GEVREY REGULARITY FOR A CLASS OF DISSIPATIVE EQUATIONS WITH ANALYTIC NONLINEARITY

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ABSTRACT. In this paper, we establish Gevrey class regularity of solutions to a class of dissipative equations on the whole space $\mathbb{R}^d$, for initial data in certain potential spaces. The equation we consider has an analytic nonlinearity and the dissipation operator is a power (possibly fractional) of the Laplacian. This generalizes the results in [15] to the L^p setting, where the space periodic case was considered. Additionally, we allow for rougher initial data and extend the results to the case of the dissipation operator being a fractional Laplacian. The main tool is a generalization of the Kato-Ponce inequality ([28]) to Gevrey spaces. As an application, we obtain temporal decay of solutions in higher Sobolev norms for a large class of equations including the Navier-Stokes equations, the subcritical quasi-geostrophic equations, a variant of the Burger's equation with a polynomial nonlinearity, and the generalized Cahn-Hilliard equation.

1. INTRODUCTION

It is well-known that regular solutions of many dissipative equations, such as the Navier-Stokes equations(NSE), the Kuramoto-Sivashinsky equation, the surface quasi-geostrophic equation and the Smoluchowski equation are in fact analytic, in both space and time variables ([33], [17], [3], [14], [45]). In fluid-dynamics, the space analyticity radius has an important physical interpretation: at this length scale the viscous effects and the (nonlinear) inertial effects are roughly comparable. Below this length scale the Fourier spectrum decays exponentially ([16], [25], [26], [13]). In other words, the space analyticity radius yields a Kolmogorov type length scale encountered in turbulence theory. This fact concerning exponential decay can be used to show that the finite dimensional Galerkin approximations converge exponentially fast in these cases. For instance, in the case of the complex Ginzburg-Landau equation, Doelman and Titi ([12]) used radius of analyticity estimates to rigorously explain numerical observation that the solutions to this equation can be accurately represented by a very low-dimensional Galerkin approximation. Other applications of analyticity radius occur in establishing sharp temporal decay rates of solutions in higher Sobolev norms ([36]), establishing geometric regularity criteria for the Navier-Stokes equations, and in measuring the spatial complexity of fluid flow (see [30], [29], [23]).

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In this paper, we consider a nonlinear evolution equation on a Banach space $\mathcal{L}$ of the form

$$\mathbf{u}_t + \Lambda^\kappa \mathbf{u} = G(\mathbf{u}), \quad (\kappa > 1, t \geq 0), \quad \mathbf{u}_0 \in \mathcal{L}. \tag{1.1}$$

The Banach space $\mathcal{L}$ is a potential space of adequate (positive or negative) regularity. The operator $\Lambda = (-\Delta)^{1/2}$ in our case, and is assumed to be defined on dense subspace $\mathcal{D}(\Lambda) \subset \mathcal{L}$. The term $G(\mathbf{u})$ is a possibly nonlocal, analytic function of $\mathbf{u}$; see equation (2.3) for a precise description. We will also assume that $\mathcal{L}$ is a Banach space contained in the Schwartz space of distributions $\mathcal{S}'(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ denotes the usual Schwartz space of test functions. The equation (1.1) will model dissipative evolutionary partial differential equations on the whole space $\mathbb{R}^d$, either with no boundary or with the space periodic boundary condition.

In this setting, we establish analyticity of solutions to (1.1) and provide a (sharp) lower estimate for the time evolution of the space analyticity radius. It should be noted that the class of equations we consider encompasses many nonlocal equations (for instance, the Navier-Stokes and the quasi-geostrophic equations). Therefore even the fact that the solutions are analytic for positive times, do not follow from Cauchy-Kovaleskaya type theorems. In case the dissipation operator is $\Lambda^\kappa = \Lambda^2 = (-\Delta)$ (i.e., $\kappa = 2$), equation (1.1) with an analytic nonlinearity was first studied by Ferrari and Titi ([15]) and Cao, Rammaha, and Titi ([6]) in the space periodic case and the sphere case respectively. Their approach was based on energy technique and their space of initial data comprised of sufficiently smooth functions belonging to adequate (L^2-based) Sobolev spaces. By contrast, we work on certain L^p-based Banach spaces of initial data which allows us to consider much rougher (even distributional) initial data in certain situations. We take the mild solution approach initiated by Fujita and Kato ([21]), Giga and Miyakawa ([22]) and Weissler ([46]) for the Navier-Stokes equations.

Following Foias and Temam ([18]), we study the evolution of Gevrey norms $\|e^{t^\gamma \Lambda}\mathbf{u}\|_{\mathcal{L}}$ for a suitable Banach spaces $\mathcal{L}$ and $\gamma > 0$. The use of Gevrey norms was pioneered by Foias and Temam ([18]) for estimating space analyticity radius for the Navier-Stokes equations and was subsequently used by many authors (see [5], [15], [2], [4], and the references there in); a closely related approach can be found in [24]. This approach enables one to avoid cumbersome recursive estimation of higher order derivatives. We show that solutions to (1.1) are *Gevrey regular* (i.e., they satisfy the estimate $\sup_{0<t<T} \|e^{t^\gamma \Lambda}\mathbf{u}\|_{\mathcal{L}} < \infty$), locally in time for initial data of arbitrary size, and globally in time if the initial data is small in "critical" regularity spaces. In many cases, this critical regularity space precisely corresponds to a scale invariance property of the equation. Our key tool in this endeavor is a generalization of the Kato-Ponce inequality ([28]) to Gevrey spaces.

One of our main emphasis in this paper is to obtain global (in time) Gevrey regular solutions to (1.1) for small initial data in critical regularity spaces. This has several applications in the study of long term dynamics. It turns out (as we show here) that many of the equations encountered in fluid dynamics has the property that for large times, the solutions have small norm in these critical regularity spaces. Thus, as a consequence of our result, we obtain exponential decay of Fourier coefficients in the periodic case and algebraic decay of higher order L^p based Sobolev norms for a wide class of equations including the Navier-Stokes equations, sub-critical quasi-geostrophic equation, a variant of Burger's equation with a higher order polynomial nonlinearity and the generalized Cahn-Hilliard equation. In some cases (example, Navier-Stokes and the subcritical quasigeostrophic equations) this generalizes known results while in some others (example, Burger's and Cahn-Hilliard), the results are new to the best of our knowledge. For instance, following Gevrey class technique, Oliver and Titi ([36]) established sharp upper and lower bounds for the (time) decay of L^2-based higher order Sobolev norms for the Navier-Stokes equations in the whole space $\mathbb{R}^d$, for a certain class of initial data. Our result yields decay for L^p-based ($p > 1$) higher order (homogeneous) Sobolev norms for the Navier-Stokes equations, and for a larger class of initial data. As another corollary of our general result, we also recover a similar decay result of Dong and Li ([14]) for the quasi-geostrophic equations with a proof that avoids the iterative estimation of higher order derivatives, and consequently the intricate combinatorics present there. In summary, our main results provide an unified approach to a variety of decay results ([36], [37], [40], [14]) thus generalizing them to: a wider class of equations; to L^p decay; and allowing for a larger class of initial data. A more detailed comparison of our results *vis a vis* some known results is made in the remarks subsequent to the relevant theorems.

The organization of this paper is as follows. In Section 2, we state our main results while in Section 3, we state our main applications concerning decay. Sections 4 and 5 are devoted to the proofs of these results while in the Appendix, we have included some requisite background on Littlewood-Paley decomposition of functions.

2. Main results

Before describing our main results, we start by establishing some notation and concepts.

A function $\mathbf{u}(\cdot) \in C([0,T];\mathcal{L})$ is said to be a *strong* solution of (1.1) if $\partial_t \mathbf{u}$ exists and $\mathbf{u}(\cdot) \in \mathcal{D}(\Lambda^\kappa)$ *a.e.* and the equation (1.1) is satisfied *a.e.* A *mild* solution of (1.1) is a solution of the corresponding integral equation

$$\mathbf{u}(t) = (S\mathbf{u})(t), \quad (S\mathbf{u})(t) := e^{-t\Lambda^\kappa}\mathbf{u}_0 + \int_0^t e^{-(t-s)\Lambda^\kappa} G(\mathbf{u}(s))ds, \tag{2.1}$$

where $\mathbf{u}(\cdot)$ is assumed to belong to $C([0,T];\mathcal{L})$. The integral on the right hand side of (2.1) is interpreted as a Bochner integral. Henceforth, by a solution to (1.1) we will mean a mild solution, which is a fixed point of the map S. For a discussion on the connection between weak, strong, mild and classical solutions see [41].

With $\Lambda = (-\Delta)^{1/2}$, the homogeneous potential spaces are defined by

$$\dot{\mathbb{H}}^\alpha_p = \{f \in \mathcal{S}'(\mathbb{R}^d) : \Lambda^\alpha f \in L^p(\mathbb{R}^d), \|f\|_{\dot{\mathbb{H}}^\alpha_p} := \|\Lambda^\alpha f\|_{L^p} < \infty\}, \quad \alpha \in \mathbb{R}.$$

For each $i = 0, \cdots, N$, let T_i be a bounded operator mapping $\dot{\mathbb{H}}^{\alpha+\alpha_{T_i}}_p, \alpha_{T_i} \geq 0$ to $\dot{\mathbb{H}}^\alpha_p$ such that T_i commutes with Λ and its operator norm is bounded uniformly with respect to α and p. More precisely, we will assume that there exists constants $C > 0$ and $\alpha_{T_i} \geq 0$ such that for all $\alpha \in \mathbb{R}, 1 < p < \infty$ and $\mathbf{v} \in \dot{\mathbb{H}}^\alpha_p$, we have

$$\|T_i \mathbf{v}\|_{\dot{\mathbb{H}}^\alpha_p} \leq C\|\mathbf{v}\|_{\dot{\mathbb{H}}^{\alpha+\alpha_{T_i}}_p}, \quad T_i\Lambda\mathbf{v} = \Lambda T_i \mathbf{v}. \tag{2.2}$$

Examples of such operators are Fourier multipliers with symbols

$$m_i(\xi) = \Omega_i(\xi)\xi^{\vec{\alpha}_{T_i}} \quad \text{or} \quad m_i(\xi) = \Omega_i(\xi)|\xi|^{\alpha_{T_i}},$$

where $\Omega_i(\cdot)$ are bounded homogeneous functions of degree zero (i.e., $\Omega_i(c\xi) = \Omega(\xi), c \in \mathbb{R}$) and $\vec{\alpha}_{T_i} = (\alpha^{(1)}_{T_i}, \cdots, \alpha^{(d_1)}_{T_i})$, $\alpha^{(j)}_{T_i} \geq 0$. Other examples also include shift operators on the space of distributions (in which case $\alpha_{T_i} = 0$).

