quad c_{13} = u_4 + iv_4.$$

It is not difficult to show that the most general real system of the form (1) whose complexification has the form (17) has the form

$$\begin{aligned} \dot{u} &= -v + au^2 + av^2 + cuw + dvw \\ \dot{v} &= u + bu^2 + bv^2 + euw + fvw \\ \dot{w} &= -w + Su^2 + Sv^2 + Tuw + Uvw\,, \end{aligned} \tag{18}$$

and that

$$\begin{gathered} a = u_1, \quad b = v_1, \quad c = u_3 + u_2, \quad d = v_2 - v_3, \quad e = v_3 + v_2, \quad f = u_3 - u_2 \\ S = c_{12}, \quad T = 2u_4, \quad U = -2v_4. \end{gathered} \tag{19}$$

The first nontrivial focus quantity for the full family (17) is

$$g_{220} = 2c_{12}u_3 = S(c + f).$$

We will not list additional focus quantities, since g_{330} has 37 terms, g_{440} about 200, and g_{550} about 800. They are posted on the website of the fourth author (http://www.math.uncc.edu/∼dsshafer).

Proposition 10. *A system of the form* (18) *for which* $S = 0$ *has a center on the local center manifold at the origin.*

Proof. If $S = 0$ then the local center manifold W^c is the invariant (u, v)-plane. By Theorem 2 the origin is a center for $\mathfrak{X}|W^c$. □

Based on the Proposition and the form of g_{220} we continue on the assumption that $S \neq 0$ and $u_3 = c + f = 0$. By rescaling the coordinate w by $1/S$ we may also assume that $S = 1$. However, to make further progress (in particular, in order for SINGULAR to be able to perform the necessary decomposition) we must impose additional restrictions. We will do this in two separate ways, first by requiring that $u_1 = v_1 = 0$ (equivalent to $a = b = 0$) and second by requiring that $u_2 = v_2 = 0$ (equivalent to $c - f = d + e = 0$).

Theorem 11. *A system of the form* (18) *for which* $a = b = c + f = 0$ *and* $S = 1$ *has a center on the local center manifold at the origin if and only if at least one of the following two sets of conditions holds:*
1. $8c + T^2 - U^2 = 4(e - d) - T^2 - U^2 = 2(e + d) + TU = 0$*;*
2. $c = d + e = 0$.

Proof. For systems of the form (18) for which $a = b = c + f = 0$ and $S = 1$ a computation with *minAssGTZ* of SINGULAR ([7, 10]) shows the minimal associated prime ideals of the primary ideals in a primary decomposition of the ideal $\langle g_{110}, \ldots, g_{550}\rangle$ to be
1. $I_1 = \langle v_4^2 - v_3 - u_2, u_4v_4 - v_2, u_4^2 + v_4^2 - 2v_3, v_2u_4 - v_3v_4 + u_2v_4,$
$\qquad v_3u_4 + u_2u_4 - v_2v_4, v_3^2 - u_2^2 - v_2^2\rangle$
2. $I_3 = \langle u_2, v_2\rangle$
3. $I_2 = \langle u_4^2 + v_4^2, v_2u_4 + u_2v_4, u_2u_4 - v_2v_4, u_2^2 + v_2^2\rangle$

A necessary condition for a center is that all the generators in at least one of these ideals vanish. Clearly the third set of conditions is a special case of the second, and simplifying the first set of conditions we reduce to
$1'$. $\langle 2u_2 + u_4^2 - v_4^2, 2v_3 - u_4^2 - v_4^2, v_2 - u_4v_4\rangle$
$2'$. $\langle u_2, v_2\rangle$

Using (19) we obtain the two sets of conditions in the theorem as necessary for existence of a center on W^c. We must show sufficiency.

Suppose condition (1) of the theorem holds. Applying the corresponding expressions in $(1')$ in (19) and letting $\alpha = u_4 + v_4$ and $\beta = u_4 - v_4$ system (18) is

$$\begin{aligned} \dot u &= -v - \tfrac{1}{2}\alpha\beta uw - \tfrac{1}{2}\beta^2 vw \\ \dot v &= \quad u + \tfrac{1}{2}\alpha^2 uw + \tfrac{1}{2}\alpha\beta vw \\ \dot w &= -w + u^2 + v^2 + (\alpha + \beta)uw + (\beta - \alpha)vw\,. \end{aligned} \tag{20}$$

A search for invariant algebraic surfaces led to the discovery that W^c is given by the equation $u^2 + v^2 + \alpha uw + \beta vw - w = 0$. (The cofactor is $\alpha u + \beta v - 1$; see for example [17, §3.6].) Solving for w and inserting into the first two equations in (20) we obtain that, in local coordinates near the origin on W^c, $[\alpha u + \beta v - 1]\mathfrak{X}|W^c$ is

$$\begin{aligned} \dot u &= \quad v - (\alpha u + \beta v)v - \tfrac{1}{2}\beta(\alpha u + \beta v)(u^2 + v^2) \\ \dot v &= -u + (\alpha u + \beta v)u + \tfrac{1}{2}\alpha(\alpha u + \beta v)(u^2 + v^2)\,. \end{aligned} \tag{21}$$