The nonlinearity G that we consider has the form

$$G(\mathbf{u}) = T_0 F(T_1\mathbf{u}, T_2\mathbf{u}, \cdots, T_N\mathbf{u}), \quad F(z_1, \cdots, z_N) = \sum_{\alpha \in \mathbb{Z}^N_+} a_\alpha z^\alpha, \tag{2.3}$$

where F above is an analytic function defined on a neighborhood of the origin in $\mathbb{R}^n$. More specific assumptions on F will be made in subsequent sections.

2.1. **A degree n nonlinearity.** In this section, we will assume that F is a monomial of degree n, i.e.,

$$G(\mathbf{u}) = T_0 F(T_1\mathbf{u}, T_2\mathbf{u}, \cdots, T_n\mathbf{u}), \quad F(z_1, \cdots, z_n) = z_1 \cdots z_n, \tag{2.4}$$

for some $n \geq 2, n \in \mathbb{N}$. However, see Remark 2.2 for an extension.

For $\xi \in \mathbb{R}^d$, denote $|\xi|_1 = \sum_{i=1}^d |\xi_i|$ while $|\xi| = (\sum_i \xi_i^2)^{1/2}$ denotes the usual Euclidean norm on $\mathbb{R}^d$. Recall that the norms $|\cdot|_1$ and $|\cdot|$ on $\mathbb{R}^d$ are equivalent. Let Λ_1 be a Fourier multiplier whose symbol is given by $m_{\Lambda_1}(\xi) = |\xi|_1$. Choose (and fix) a constant $c > 0$ such that $c|\xi|_1 < \frac{1}{4}|\xi|$ for all $\xi \in \mathbb{R}^d$. This is possible since all norms on $\mathbb{R}^d$ are equivalent. With this notation, we define the Gevrey norm (in $\dot{\mathbb{H}}^\beta_p$) to be

$$\|\mathbf{v}\|_{Gv(s,\beta,p)} = \left\|e^{cs^{1/\kappa}\Lambda_1}\Lambda^\beta \mathbf{v}\right\|_{L^p} = \left\|e^{cs^{1/\kappa}\Lambda_1}\mathbf{v}\right\|_{\dot{\mathbb{H}}^\beta_p}, \tag{2.5}$$

where $s \geq 0$, $\beta \in \mathbb{R}$, and $1 \leq p \leq \infty$.

Before stating our result here, we need to determine an appropriate function space for the initial data. The idea is to choose a function space where the linear term $\mathbf{u}_t + \Lambda^\kappa \mathbf{u}$ and the nonlinear term $G(\mathbf{u})$ have the same regularity. This can be interpreted in terms of the scaling invariance. The equation (1.1) with the nonlinearity (2.4) satisfies the following scaling. Assume that $\mathbf{u}$ is a solution of (1.1). Then, the same is true for the rescaled functions

$$\mathbf{u}_\lambda(t,x) = \lambda^s u(\lambda^\kappa t, \lambda x), \quad s = \frac{\kappa - \sum_{i=0}^n \alpha_{T_i}}{n-1}.$$

Therefore, $\dot{\mathbb{H}}^{\beta_c}_p$ is the scaling invariant space for initial data, with

$$\beta_c = \frac{d}{p} - \frac{\kappa - \sum_{i=0}^n \alpha_{T_i}}{n-1}, \tag{2.6}$$

and we can expect the local existence for $\beta > \beta_c$ for large data and the global existence for $\beta = \beta_c$ for small data.

Theorem 2.1. *Let G be a nonlinearity as in (2.4). Let $\mathbf{u}_0 \in \dot{\mathbb{H}}^{\beta_0}_p$, with*

$$\frac{d}{p} - \frac{\kappa - \sum_{i=0}^n \alpha_{T_i}}{n-1} = \beta_c \leq \beta_0 < \frac{d}{p} + \min_{1\leq i \leq n} \alpha_{T_i}.$$

Assume that the following condition holds:

$$\min_{1\leq i\leq n} \alpha_{T_i} > \max\left\{\frac{\sum_{i=1}^n \alpha_{T_i}}{n} - \frac{d}{np}, \ \frac{\sum_{i=0}^n \alpha_{T_i} - \kappa}{n-1}\right\}. \tag{2.7}$$

Then there exists an adequate $T = T(\mathbf{u}_0)$ and $\beta > 0$ with $\beta_0 + \beta > 0$, and a solution of (2.1) belonging to the space $C([0,T]; \dot{\mathbb{H}}_p^{\beta_0})$ which additionally satisfies

$$\max\Big\{ \sup_{o<t<T} \|\mathbf{u}(t)\|_{Gv(t,\beta_0,p)},\ \sup_{t>0} t^{\beta/\kappa}\|\mathbf{u}(t)\|_{Gv(t,\beta_0+\beta,p)} \Big\} \le 2\|\mathbf{u}_0\|_{\dot{\mathbb{H}}_p^{\beta_0}}. \tag{2.8}$$

Moreover, if $\beta_0 > \beta_c$, then the time of existence T is given by

$$T \ge \frac{C}{\|\mathbf{u}_0\|_{\dot{\mathbb{H}}_p^{\beta_0}}^{(\beta_0-\beta_c)/\kappa}},$$

for some constant C independent of $\mathbf{u}_0$. On the other hand, in case $\mathbf{u}_0 \in \dot{\mathbb{H}}_p^{\beta_c}$, there exists $\epsilon > 0$ such that whenever $\|\mathbf{u}_0\|_{\dot{\mathbb{H}}_p^{\beta_c}} < \epsilon$, we can take $T = \infty$.

Remark 2.2. The above theorem can be readily generalized to the case of a nonlinearity where F in (2.4) is a homogeneous polynomial of degree n with the following property: there exists an $\alpha \in \mathbb{R}$ such that

$$F(z_1, \cdots, z_N) = \sum_{\alpha\in\mathcal{S}} a_\alpha z^\alpha,$$

where

$$\mathcal{S} = \{\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{Z}_+^N : \sum_{i=1}^N \alpha_{T_i}\chi_{\{\alpha_i\neq 0\}} = \alpha \text{ and } \sum_{i=1}^N \alpha_i = n\}.$$

This for instance is satisfied if F is a homogeneous polynomial of degree n and $\alpha_{T_i} = \alpha_{T_j}$ for all $i, j \neq 0$.

Remark 2.3. A version of Theorem 2.1, for the special case of a quadratic nonlinearity, was established in [1]. In contrast to the set up of the real space here, the norms on the initial data space there were defined in the Fourier space. This enables one to completely avoid the detailed harmonic analysis machinery used in the proof here. However, due to the Hausdorff-Young inequality, even restricted to the quadratic nonlinearity case, our consideration here yields a larger space of initial data in several applications. As we will see later, due to this, we can obtain decay of L^p ($p > 1$) based higher Sobolev norms (for instance for the Navier-Stokes equations) not available in [1].

2.2. **Analytic Nonlinearity.** In this section, we will consider the more general case of an analytic nonlinearity. Let $F(z) = \sum_{k\in\mathbb{Z}_+^n} a_k z^k$ be a real analytic function in a neighborhood of the origin. Here $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$ and we employ the multi-index convention $z^k = z_1^{k_1} \cdots z_n^{k_n}$ for $k = (k_1, \cdots, k_n)$. The "majorizing function" for F is defined to be

$$F_M(r) = \sum_{k\in\mathbb{Z}_+^d} |a_k| r^{|k|}, \quad r < \infty, \tag{2.9}$$

where for any multi-index $k \in \mathbb{Z}_+^d$, $|k| = k_1 + \cdots + k_d$. The functions F and F_M are clearly analytic in the open balls (in $\mathbb{R}^d$ and $\mathbb{R}$ respectively) with center zero and radius

$$R_M = \sup\{r : F_M(r) < \infty\}. \tag{2.10}$$

We will assume that the set in the right hand side of (2.10) is nonempty. The derivative of the function F_M, denoted by F'_M, is also analytic in the ball of radius R_M.

The nonlinearity G is of the type

$$G(\mathbf{u}) = T_0 F\big(T_1\mathbf{u}, \cdots, T_n\mathbf{u}\big), \tag{2.11}$$

where T_i are as defined in (2.2).

We consider the inhomogeneous Sobolev space

$$\mathbb{H}^{\alpha}_p = \{f : \mathbb{R}^d \to \mathbb{R}^{d_1} : \|f\|_{\mathbb{H}^{\alpha}_p} := \|(I+\Lambda)^{\alpha} f\|_{L^p} < \infty\},$$

with $\Lambda = (-\Delta)^{1/2}$. We recall that from the standard Sobolev inequalities and (4.4), we have

$$\|f\|_{L^\infty} \le \|f\|_{\mathbb{H}^{\beta}_p}, \quad \|fg\|_{\mathbb{H}^{\beta}_p} \le C\|f\|_{\mathbb{H}^{\beta}_p}\|g\|_{\mathbb{H}^{\beta}_p} \quad \text{for } \beta > \frac{d}{p},\ \ 1 < p < \infty. \tag{2.12}$$

The Gevrey norm here is defined as

$$\|\mathbf{v}\|_{Gv(s,\beta,p)} = \|e^{\frac{1}{2}s^{1/\kappa}\Lambda_1}(1+\Lambda)^{\beta}\mathbf{v}\|_{L^p}. \tag{2.13}$$

We will show that the Gevrey space $Gv(s,\beta,p)$ with $\beta > \frac{d}{p}$ is a Banach algebra. We are now ready to state our main result concerning analytic nonlinearity.

Theorem 2.4. *Let $\beta > \frac{d}{p}$ and assume that $\frac{\alpha_{T_0}+\alpha}{\kappa} < 1$, where $\alpha := \max\limits_{1\le i\le n}\{\alpha_{T_i}\}$. Let $\|\mathbf{u}_0\|_{\mathbb{H}^{\beta+\alpha}_p} < R$. Assume that $2RC < R_M$ where C as in Lemma 4.9 and R_M as in (2.10). There exists a time $T > 0$ and a solution $\mathbf{u}$ of (2.1) such that*

$$\sup_{s\in(0,T)} \|\mathbf{u}(s)\|_{Gv(s,\beta,p)} < \infty.$$

Note that unlike Theorem 2.1, we do not get a global existence result here even in case of small initial data. However, in case of the periodic boundary condition, we do obtain a global existence result for small data. We need to assume however that the nonlinearity G has the property that it leaves the space of mean zero periodic functions invariant (this happens for instance if $T_0 = \nabla$). This is due to the fact that in the periodic case, Λ has a minimum eigenvalue, denoted by $\lambda_0 > 0$, and the Fourier spectrum of all periodic functions with space average zero is contained in the complement of a ball with radius λ_0. More precisely, we have the following result.

Theorem 2.5. *Consider the equation (1.1) with space periodic boundary condition. Assume that the space of mean zero functions are invariant to the analytic nonlinearity G which moreover satisfies*

$$a_0 = 0, \qquad \sum_{k\in\mathbb{Z}^d_+, |k|=1} |a_k| < \delta,$$

where $\delta \ge 0$ is suitably small. Then, there exists $\epsilon > 0$ such that if $\|\mathbf{u}_0\|_{\mathbb{H}^{\beta}_p} < \epsilon$, with $\beta > \frac{d}{p}$, and $1 < p < \infty$, we can obtain an unique solution to (2.1) satisfying $\sup_{0<t<\infty}\|\mathbf{u}\|_{Gv(t,\beta,p)} < \infty$.

2.3. **Equation in Fourier space.** In Subsection 2.2, we worked under the restriction $1 < p < \infty$. In this section, we consider the case $p = \infty$. The harmonic analysis tools used in the previous sections do not work here since the singular integrals are not bounded in L^∞. We therefore resort to working in the frequency space using th Fourier transform. Recall that if the Fourier transform of a distribution in $\mathcal{S}'(\mathbb{R}^d)$ is in L^1 in the Fourier spaces, then it is an L^∞ function. The development here is in the spirit of [3] and [4]. The other borderline case of $p = \infty$ (in space variables) is similar and discussed in Remark 2.7.

We will denote by $\mathcal{F}$ the Fourier transform (in the space variables) and by $\mathcal{F}^{-1}$ its inverse. By a notational abuse, letting $\mathbf{u} = \mathcal{F}(\mathbf{u})$, we can reformulate (1.1) as

$$\mathbf{u}_t(\xi, t) + |\xi|^\kappa \mathbf{u}(\xi, t) = G\big(\mathbf{u}(\cdot, t)\big)(\xi), \quad \mathbf{u}(\xi, 0) = \mathbf{u}_0(\xi), \tag{2.14}$$

where G is as in (2.11). Recalling that the Fourier transform converts products in real space to convolutions in the frequency space, the analytic function F in (2.9) takes the form

$$F(\mathbf{v}) = \sum_n a_n v_1^{*n_1} \cdots v_d^{*n_d}, \mathbf{v} = (v_1, \cdots, v_d)$$

where $*$ denotes convolution. Denoting by $\mathfrak{D}$ the multiplication operator $(\mathfrak{D}\mathbf{v})(\xi) = |\xi|^\kappa \mathbf{v}(\xi)$, we can write the mild formulation of (1.1) as

$$\mathbf{u}(t) = e^{-t\mathfrak{D}}\mathbf{u}_0 + \int_0^t e^{-(t-s)\mathfrak{D}} G\big(\mathbf{u}(\cdot, s)\big) ds. \tag{2.15}$$

In this section, we will denote $\|\cdot\|$ the L^1 norm in the Fourier space, i.e.,

$$\|\mathbf{v}\| = \int_{\mathbb{R}^{d_1}} |\mathbf{v}(\xi)| d\xi.$$

In case $\mathbf{v}(\xi)$ is a vector, $|\mathbf{v}(\xi)|$ will denote its usual Euclidean norm. We also recall that L^1 is a Banach algebra under convolution and

$$\|u * v\| \le \|u\| \|v\|.$$

For $s \ge 0$ and $\beta \in \mathbb{R}$, we will now introduce the Gevrey norms as

$$\|\mathbf{v}\|_{Gv(s,\beta)} = \int_{\mathbb{R}^{d_1}} e^{\frac{1}{2}s^{1/\kappa}|\xi|} |\xi|^\beta |\mathbf{v}(\xi)| d\xi. \tag{2.16}$$

For notational simplicity, we will suppress the dependence of the Gevrey norm on κ (since it is fixed), and when $s = 0$, we denote $\|\mathbf{v}\|_{Gv(0,\beta)} = \|\mathbf{v}\|_\beta$ and when $\beta = 0$, we will write $\|\mathbf{v}\|_{Gv(s,0)} = \|\mathbf{v}\|_{Gv(s)}$. Also, denote

$$\mathbb{V}_\beta = \{\mathbf{v} : \|\mathbf{v}\|_\beta < \infty\}.$$

When $\beta = 0$, we will simply write $\mathbb{V} = \mathbb{V}_0$.

For $\mathbf{u} = (u_1, \cdots, u_{d_1}), \mathbf{v} = (v_1, \cdots, v_{d_1})$ vector valued functions, we denote

$$\mathbf{u} * \mathbf{v} = (\sum_{ij} b_{ijk} u_i * v_j)_{k=1}^d, \quad b_{ijk} \in \mathbb{R}.$$

It is easy to see, applying the Cauchy-Schwartz inequality, that $|(\mathbf{u} * \mathbf{v})(\xi)| \le C(|\mathbf{u}| * |\mathbf{v}|)(\xi)$, where $C = \max_{i,j,k} |b_{ijk}|$.

We now state the existence of a solution to (2.14) with respect in the Gevrey spaces (2.16).

Theorem 2.6. *Let* $\alpha = \max_{1\le i\le n}\{\alpha_{T_i}\}$ *and assume that*

$$\frac{\alpha_{T_0}+\alpha}{\kappa} < 1 \text{ and } \max\{\|\mathbf{u}_0\|, \|\mathbf{u}_0\|_\alpha\} < R \text{ with } 2RC < R_M,$$

where C *is as in Lemma 4.11 and* R_M *as in (2.10). There exists a time* $T > 0$ *and a solution* $\mathbf{u}$ *of (2.14) such that*

$$\sup_{s\in(0,T)} \|\mathbf{u}(s)\|_{Gv(s)} < \infty. \tag{2.17}$$

Additionally, suppose that we are in the space periodic setting and the nonlinearity G *leaves the space of functions with zero space average invariant. Then, if we consider (2.14) on that subspace, there exists* $\epsilon > 0$ *such that we can take* $T = \infty$ *in the above inequality provided* $\max\{\|\mathbf{u}_0\|, \|\mathbf{u}_0\|_\alpha\} < \epsilon$.

Remark 2.7. In this subsection, we worked with L^1 norm of the Fourier transform of the function. Due to the Hausdorff-Young inequality, this "corresponds" to the L^∞ norm in the space variable. However, the other borderline case of Theorem 2.4 occurs when we have L^1 norm in the space variable. In the frequency space, this corresponds to the L^∞ norm (in the sense that $\|\mathcal{F}(\mathbf{u})\|_{L^\infty} \le \|\mathbf{u}\|_{L^1}$). We can obtain an analogue of Theorem 2.6 if instead of the L^1 norm, we let

$$|||\mathbf{u}||| = \sup_{\xi\in\mathbb{R}^d} (1+|\xi|)^{\alpha_0}|\mathbf{u}(\xi)|, \alpha_0 > d.$$

The conditions with $\max\{\|\mathbf{u}_0\|, \|\mathbf{u}_0\|_\alpha\}$ in Theorem 2.5 will simply be replaced with analogous conditions with $|||\mathbf{u}_0|||_{\alpha+\alpha_0}$. The proof of this fact is very similar to the proof of Theorem 2.6 once one notes that like L^1, the Gevrey space based on this norm is also a Banach algebra under convolution (see [4]).

Remark 2.8. **Exponential decay of Fourier coefficients:** In the space periodic setting (with period L), the operator Λ restricted to the subspace of functions with space average zero, has a discrete spectrum with its lowest eigenvalue being $\frac{2\pi}{L}$. Then, provided $\max\{\|\mathbf{u}_0\|, \|\mathbf{u}_0\|_\alpha\}$ in small enough (or in view of Remark 2.7, if $|||\mathbf{u}_0|||_{\alpha_0+\alpha}$ is small enough), we have the exponential decay of the Fourier coefficients

$$|\mathcal{F}(\mathbf{u})(\xi,t)| \le Ce^{-t^{1/\kappa}|\xi|}.$$

This is a generalization of the results in [16] and [13] for the special case of the 3d Navier-Stokes equations. In fact, for the 3d Navier-Stokes equations, following similar techniques as is presented here, one can sharpen this decay result (see [3]) by demanding only that

$$\sup_{k\in\mathbb{Z}^3_+} |k|^2|\mathcal{F}(\mathbf{u}_0)(k)| < \epsilon.$$

This fact was proven in [3] using similar methods, and independently in [42].

3. Applications: Decay of Sobolev Norms

Theorem 2.1 tells us that if the initial data is small in the corresponding critical space (i.e. in $\dot{\mathbb{H}}^{\beta_c}_p$), then the solution is globally in the Gevrey class. Due to Lemma 4.1 (or Lemma 4.2), this allows us to obtain the following time decay of (homogeneous) Sobolev norms as follows:

$$\begin{aligned}\|\Lambda^{\zeta}\mathbf{u}(t)\|_{L^p} &= \|\Lambda^{\zeta-\beta_c}e^{-ct^{1/\kappa}\Lambda_1}e^{ct^{1/\kappa}\Lambda_1}\Lambda^{\beta_c}\mathbf{u}(t)\|_{L^p}\\ &\le C_\zeta t^{-\frac{1}{\kappa}(\zeta-\beta_c)}\|\mathbf{u}(t)\|_{Gv(t,\beta_c,p)}, \quad (C_\zeta\sim C^\zeta\zeta^\zeta, \zeta>\beta_c).\end{aligned} \tag{3.1}$$

If we can show that a solution $\mathbf{u}(\cdot)$ of (1.1) satisfies

$$\liminf_{t\to\infty}\|\mathbf{u}(t)\|_{\dot{\mathbb{H}}^{\beta_c}_p}=0, \tag{3.2}$$

then due to Theorem 2.1, after a certain transient time t_0, we have

$$\sup_{t>t_0}\|\mathbf{u}(t)\|_{Gv(t-t_0,\beta_c,p)}<\infty.$$

Consequently, we obtain

$$\|\Lambda^{\zeta}\mathbf{u}(t)\|_{L^p}\le C_\zeta\|\mathbf{u}(t_0)\|_{\dot{\mathbb{H}}^{\beta_0}_p}(t-t_0)^{-\frac{1}{\kappa}(\zeta-\beta_c)}, \quad \zeta>\beta_c \tag{3.3}$$

where $\|\mathbf{u}(t_0)\|_{\dot{\mathbb{H}}^{\beta_c}_p}$ is sufficiently small to apply Theorem 2.1. In the next subsections, we will provide several examples where this can be achieved.

3.1. **Navier-Stokes equations.** The first application of Theorem 2.1 is the three dimensional Navier-Stokes equations, with $\beta_c=\frac{3}{p}-1$, $\max\left\{0,1-\frac{3}{2p}\right\}<\beta<1$. Compared with previous works by [38, 39, 37, 36, 40, 35] and others, where decay of L^2-based Sobolev norms have been achieved, we will provide decay of L^p-based Sobolev norms.