The form of the system suggests a symmetry with respect to the radial line orthogonal to $\alpha u + \beta v = 0$. Under the rotation

$$u = (\tfrac{\alpha}{\Delta})u' - (\tfrac{\beta}{\Delta})v', \quad v = (\tfrac{\beta}{\Delta})u' + (\tfrac{\alpha}{\Delta})v' \qquad (\Delta = \sqrt{\alpha^2 + \beta^2}\,)$$

system (21) becomes

$$\dot{u}' = v' - \Delta u'v', \quad \dot{v}' = -u' + \Delta u'^2 + \tfrac{\Delta^2}{2}u'(u'^2 + v'^2)$$

which is invariant under the involution $(u', v', t) \to (u', -v', -t)$, hence is time-reversible, and so the origin is a center.

When the second condition of the theorem holds system (18) reduces to

$$\begin{aligned}\dot{u} &= -v + dvw\\ \dot{v} &= u - duw\\ \dot{w} &= -w + u^2 + v^2 + Tuw + Uvw\,.\end{aligned}$$

A first integral is $F(u, v, w) = u^2 + v^2$. □

As stated before the second subfamily of family (18) we study arises from the reduction $u_2 = v_2 = 0$, equivalent to $c - f = d + e = 0$. Recall that in view of Proposition 10 we have reduced without loss of generality to the situation $S = 1$ and $c + f = 0$, so that in fact $c = f = 0$ in this case.

Theorem 12. *A system of the form* (18) *for which* $d + e = c = f = 0$ *and* $S = 1$ *has a center on the local center manifold at the origin if and only if at least one of the following three sets of conditions holds:*
1. $a = b = 0$*;*
2. $T - 2a = U - 2b = 0$.
3. $d = e = 0$.

Proof. For systems of the form (18) for which $d + e = c = f = 0$ and $S = 1$ a computation with *minAssGTZ* of SINGULAR ([7, 10]) shows the minimal associated prime ideals of the primary ideals in a primary decomposition of the ideal $\langle g_{110}, \ldots, g_{550}\rangle$ to be
1. $\langle v_1, u_1\rangle$
2. $\langle v_1 + v_4, u_1 - u_4\rangle$
3. $\langle v_3\rangle$
4. $\langle u_4^2 + v_4^2, v_1u_4 + u_1v_4, u_1u_4 - v_1v_4, u_1^2 + v_1^2\rangle$

The fourth set of conditions is a special case of the first. Using (19) we obtain the three sets of conditions in the theorem as necessary for existence of a center on W^c. We must show sufficiency.

When the first condition of the theorem holds system (18) reduces to

$$\begin{aligned}\dot{u} &= -v + dvw\\ \dot{v} &= u - dvw\\ \dot{w} &= -w + u^2 + v^2 + Tuw + Uvw\,,\end{aligned}$$

A first integral is $F(u, v, w) = u^2 + v^2$.

When the second condition of the theorem holds system (18) reduces to

$$\begin{aligned}\dot{u} &= -v + au^2 + av^2 + dvw\\ \dot{v} &= u + bu^2 + bv^2 - duw\\ \dot{w} &= -w + u^2 + v^2 + 2auw + 2bvw\,,\end{aligned}$$

The local center manifold W^c is $u^2+v^2-w=0$ (with cofactor $-1+2au+2bv$). On it the system is

$$\begin{aligned}\dot{u} &= -v+(a+dv)(u^2+v^2)\\ \dot{v} &= \quad u+(b-du)(u^2+v^2)\,.\end{aligned}$$

Under the rotation

$$u=(\tfrac{a}{\Delta})u'-(\tfrac{b}{\Delta})v',\quad v=(\tfrac{b}{\Delta})u'+(\tfrac{a}{\Delta})v' \qquad (\Delta=\sqrt{a^2+b^2}\,)$$

the system becomes

$$\dot{u}'=-v'+(\Delta+dv')(u'^2+v'^2),\quad \dot{v}'=u'-du'(u'^2+v'^2)$$

which is invariant under the involution $(u',v',t)\to(-u',v',-t)$, hence is time-reversible, and so the origin is a center.

When the third condition of the theorem holds system (18) reduces to

$$\begin{aligned}\dot{u} &= -v+au^2+av^2\\ \dot{v} &= \quad u+bu^2+bv^2\\ \dot{w} &= -w+u^2+v^2+Tuw+Uvw\,.\end{aligned}\tag{22}$$

By Theorem 2 a planar quadratic system of the form

$$\dot{u}=-v+au^2+av^2,\quad \dot{v}=u+bu^2+bv^2$$

has a center at the origin, hence by Theorem 1 admits a local analytic first integral of the form $\Psi(u,v)=u^2+v^2+\cdots$. Then $F(u,v,w)=\Psi(u,v)$ is a local analytic first integral for (22). □

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