Theorem 3.1. *Let $1<p<\infty$ and $\mathbf{u}$ be a weak solution of the three dimensional Navier-Stokes equations where $\mathbf{u}_0\in L^2$. In case $1<p<2$, we additionally assume that $\omega_0=\nabla\times\mathbf{u}_0\in L^1$. Let $\beta_c=\frac{3}{p}-1$. We then have the following decay estimate:*

$$\|\Lambda^{\zeta}\mathbf{u}(t)\|_{L^p}\le C_\zeta\epsilon(t-t_0)^{-\frac{1}{2}(\zeta-\beta_c)}, \quad \zeta>\max\{0,\beta_c\}, \tag{3.4}$$

where $\epsilon=\|\mathbf{u}(t_0)\|_{\dot{\mathbb{H}}^{\beta_c}_p}$ is sufficiently small to apply Theorem 2.1, and $C_\zeta\sim C^\zeta\zeta^\zeta$ for $p=2$.

Remark 3.2. Existence of solutions to the NSE in Gevrey classes was first proven for the periodic boundary condition by Foias and Temam ([18]) (for initial data in $\mathbb{H}^1$) and subsequently by Oliver and Titi on the whole space, with initial data in $\mathbb{H}^s, s>n/2, n=2,3$ (see also [31] for initial data in $\dot{\mathbb{H}}^{1/2}$ for 3D NSE). By following a slightly different approach, namely interpolating the L^p norms of the solution and its analytic extension, Grujic and Kukavica ([24]) proved analyticity of solutions to the 3D NSE for initial data in $L^p, p>3$ (see also [32] for a different proof). Analyticity of solutions for initial data in homogeneous potential spaces $\dot{\mathbb{H}}^{\alpha}_p, 1<p<\infty, \alpha\ge\frac{3}{p}-1$, which includes the above mentioned L^p spaces, follows from Theorem 2.1.

The decay in L^2-based (homogeneous) Sobolev norms $\|\mathbf{u}\|_{\dot{\mathbb{H}}^\zeta}$ for the NSE were, to the best of our knowledge, first given in [37] and [40]. However, the constants C_ζ were not explicitly identified there. The sharp (and optimal, in the sense of providing lower bounds as well) decay results were provided by Oliver and Titi ([36]) following the Gevrey class approach. The constants C_ζ in (3.4) is of the same order as in [36]. Thus, (3.4) is a L^p version of the sharp decay result in [36]. In [37] and [36], there is also an assumption of the decay of the L^2 norm of the solution. This is circumvented here due to our working in the "critical" space $\dot{\mathbb{H}}^{1/2}$.

3.2. **Subcritical dissipative surface quasi-geostrophic equations.** These equations are given by

$$\begin{cases} \eta_t + u\cdot\nabla\eta + \Lambda^\kappa \eta = 0, & (x,t)\in\mathbb{R}^2\times(0,\infty), \\ \eta(x,0) = \eta_0(x), & x\in\mathbb{R}^2, \end{cases} \tag{3.5}$$

where,

$$u = (-\mathcal{R}_2\eta, \mathcal{R}_1\eta) := (-\partial_{x_2}\Lambda\eta, \partial_{x_1}\Lambda\eta).$$

We will consider the sub-critical case when the parameter κ satisfies $\kappa\in(1,2]$. For initial data in $L^p, p\ge \frac{2}{\kappa-1}$, the global existence of a unique, regular solution to (3.5) in the sub-critical case is known (see [7] and the references there in). The following theorem concerning the long time behavior of higher Sobolev norms was first given in [14]. The proof in [14] involves iterative estimation of higher order derivatives involving elaborate combinatorial arguments. We obtain this result as an application of Theorem 2.1.

Theorem 3.3. *Let $\kappa\in(1,2]$ in (3.5) and denote $p_0 = \frac{2}{\kappa-1}$. Let η be the unique regular solution to for initial data $\eta_0\in L^{p_0}(\mathbb{R}^2)$. The following estimate then holds with a constant*

$$\|\eta\|_{\dot{\mathbb{H}}^{\alpha}_{p_0}} \le \frac{C^\alpha \alpha^\alpha}{t^{\frac{\alpha}{\kappa}}} \text{ for all } t>0, \alpha>0, \tag{3.6}$$

where the constant C may depend on η_0, but is independent of t,α.

Remark 3.4. The higher order decay result in Theorem 3.3 was first proven in [14]. The approach there was an iterative estimation of higher order derivatives. The use of Gevrey class technique presented here eliminates the necessity of involved combinatorial arguments which is now encoded in the Gevrey norm.

3.3. **Burger type equation with higher order nonlinearity.** We now give an application to decay for a viscous Burger's equation on $\mathbb{R}$ of the form

$$\partial_t u - \Delta u = \partial_x(u^n), \quad x\in\mathbb{R}, \quad n\ge 3. \tag{3.7}$$

Here we take $p=2$ and, as is customary, we denote $\dot{\mathbb{H}}^\beta_2 = \dot{\mathbb{H}}^\beta, \beta\in\mathbb{R}, \beta\ne 0$ and $\dot{\mathbb{H}}^0 = L^2$. For $p=2, \kappa=2, \alpha_{T_0}=1$ and $\alpha_{T_i}=0, 1\le i\le n$, the critical space for (3.7) is $\dot{\mathbb{H}}^{\frac12-\frac{1}{n-1}}$, i.e., $\beta_c = \frac12 - \frac{1}{n-1}$.

Theorem 3.5. *Let $u_0\in\dot{\mathbb{H}}^{-1}\cap L^2$ for $n=3$ and $u_0\in L^2$ for $n\ge 4$. Then, for any weak solution u of (3.7), there exists $t_0>0$ such that*

$$\|\Lambda^\zeta u(t)\|_{L^2} \le C_\zeta \epsilon (t-t_0)^{-\frac12(\zeta-\beta_c)}, \quad \zeta>\beta_c := \frac12 - \frac{1}{n-1}.$$

Here, $C_\zeta\sim\zeta^\zeta C^\zeta$ and $\epsilon = \|u(t_0)\|_{\dot{\mathbb{H}}^{\beta_c}}$ is sufficiently small to apply Theorem 2.1.

Remark 3.6. In case $n=2$, the critical space is $\dot{\mathbb{H}}^{-\frac12}$. If

$$\liminf_{t\to\infty} \|u\|_{\dot{\mathbb{H}}^{-\frac12}} = 0, \tag{3.8}$$

then, decay of higher Sobolev norms hold.

3.4. **Cahn-Hilliard equation (special case).** The Cahn-Hilliard equation is given by

$$u_t = -\Delta^2 u - \alpha\Delta u + \beta\Delta(u^3), \quad x \in \mathbb{R}^d, \beta > 0, \alpha \geq 0. \tag{3.9}$$

Here we will consider the special case when $\alpha = 0$ and $\beta > 0$. Here, in the notation of Theorem 2.1, we have $n = 3, \kappa = 4, \alpha_{T_0} = 2, \alpha_{T_i} = 0, i = 1, 2, 3,$ from (2.6), $\beta_c := \frac{d}{2} - 1$. We have the following decay result.

Theorem 3.7. *Let u be a (weak) solution of (3.9)with initial data $u_0 \in L^2$ in space dimensions $d \geq 1$. Then, there exists $t_0 > 0$ such that*

$$\|\Lambda^\zeta \mathbf{u}(t)\|_{L^2} \leq C_\zeta \epsilon (t - t_0)^{-\frac{1}{2}(\zeta - \beta_c)}, \quad \zeta > \beta_c := \frac{d}{2} - 1.$$

Here, $C_\zeta \sim \zeta^\zeta C^\zeta$ when $p = 2$ and $\epsilon = \|u(t_0)\|_{\dot{\mathbb{H}}^{\beta_c}}$ is sufficiently small to apply Theorem 2.1.

The generalized Cahn-Hilliard equation has been studied in [44]. Due to Theorem 2.5, in the periodic case, we can improve the previous decay result to include the generalized Cahn-Hilliard equation.

Theorem 3.8. *Let $u : \mathbb{R}^d \to \mathbb{R}, d = 1, 2, 3$ be a space periodic solution of the general Cahn-Hilliard equation*

$$u_t - \Delta^2 u = \Delta f(u), \quad f(u) = \sum_{j=1}^{2N-1} a_j u^j \quad \text{and} \quad a_{2N-1} = A > 0.$$

Assume further that

$$\sum_{j=1}^{2N-2} j|a_j| < \delta, \tag{3.10}$$

for δ suitably small and let $u_0 \in \dot{\mathbb{H}}^\beta$ with $\frac{d}{2} < \beta < 2$. Then, any solution u satisfies the decay estimate

$$\|\Lambda^\zeta u(t)\|_{L^2} \leq C_\zeta \|u(t_0)\|_{\dot{\mathbb{H}}^\beta} (t - t_0)^{-\frac{\zeta - \beta}{\kappa}}, \quad \zeta > \beta,$$

where $\|u(t_0)\|_{\dot{\mathbb{H}}^\beta}$ is sufficiently small to apply Theorem 2.5.

Remark 3.9. The decay results presented in Theorems 3.5, 3.7 and 3.8 are new to the best of our knowledge.

4. Proofs of Main Results

4.1. **Degree n nonlinearity.** We will start with some preparatory results. Here we will follow the notation in Section 2.1. The following lemma is well-known. We provide a proof for completeness and to give precise estimates of the constants involved.

Lemma 4.1. *Let $\beta, t \geq 0$ and $\kappa > 0$. The Fourier multipliers corresponding to the symbols $m_1(\xi) = |\xi|^\beta e^{-t|\xi|_1^\kappa}$ and $m_2(\xi) = |\xi|^\beta e^{-t|\xi|^\kappa}$ are given by convolution with corresponding kernels k_1 and k_2 both of which are L^1 functions with*

$$\|k_i\|_{L^1} \leq \frac{2C^{\beta/\kappa}\beta^{\beta/\kappa}}{t^{\beta/\kappa}}, i = 1, 2,$$

where C is a constant independent of β and κ.

Proof. We only need to obtain the desired estimate for m_1 and m_2 with $t = 1$. Then, time dependent bounds can be obtain by the scaling: $\xi \mapsto t^{\frac{1}{\kappa}}\xi$. To estimate m_i at $t = 1$, we will use the Littlewood-Paley decomposition (see Appendix for the definition of this decomposition). Since the estimation is the same for m_1 and m_2, we only estimate m_2. We apply $\triangle_j$ to m_2 and take the L^1 norm.

$$\|\triangle_j m_2\|_{L^1} \le C2^{j\beta}\|\triangle_j e^{-|\xi|^\kappa}\|_{L^1} \le C2^{j\beta}e^{-2^{j\kappa}},$$

where the second inequality can be found in [27]. Since

$$\|m_2\|_{L^1} \le C\sum_{j\in Z}\|\triangle_j m_2\|_{L^1},$$

we have

$$\|m_2\|_{L^1} \le C\sum_{j\in Z} 2^{j\beta}e^{-2^{j\kappa}}. \tag{4.1}$$

The series $\sum_{j\in Z} 2^{j\beta}e^{-2^{j\kappa}}$ is readily seen to be bounded by $2C^{\beta/\kappa}\beta^{\beta/\kappa}$. □

From this lemma and the definitions of the operators T_i and the Gevrey norms, we immediately have the following estimates concerning the Gevrey norms.

Lemma 4.2. *Denote $C_\beta = C^\beta\beta^\beta$. For any $s, t, \beta' \ge 0$, we have*

$$\|T_i\mathbf{v}\|_{Gv(s,\beta,p)} \le C\|\mathbf{v}\|_{Gv(s,\beta+\alpha_{T_i},p)}, \quad \|e^{-\frac{1}{2}t\Lambda^k}\mathbf{v}\|_{Gv(s,\beta+\beta',p)} \le \frac{C_{\beta'}}{t^{\beta'/\kappa}}\|\mathbf{v}\|_{Gv(s,\beta,p)},$$

where α_{T_i} is as defined in (2.2).

We will need the following elementary inequality:

$$(x+y)^\gamma \le x^\gamma + y^\gamma, \quad x, y \ge 0, \quad \gamma \in [0,1]. \tag{4.2}$$

In the inequality above, by convention, we take $0^0 = 1$ when $\gamma = 0$ and $\min(x, y) = 0$. We also have the following result concerning the semigroup.

Lemma 4.3. *Let $0 \le s \le t < \infty$. Let*

$$E = e^{-c((t-s)^{1/\kappa}+s^{1/\kappa}-t^{1/\kappa})\Lambda_1}$$

The operator E is either the identity operator or is a Fourier multiplier with an L^1 kernel and its L^1 norm is bounded independent of s, t.

Proof. Note that since $\kappa \ge 1$, by (4.2) we have $t^{1/\kappa} \le s^{1/\kappa} + (t-s)^{1/\kappa}$. Thus, $E = e^{-a\Lambda_1}$ where $a = c\{(t-s)^{1/\kappa}+s^{1/\kappa}-t^{1/\kappa}\} \ge 0$. In case $a = 0$, E is the identity operator while if $a > 0$, E is a Fourier multiplier with symbol $m_E(\xi) = \prod_{i=1}^d e^{-a|\xi_i|}$. Thus, the kernel of E is given by the product of one dimensional Poisson kernels $\prod_{i=1}^d \frac{a}{\pi(a^2+x_i^2)}$. The L^1 norm of this kernel is bounded by a constant independent of a. □

We will also need the following lemma to proceed.

Lemma 4.4. *Let $\kappa \ge 1$. The operator $\tilde{E} = e^{-\frac{1}{2}a\Lambda^\kappa + a^{1/\kappa}c\Lambda_1}$ is a Fourier multiplier which acts as a bounded operator on all L^p spaces ($1 < p < \infty$) and its operator norm is uniformly bounded with respect to $a \ge 0$.*

Proof. When $a = 0$, $\tilde{E}$ is the identity operator. On the other hand, if $a > 0$, then $\tilde{E}$ is Fourier multiplier with symbol $m_{\tilde{E}}(\xi) = e^{-\frac{1}{2}|a^{1/\kappa}\xi|^\kappa + c|a^{1/\kappa}\xi|_1}$. Since $m_{\tilde{E}}(\xi)$ is uniformly bounded for all ξ the claim now follows from Hormander's multiplier theorem (see [43]). □

We also have the following lemma concerning the linear term.

Lemma 4.5. *Let $\mathbf{u}_0 \in \dot{\mathbb{H}}_p^{\beta_0}$ for some $\beta_0 \in \mathbb{R}$ and $0 < T \le \infty, \beta > 0$. Denote*

$$M = \sup_{0<t<T} t^{\beta/\kappa} \|e^{-t\Lambda^\kappa}\mathbf{u}_0\|_{Gv(t,\beta_0+\beta,p)}. \tag{4.3}$$

Then $M \le C_\beta \|\mathbf{u}_0\|_{\dot{\mathbb{H}}_p^{\beta_0}}$ and $M \to 0$ as $T \to 0$.

Proof. Due to Lemmas 4.2 and 4.4, for any $0 < t \le T$, we have

$$\begin{aligned}
\|e^{-t\Lambda^\kappa}\mathbf{u}_0\|_{Gv(t,\beta_0+\beta,p)} &= \|e^{ct^{1/\kappa}\Lambda_1 - \frac{1}{2}t\Lambda^\kappa}\Lambda^\beta e^{-\frac{1}{2}t\Lambda^\kappa}\Lambda^{\beta_0}\mathbf{u}_0\|_{L^p} \\
&\le C\|\Lambda^\beta e^{-\frac{1}{2}t\Lambda^\kappa}\Lambda^{\beta_0}\mathbf{u}_0\|_{L^p} \le C_\beta t^{-\beta/\kappa}\|\Lambda^{\beta_0}\mathbf{u}_0\|_{L^p} \\
&= C_\beta t^{-\beta/\kappa}\|\mathbf{u}_0\|_{\dot{\mathbb{H}}_p^{\beta_0}},
\end{aligned}$$

which proves the first assertion. Concerning the second, note that there exists $\mathbf{u}_0' \in \dot{\mathbb{H}}_p^{\beta+\beta_0}$ such that $\|\mathbf{u}_0 - \mathbf{u}_0'\|_{\dot{\mathbb{H}}_p^{\beta_0}} < \epsilon$ for any $\epsilon > 0$. Proceeding as above, and using the fact that $\mathbf{u}_0' \in \dot{\mathbb{H}}_p^{\beta+\beta_0}$, we obtain

$$\begin{aligned}
&t^{\beta/\kappa}\|e^{-t\Lambda^\kappa}\mathbf{u}_0\|_{Gv(t,\beta_0+\beta,p)} \\
&\le t^{\beta/\kappa}\|e^{-t\Lambda^\kappa}(\mathbf{u}_0 - \mathbf{u}_0')\|_{Gv(t,\beta_0+\beta,p)} + t^{\beta/\kappa}\|e^{-t\Lambda^\kappa}\mathbf{u}_0'\|_{Gv(t,\beta_0+\beta,p)} \\
&\le C\epsilon + Ct^{\beta/\kappa}\|\mathbf{u}_0'\|_{\dot{\mathbb{H}}_p^{\beta+\beta_0}}.
\end{aligned}$$

Letting $T \to 0$ and noting that $\epsilon > 0$ is arbitrary, the claim follows. □

We will need the following versions of the Kato-Ponce inequality ([28]). For $\Gamma = \Lambda$ or $\Gamma = (I + \Lambda)$, we have

$$\|\Gamma^\beta(\varphi\psi)\|_{L^p} \le C\left[\|\Gamma^\beta\varphi\|_{L^{p_1}}\|\psi\|_{L^{q_1}} + \|\varphi\|_{L^{p_2}}\|\Gamma^\beta\psi\|_{L^{q_2}}\right], \tag{4.4}$$

where $\beta \ge 0$ and $1 < p, p_i, q_i < \infty$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. For completeness, a proof is provided in the Appendix.

The following lemma, which shows that the Kato-Ponce inequality holds for Gevrey norms, is crucial to estimate the nonlinear term in (1.1).

Lemma 4.6. *Let $t, \beta \ge 0$ and $1 < p, p_i, q_i < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Then, we have the following estimate:*

$$\|fg\|_{Gv(t,\beta,p)} \le C\Big[\|f\|_{Gv(t,\beta,p_1)}\|g\|_{Gv(t,0,q_1)} + \|f\|_{Gv(t,0,p_2)}\|g\|_{Gv(t,\beta,q_2)}\Big].$$

Proof. For notational convenience, denote

$$a = ct^{1/\kappa} \text{ and } \varphi = e^{a\Lambda_1}f, \psi = e^{a\Lambda_1}g. \tag{4.5}$$

By a density argument, it will be enough to prove the result for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$. Note that, using the Fourier inversion formula, we have

$$\begin{aligned}
B_a(f,g) &:= e^{a\Lambda_1}(f \cdot g) = e^{a\Lambda_1}((e^{-a\Lambda_1}\varphi) \cdot (e^{-a\Lambda_1}\psi)) \\
&= \frac{1}{(2\pi)^d}\int\int e^{\imath x\cdot(\xi+\eta)}e^{a(\|\xi+\eta\|_1 - \|\xi\|_1 - \|\eta\|_1)}\hat{f}(\xi)\hat{g}(\eta)\,d\xi\,d\eta.
\end{aligned}$$

Recall that for a vector $\xi = (\xi_1, \cdots, \xi_d)$, we denoted $\|\xi\|_1 = \sum_{i=1}^d |\xi_i|$. For $\xi = (\xi_1, \cdots, \xi_d), \eta = (\eta_1, \cdots, \eta_d)$, we now split the domain of integration of the above integral into sub-domains depending on the sign of ξ_j, η_j and $\xi_j + \eta_j$. In order to do so, we introduce the operators acting on one variable (see page 253 in [31]) by

$$K_1 f = \frac{1}{2\pi} \int_0^\infty e^{\imath x\xi} \hat{f}(\xi)\, d\xi, \quad K_{-1} f = \frac{1}{2\pi} \int_{-\infty}^0 e^{\imath x\xi} \hat{f}(\xi)\, d\xi.$$

Let the operators $L_{a,-1}$ and $L_{a,1}$ be defined by

$$L_{a,1} f = f, \quad L_{a,-1} f = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\imath x\xi} e^{-2a|\xi|} \hat{f}(\xi)\, d\xi.$$

For $\vec{\alpha} = (\alpha_1, \cdots, \alpha_d), \vec{\beta} = (\beta_1, \cdots, \beta_d) \in \{-1, 1\}^d$, denote the operator

$$Z_{a,\vec{\alpha},\vec{\beta}} = K_{\beta_1} L_{t,\alpha_1\beta_1} \otimes \cdot \otimes K_{\beta_d} L_{t,\alpha_d\beta_d} \text{ and } K_{\vec{\alpha}} = k_{\alpha_1} \otimes \cdots \otimes K_{\alpha_d}.$$

The above tensor product means that the $j-$th operator in the tensor product acts on the $j-$th variable of the function $f(x_1, \cdots, x_d)$. A tedious (but elementary) calculation now yields the following identity:

$$B_a(f, g) = \sum_{\vec{\alpha},\vec{\beta},\vec{\gamma} \in \{-1,1\}^d} K_{\vec{\alpha}}((Z_{a,\vec{\alpha},\vec{\beta}} f) \cdot (Z_{a,\vec{\alpha},\vec{\gamma}} g)).$$

We now note that the operators $K_{\vec{\alpha}}, Z_{a,\vec{\alpha},\vec{\beta}}$ defined above, being linear combinations of Fourier multipliers (including Hilbert transform) and the identity operator, commute with Λ_1 and Λ. Moreover, they are bounded linear operators on $L^p, 1 < p < \infty$ and the corresponding operator norm of $Z_{a,\vec{\alpha},\vec{\beta}}$ is bounded independent of $a \geq 0$. Thus, we can write

$$\begin{aligned}
&\|fg\|_{Gv(t,\beta,p)} = \|\Lambda^\beta e^{a\Lambda_1}(fg)\|_{L^p} = \|\Lambda^\beta e^{a\Lambda_1}((e^{-a\Lambda_1}\varphi)(e^{-a\Lambda_1}\psi))\|_{L^p} \\
&= \Big\|\Lambda^\beta \Big[\sum_{\vec{\alpha},\vec{\beta},\vec{\gamma} \in \{-1,1\}^d} K_{\vec{\alpha}} \Big(\big(Z_{a,\vec{\alpha},\vec{\beta}}\varphi\big)\big(Z_{a,\vec{\alpha},\vec{\gamma}}\psi\big)\Big)\Big]\Big\|_{L^p} \\
&= \Big\|\Big[\sum_{\vec{\alpha},\vec{\beta},\vec{\gamma} \in \{-1,1\}^d} K_{\vec{\alpha}} \Lambda^\beta \Big(\big(Z_{a,\vec{\alpha},\vec{\beta}}\varphi\big)\big(Z_{a,\vec{\alpha},\vec{\gamma}}\psi\big)\Big)\Big]\Big\|_{L^p} \\
&\leq C \Big\|\Lambda^\beta \Big(\big(Z_{a,\vec{\alpha},\vec{\beta}}\varphi\big)\big(Z_{a,\vec{\alpha},\vec{\gamma}}\psi\big)\Big)\Big\|_{L^p} \\
&\leq C \Big[\big\|\Lambda^\beta Z_{a,\vec{\alpha},\vec{\beta}}\varphi\big\|_{L^{p_1}} \big\|Z_{a,\vec{\alpha},\vec{\gamma}}\psi)\big\|_{L^{q_1}} + \big\|Z_{a,\vec{\alpha},\vec{\beta}}\varphi\big\|_{L^{p_2}} \big\|\Lambda^\beta Z_{a,\vec{\alpha},\vec{\gamma}}\psi)\big\|_{L^{q_2}} \Big] \\
&\leq C \Big[\big\|Z_{a,\vec{\alpha},\vec{\beta}}\Lambda^\beta\varphi\big\|_{L^{p_1}} \big\|Z_{a,\vec{\alpha},\vec{\gamma}}\psi\big\|_{L^{q_1}} + \big\|Z_{a,\vec{\alpha},\vec{\beta}}\varphi\big\|_{L^{p_2}} \big\|Z_{a,\vec{\alpha},\vec{\gamma}}\Lambda^\beta\psi\big\|_{L^{q_2}} \Big] \\
&\leq C \Big[\|f\|_{Gv(t,\beta,p_1)} \|g\|_{Gv(t,0,q_1)} + \|f\|_{Gv(t,0,p_2)} \|g\|_{Gv(t,\beta,q_2)} \Big],
\end{aligned}$$

where to obtain the above relations, we used the commutativity of Λ^β with the operators $K_{\vec{\alpha}}, Z_{a,\vec{\alpha},\vec{\beta}}$, the L^p-boundedness of these operators (with uniformly bounded operator norms with respect to $a \geq 0$) and (4.4). □

We will also need the following lemma.

Lemma 4.7. *All functions are defined on $\mathbb{R}^d$. Let $s \geq 0$, $1 < p < \infty$ and $0 < \alpha, \beta < \frac{d}{p}$ and $\alpha + \beta > \frac{d}{p}$. We have*

$$\|fg\|_{Gv(s,\gamma,p)} \leq C \|f\|_{Gv(s,\alpha,p)} \|g\|_{Gv(s,\beta,p)}, \quad \gamma = \alpha + \beta - \frac{d}{p}. \tag{4.6}$$

Proof. Applying Lemma 4.6, we have

$$\|fg\|_{Gv(s,\gamma,p)} \le C(\|f\|_{Gv(s,\gamma,r_1)}\|g\|_{Gv(s,0,r_2)} + \|f\|_{Gv(s,0,r_3)}\|g\|_{Gv(s,\gamma,r_4)}), \tag{4.7}$$

where $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4} = \frac{1}{p}$. We now need the Sobolev inequalities ([31])

$$\|f\|_{L^q} \le C\|\Lambda^\delta f\|_{L^p}, \quad 0 \le \delta < \frac{d}{p}, \quad \delta = \frac{d}{p} - \frac{d}{q}. \tag{4.8}$$

By taking $r_i, i = 1, \cdots, 4$ in (4.7) with

$$\frac{1}{r_1} = \frac{1}{p} - \frac{\alpha-\gamma}{d}, \quad \frac{1}{r_2} = \frac{1}{p} - \frac{\beta}{d}, \quad \frac{1}{r_3} = \frac{1}{p} - \frac{\alpha}{d}, \quad \frac{1}{r_4} = \frac{1}{p} - \frac{\beta-\gamma}{d},$$

and applying the Sobolev inequalities given above, we obtain (4.6). □

An iterative application of this lemma yields the following:

$$\Big\|\prod_{i=1}^n f_i\Big\|_{Gv(s,\gamma,p)} \le C\prod_{i=1}^n \|f_i\|_{Gv(s,\alpha_i,p)} \tag{4.9}$$

provided

$$\gamma = \sum_{i=1}^n \alpha_i - \frac{(n-1)d}{p} > 0, \quad \max_{1\le i\le n}\{\alpha_i\} < \frac{d}{p}.$$

Proof of Theorem 2.1. Let $\gamma = \beta_0 + \beta$ for adequate $\beta > 0$ to be specified later and define the Banach space

$$E = \left\{\mathbf{u} \in C([0,T); \dot{\mathbb{H}}_p^{\beta_0}) : \|\mathbf{u}(\cdot)\|_E := \max\{\|\mathbf{u}(\cdot)\|_{E_1}, \|\mathbf{u}(\cdot)\|_{E_2}\} < \infty\right\},$$

where

$$\|\mathbf{u}(\cdot)\|_{E_1} := \sup_{s\in(0,T)} \|\mathbf{u}(s)\|_{Gv(s,\beta_0,p)}, \quad \|\mathbf{u}(\cdot)\|_{E_2} := s^{\beta/\kappa}\|\mathbf{u}(s)\|_{Gv(s,\gamma,p)}.$$

Additionally,

$$\mathcal{E} = \{\mathbf{u} \in E : \|\mathbf{u} - e^{-t\Lambda^\kappa}\mathbf{u}_0\|_E \le M\},$$

where M is as in (4.3). It is easy to see that $\mathcal{E}$ is a compete metric space (it is a closed, bounded subset of the Banach space E). For S as in (2.1), note that $(S\mathbf{u})(t) = e^{-t\Lambda^\kappa}\mathbf{u}_0 + (B\mathbf{u})(t)$ where

$$(B\mathbf{u})(t) = \int_0^t e^{-(t-s)\Lambda^\kappa} G(\mathbf{u}(s))\, ds. \tag{4.10}$$

We will prove an inequality of the form

$$\|(B\mathbf{u})\|_E \le CT^\mu \|\mathbf{u}\|_{E_2}^n \tag{4.11}$$

for adequate $\mu \ge 0$ satisfying $\mu = 0$ if and only if $\beta_0 = \beta_c$. Note that this implies

$$\|S\mathbf{u} - e^{-t\Lambda^\kappa}\mathbf{u}_0\|_E \le CT^\mu M^n, \quad \|S\mathbf{u} - S\mathbf{v}\|_E \le nCT^\mu M^{n-1}\|\mathbf{u}-\mathbf{v}\|_E.$$

The first inequality immediately follows from (4.11) while the second follows by noting

$$G(\mathbf{u}) - G(\mathbf{v}) = \sum_{i=1}^n T_0 F(T_1\mathbf{u}, \cdots, T_{i-1}\mathbf{u}, T_i(\mathbf{u}-\mathbf{v}), T_{i+1}\mathbf{v}, \cdots, T_n(\mathbf{v})). \tag{4.12}$$

If $\beta_0 > \beta_c$, then $\mu > 0$ and S is a contractive self map of $\mathcal{E}$ provided T is sufficiently small. On the other hand, in case $\beta_0 = \beta_c$ and $\mu = 0$, in view of Lemma 4.5, we can either choose T small in case the initial data is arbitrary or the initial data sufficiently small if $T = \infty$, to reach the same conclusion.

We now proceed to estimate $\|B\mathbf{u}\|_{E_2}$, the estimate for $\|B\mathbf{u}\|_{E_1}$ being similar. Note that

$$\begin{aligned}
&\|e^{-(t-s)\Lambda^\kappa} G(\mathbf{u}(s))\|_{Gv(t,\gamma,p)} \\
&= \|e^{ct^{1/\kappa}\Lambda_1}\Lambda^\gamma e^{-(t-s)\Lambda^\kappa} G(\mathbf{u}(s))\|_{L^p} \\
&\le C\|e^{c(t-s)^{1/\kappa}\Lambda_1} e^{cs^{1/\kappa}\Lambda_1} e^{-(t-s)\Lambda^\kappa}\Lambda^\gamma G(\mathbf{u}(s))\|_{L^p} \qquad (4.13)\\
&= \|e^{c(t-s)^{1/\kappa}\Lambda_1 - \frac{1}{2}(t-s)\Lambda^\kappa} e^{cs^{1/\kappa}\Lambda_1}\Lambda^\gamma e^{-\frac{1}{2}(t-s)\Lambda^\kappa} G(\mathbf{u}(s))\|_{L^p} \\
&\le C\|\Lambda^\gamma e^{cs^{1/\kappa}\Lambda_1} e^{-\frac{1}{2}(t-s)\Lambda^\kappa} G(\mathbf{u}(s))\|_{L^p}.
\end{aligned}$$

By (4.9) with $\alpha_i = \gamma - \alpha_{T_i}$ and Lemma 4.2 with $\beta' = \sum_{i=0}^n \alpha_{T_i} + (n-1)\frac{d}{p} - (n-1)\gamma$, we have

$$\begin{aligned}
&t^{\beta/\kappa}\|(B\mathbf{u})(t)\|_{Gv(t,\gamma,p)} \\
&\le C\|\mathbf{u}\|_{E_2}^n t^{\beta/\kappa}\int_0^t \frac{1}{s^{(n\beta)/\kappa}} \frac{1}{(t-s)^{\frac{1}{\kappa}[\sum_{i=0}^n \alpha_{T_i} + (n-1)\frac{d}{p} - (n-1)\gamma]}} ds \qquad (4.14)\\
&\le C\|\mathbf{u}\|_{E_2}^n t^\mu,
\end{aligned}$$

where

$$\mu = \frac{(n-1)(\beta_0 - \beta_c)}{\kappa}.$$

In order to apply (4.9) and Lemma 4.2, and to ensure the finiteness of the integral in (4.14), we need

$$\beta' = \sum_{i=0}^n \alpha_{T_i} + (n-1)\Big(\frac{d}{p} - \gamma\Big) \ge 0, \quad \gamma - \min_{1\le i\le n}\alpha_{T_i} < \frac{d}{p}, \quad n\gamma - \sum_{i=1}^n \alpha_{T_i} - (n-1)\frac{d}{p} > 0.$$

For the convergence of the integral in (4.14), we also need

$$n\beta < \kappa, \quad \beta' < \kappa.$$

A choice of $\beta > 0$ satisfying all these conditions can be made provided the conditions on the parameters stated in the theorem hold.

4.2. **Analytic Nonlinearity.** Here we follow the notation in subsection .. We will once again start with some auxiliary results.

Lemma 4.8. *Let $\beta > \frac{d}{p}$. For any $t \ge 0$, we have*

$$\|fg\|_{Gv(t,\beta,p)} \le C\|f\|_{Gv(t,\beta,p)}\|g\|_{Gv(t,\beta,p)}.$$

Proof. For notational convenience, denote $a = ct^{1/\kappa}$ and $\varphi = e^{a\Lambda_1}f, \psi = e^{a\Lambda_1}g$. Proceeding as in the proof of Lemma 4.6 (with the notation there in) and using (2.12), we obtain

$$\begin{aligned}
&\|fg\|_{Gv(t,\beta,p)} \\
&\le C\left[\left\|Z_{a,\vec\alpha,\vec\beta}(I+\Lambda)^\beta\varphi\right\|_{L^p}\left\|Z_{a,\vec\alpha,\vec\gamma}\psi\right\|_{L^\infty} + \left\|Z_{a,\vec\alpha,\vec\beta}\varphi\right\|_{L^\infty}\left\|Z_{a,\vec\alpha,\vec\gamma}(I+\Lambda)^\beta\psi\right\|_{L^p}\right] \\
&\le C\left[\left\|(I+\Lambda)^\beta\varphi\right\|_{L^p}\left\|(I+\Lambda)^\beta Z_{a,\vec\alpha,\vec\gamma}\psi\right\|_{L^p} + \left\|(I+\Lambda)^\beta Z_{a,\vec\alpha,\vec\beta}\varphi\right\|_{L^p}\left\|(I+\Lambda)^\beta\psi\right\|_{L^p}\right] \\
&\le C\|\varphi\|_{\mathbb{H}_p^\beta}\|\psi\|_{\mathbb{H}_p^\beta} \le C\|f\|_{Gv(t,\beta,p)}\|g\|_{Gv(t,\beta,p)},
\end{aligned}$$

where we have used the fact that $(I+\Lambda)$ commutes with $Z_{a,\vec{\alpha},\vec{\beta}}$. □

Lemma 4.9. *Let $\beta > \frac{d}{p}, s \geq 0$ and $\mathbf{u}, \mathbf{v}$ be such that*

$$\max_{1\leq i\leq n} \{\|T_i\mathbf{u}\|_{Gv(s,\beta,p)}, \|T_i\mathbf{v}\|_{Gv(s,\beta,p)}\} \leq R.$$

For notational simplicity, denote $F(T_1\mathbf{u}, \cdots, T_n\mathbf{u}) = F(\mathbf{u})$. There exists a constant C independent of $s, \mathbf{u}, \mathbf{v}$ such that

$$\begin{aligned} &\|F(\mathbf{u})\|_{Gv(s,\beta,p)} \leq F_M(RC), \\ &\|F(\mathbf{u}) - F(\mathbf{v})\|_{Gv(s,\beta,p)} \leq CF'_M(RC) \max_{1\leq i\leq n} \|T_i(\mathbf{u}-\mathbf{v})\|_{Gv(s,\beta,p)}. \end{aligned} \tag{4.15}$$

Proof. The first inequality in (4.15) is an immediate consequence of Lemma 4.8. Concerning the second, proceeding as in (4.12), we have

$$\begin{aligned} \|F(\mathbf{u}) - F(\mathbf{v})\|_{Gv(s,\beta,p)} &\leq \sum_{j\in\mathbb{Z}^d_+} |a_j||j|C^{|j|}R^{|j|-1} \max_{1\leq i\leq n} \|T_i(\mathbf{u}-\mathbf{v})\|_{Gv(s,\beta,p)} \\ &\leq CF'_M(RC) \max_{1\leq i\leq n} \|T_i(\mathbf{u}-\mathbf{v})\|_{Gv(s,\beta,p)}. \end{aligned}$$

This completes the proof. □

Proof of Theorem 2.4. We recall $\alpha = \max_{1\leq i\leq n} \{\alpha_{T_i}\}$, and set

$$E = \left\{ \mathbf{u} \in C\big((0,T); \mathbb{H}^{\beta_0+\alpha}\big) : \|\mathbf{u}(\cdot)\|_E = \sup_{s\in(0,T)} \|\mathbf{u}(s)\|_{Gv(s,\alpha,p)} \leq 2R \right\}$$

Proceeding as in the proof of Theorem 2.1 and using Lemma 4.9, we obtain

$$\|e^{-(t-s)\Lambda^\kappa} G(\mathbf{u}(s))\|_{Gv(t,\alpha,p)} \leq \frac{C}{(t-s)^{(\alpha_{T_0}+\alpha)/\kappa}} F_M(2RC). \tag{4.16}$$

Thus,

$$\Big\| \int_0^t e^{-(t-s)\Lambda^\kappa} G(\mathbf{u}(s))\, ds \Big\|_E \leq CF_M(2RC) T^{1-\frac{\alpha_{T_0}+\alpha}{\kappa}}.$$

Similarly, using the second inequality in (4.15), we can also obtain

$$\|S\mathbf{u} - S\mathbf{v}\|_E \leq CF'_M(2RC) T^{1-\frac{\alpha_{T_0}+\alpha}{\kappa}} \|\mathbf{u}-\mathbf{v}\|_E.$$

The proof is now completed along the lines of the proof of Theorem 2.1.

4.3. **The periodic case.** In this case, Λ has a minimum eigenvalue, denoted by $\lambda_0 > 0$, and the Fourier spectrum of all periodic functions with space average zero is contained in the complement of a ball with radius λ_0. Thus, following [11, 27], we can show that

$$\|e^{-a\Lambda^\kappa} \mathbf{u}_0\|_{L^p} \leq Ce^{-a\lambda_0^\kappa} \|\mathbf{u}_0\|_p, \quad a > 0.$$

This fact can be easily proven for $p = 2$ using Plancheral theorem. Thus, instead of (4.16), we can obtain

$$\begin{aligned} \|e^{-(t-s)\Lambda^\kappa} G(\mathbf{u}(s))\|_{Gv(t,\beta+\alpha,p)} &= \|e^{-\frac{1}{4}(t-s)\Lambda^\kappa} e^{-\frac{3}{4}(t-s)\Lambda^\kappa} G(\mathbf{u}(s))\|_{Gv(t,\beta+\alpha,p)} \\ &\leq e^{-\frac{1}{2}\lambda_0^\kappa(t-s)} \|e^{-\frac{3}{4}(t-s)\Lambda^\kappa} G(\mathbf{u}(s))\|_{Gv(t,\beta+\alpha,p)} \\ &\leq \frac{Ce^{-b(t-s)}}{(t-s)^{(\alpha_{T_0}+\alpha)/\kappa}} F_M(2RC), \end{aligned} \tag{4.17}$$

where $b = \frac{1}{2}\lambda_0^\kappa$. Using now the elementary fact (see for instance Proposition 7.5 in [3])

$$\sup_{t>0} \int_0^t e^{-b(t-s)} s^{-a} ds \le C < \infty, \quad b > 0, \quad 0 < a < 1,$$

it follows that

$$\|S\mathbf{u}\|_E \le CF_M(2RC) \quad \text{and} \quad \|S\mathbf{u} - S\mathbf{v}\|_E \le CF_M'(2RC)\|\mathbf{u} - \mathbf{v}\|_E.$$

The proof of Theorem 2.5 follows immediately from the above discussion and by noting that

$$\lim_{R\to 0} F_M(2RC) = 0, \quad \limsup_{R\to 0} F_M'(2RC) \le \delta.$$

We can thus ensure that S is a contractive self map for $T = \infty$ provided $\|\mathbf{u}_0\|_{\dot{\mathbb{H}}_p^{\beta+\alpha}}$ and δ are small.

4.4. **Equation in Fourier space.** As before, we start with some estimates on Gevrey norms.

Lemma 4.10. *For any $s \ge 0$ and $\beta = 0$, we have*

$$\|\mathbf{u} * \mathbf{v}\|_{Gv(s)} \le C\|\mathbf{u}\|_{Gv(s)}\|\mathbf{v}\|_{Gv(s)}.$$

Proof. Let $\gamma = \frac{1}{2}s^{1/\kappa}$. By triangle inequality $|\xi| \le |\xi - \eta| + |\eta|$,

$$\begin{aligned}\|\mathbf{u} * \mathbf{v}\|_{Gv(s)} &\le C \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_1}} e^{\gamma|\xi|}|\mathbf{u}|(\xi - \eta)|\mathbf{v}|(\eta) d\xi d\eta \\ &\le C \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_1}} e^{\gamma|\xi-\eta|}|\mathbf{u}|(\xi - \eta)e^{\gamma|\eta|}|\mathbf{v}|(\eta) d\xi d\eta \le C\|\mathbf{u}\|_{Gv(s)}\|\mathbf{v}\|_{Gv(s)},\end{aligned}$$

which completes the proof. □

Lemma 4.10 tells that for any $s \ge 0$, $Gv(s)$ is a Banach algebra. Therefore, this space can be used to estimate analytic functions $G(\mathbf{u})$ as follows.

Lemma 4.11. *Let G be as in (2.11) and $\mathbf{u}, \mathbf{v}$ be such that*

$$\max_{1\le i\le n} \{\|T_i\mathbf{u}\|_{Gv(s,\beta,p)}, \|T_i\mathbf{v}\|_{Gv(s,\beta,p)}\} \le R.$$

For notational simplicity, denote $F(T_1\mathbf{u}, \cdots, T_n\mathbf{u}) = F(\mathbf{u})$. There exists a constant C independent of $s, \mathbf{u}, \mathbf{v}$ such that

$$\begin{aligned}&\|F(\mathbf{u})\|_{Gv(s)} \le C_1 F_M(RC), \\ &\|F(\mathbf{u}) - F(\mathbf{v})\|_{Gv(s)} \le CF_M'(RC) \max_{1\le i\le n} \|T_i(\mathbf{u} - \mathbf{v})\|_{Gv(s)}.\end{aligned} \tag{4.18}$$

Proof. The first inequality in (4.18) is an immediate consequence of Lemma 4.10. Concerning the second, we have

$$\begin{aligned}F(\mathbf{u}) - F(\mathbf{v}) &= \sum_{j\in\mathbb{Z}_+^d} a_j(\mathbf{u}^{*j} - \mathbf{v}^{*j}) \\ &= \sum_{j\in\mathbb{Z}_+^d} a_j \sum_{k=0}^{|j|-1} \big[\mathbf{u}^{*(|j|-|k|)} * \mathbf{v}^{*|k|} - \mathbf{u}^{*(|j|-|k|-1)} * \mathbf{v}^{*(|k|+1)}\big].\end{aligned}$$

Applying now Lemma 4.10, we readily obtain

$$\begin{aligned}\|F(\mathbf{u})-F(\mathbf{v})\|_{Gv(s)} &\le \sum_{j\ge 1}|a_j|jC^jR^{j-1}\max_{1\le i\le n}\|T_i(\mathbf{u}-\mathbf{v})\|_{Gv(s)}\\ &\le CF_M'(RC)\max_{1\le i\le n}\|T_i(\mathbf{u}-\mathbf{v})\|_{Gv(s)}.\end{aligned}$$

This completes the proof. □

Lemma 4.12. *From the assumptions on T_i, for $s,t,\beta\ge 0$,*

$$\|T_i\mathbf{v}\|_{Gv(s)}\le \|\mathbf{v}\|_{Gv(s,\alpha_{T_i})},\quad \|T_ie^{-\frac12 t\mathfrak{D}}\mathbf{v}\|_{Gv(s)}\le \frac{C_{\alpha_{T_i}}}{t^{\alpha_{T_i}/\kappa}}\|\mathbf{v}\|_{Gv(s)}. \tag{4.19}$$

Proof. By definition of the Gevery norm, the first term in (4.19) can be obtained by

$$\|T_i\mathbf{v}\|_{Gv(s)}\le \int_{\mathbb{R}^{d_1}}|\xi|^{\alpha_{T_i}}e^{\frac12 s^{\frac1\kappa}|\xi|}|\mathbf{v}(\xi)|d\xi=\|\mathbf{v}\|_{Gv(s,\alpha_{T_i})}.$$

Since for all $\alpha\ge 0$

$$|\xi|^\alpha e^{-t|\xi|^\kappa}\le C_\alpha t^{-\frac{\alpha}{\kappa}},\qquad \text{where C is independent of}\quad \xi,t,\alpha,$$

we can prove the second term in (4.20) by

$$\begin{aligned}\|T_ie^{-\frac12 t\mathfrak{D}}\mathbf{v}\|_{Gv(s)} &\le \int_{\mathbb{R}^{d_1}}|\xi|^{\alpha_{T_i}}e^{-\frac12|\xi|^\kappa t}e^{\frac12 s^{\frac1\kappa}|\xi|}|\mathbf{v}(\xi)|d\xi\\ &\le \frac{C_{\alpha_{T_i}}}{t^{\alpha_{T_i}/\kappa}}\int_{\mathbb{R}^{d_1}}e^{\frac12 s^{\frac1\kappa}|\xi|}|\mathbf{v}(\xi)|d\xi=\frac{C}{t^{\alpha_{T_i}/\kappa}}\|\mathbf{v}\|_{Gv(s)}.\end{aligned}$$

which completes the proof. □

Proof of Theorem 2.6 We set

$$E=\left\{\mathbf{u}\in C\big((0,T);\mathbb{V}\big):\|\mathbf{u}(\cdot)\|_E=\sup_{s\in(0,T)}\max\Big\{\|\mathbf{u}(s)\|_{Gv(s)},\ \|\mathbf{u}(s)\|_{Gv(s,\alpha)}\Big\}\le R\right\}$$

For $\mathbf{u}\in E$, we define

$$(S\mathbf{u})(t)=e^{-t\mathfrak{D}}\mathbf{u}_0+\int_0^t e^{-(t-s)\mathfrak{D}}G(\mathbf{u}(s))ds. \tag{4.20}$$

Note that since $\kappa\ge 1$, we have $t^{1/\kappa}\le s^{1/\kappa}+(t-s)^{1/\kappa}$. Consequently,

$$\begin{aligned}&\|e^{-(t-s)\mathfrak{D}}G(\mathbf{u}(s))\|_{Gv(t)}\\ &\le \int_{\mathbb{R}^{d_1}}e^{\frac12 t^{1/\kappa}|\xi|}e^{-(t-s)|\xi|^\kappa}\big|(T_0F(\mathbf{u}(s)))(\xi)\big|d\xi\\ &\le C\int_{\mathbb{R}^{d_1}}e^{\frac12(t-s)^{1/\kappa}|\xi|}e^{-\frac12(t-s)|\xi|^\kappa}e^{\frac12 s^{1/\kappa}}e^{-\frac{(t-s)}{2}|\xi|^\kappa}\big|(T_0F(\mathbf{u}(s)))(\xi)\big|d\xi\\ &\le C\int_{\mathbb{R}^{d_1}}e^{\frac12 s^{1/\kappa}|\xi|}e^{-\frac{(t-s)}{2}|\xi|^\kappa}\big|(T_0F(\mathbf{u}(s)))(\xi)\big|d\xi,\end{aligned} \tag{4.21}$$

where we use the fact that

$$e^{\frac12\tau^{1/\kappa}|\xi|}e^{-\frac{\tau}{2}|\xi|^\kappa}=e^{\frac12|\tau^{1/\kappa}\xi|}e^{-\frac12|\tau^{1/\kappa}\xi|^\kappa}\le C,\quad (\tau\ge 0,\xi\in\mathbb{R}^d), \tag{4.22}$$

with the constant C independent of τ, ξ. Therefore,

$$\begin{aligned}\|e^{-(t-s)\mathfrak{D}}G(\mathbf{u}(s))\|_{Gv(t)} &\le \frac{C_{\alpha_{T_0}}}{(t-s)^{\alpha_{T_0}/\kappa}}\|F(\mathbf{u})\|_{Gv(t)} \\ &\le \frac{C_{\alpha_{T_0}}}{(t-s)^{\alpha_{T_0}/\kappa}}C_1F_M(RC),\end{aligned} \tag{4.23}$$

where we used (4.19) to obtain the first inequality, and we used the first inequality in (4.18) to obtain the second inequality in (4.23). Similarly, one can obtain also the estimate

$$\|e^{-(t-s)\mathfrak{D}}G(\mathbf{u}(s))\|_{Gv(t,\alpha)} \le \frac{C_{\alpha_{T_0}+\alpha}}{(t-s)^{(\alpha_{T_0}+\alpha)/\kappa}}C_1F_M(RC). \tag{4.24}$$

Thus,

$$\begin{aligned}&\Big\|\int_0^t e^{-(t-s)\mathfrak{D}}G(\mathbf{u}(s))\,ds\Big\|_E \\ &\le C_{\alpha_{T_0}+\alpha}F_M(RC)\max\Big\{T^{1-\frac{\alpha_{T_0}+\alpha}{\kappa}}, T^{1-\frac{\alpha_{T_0}}{\kappa}}\Big\}.\end{aligned} \tag{4.25}$$

Proceeding along similar lines but using the second inequality in (4.18) in the last step, we can also obtain

$$\|S\mathbf{u}_1 - S\mathbf{u}_2\|_E \le F_M^{'}(RC)\max\Big\{T^{1-\frac{\alpha_{T_0}+\alpha}{\kappa}}, T^{1-\frac{\alpha_{T_0}}{\kappa}}\Big\}\|\mathbf{u}_1-\mathbf{u}_2\|_E. \tag{4.26}$$

Concerning the linear term, using (4.22), it is easy to see that

$$\max\Big\{\|e^{-t\mathfrak{D}}\mathbf{u}_0\|_{Gv(t)}, \|e^{-t\mathfrak{D}}\mathbf{u}_0\|_{Gv(t,\alpha)}\Big\} \le \tilde{C}\max\left\{\|\mathbf{u}_0\|, \|\mathbf{u}_0\|_\alpha\right\} \le \frac{R}{2}. \tag{4.27}$$

From (4.25), (4.26) and (4.27), it follows that S is a (strictly) contractive self map of E provided T is suitably small. Thus we can find a fixed point.

The adjustment in the above argument necessary to obtain the global result in the periodic setting is similar to Subsection 4.3 above.

5. Proof of Applications

We will now provide the proof of the decay results stated in Section 3. We will follow the notation in Section 3.

5.1. **Proof of Theorem 3.1.** It is enough to prove (3.2). For $\mathbf{u}_0 \in L^2$, we have the following energy estimate

$$\|\mathbf{u}(t)\|_{L^2}^2 + \int_0^t \|\mathbf{u}(s)\|_{\dot{\mathbb{H}}^1}^2\,ds \le \|\mathbf{u}_0\|_{L^2}^2.$$

This implies that

$$\sup_{t>0}\|u(t)\|_{L^2}^2 \le \|u_0\|_{L^2}^2, \quad \liminf_{t\to\infty}\|u(t)\|_{\dot{\mathbb{H}}^1} = 0. \tag{5.1}$$

In order to obtain the second relation in (5.1), for $\epsilon > 0$ arbitrary, choose t large so that $\frac{1}{t}\|u_0\|_{L^2}^2 < \epsilon/4$. We note that the energy inequality yields $\frac{1}{t}\int_0^t \|u(s)\|_{\dot{\mathbb{H}}^1}^2\,ds \le \frac{1}{t}\|u_0\|_{L^2}^2$. This immediately implies that there exists $t_0 \in (0,t)$ such that $\|u(t_0)\|_{\dot{\mathbb{H}}^1}^2 < \epsilon$. Recall now the interpolation inequality

$$\|\mathbf{u}\|_{\dot{\mathbb{H}}^\beta} \le \|\mathbf{u}\|_{\dot{\mathbb{H}}^{\beta_1}}^\theta \|\mathbf{u}\|_{\dot{\mathbb{H}}^{\beta_2}}^{1-\theta}, \quad \beta = \theta\beta_1 + (1-\theta)\beta_2, \quad \theta\in(0,1), \quad \beta_i \in \mathbb{R}, i = 1,2.$$

Due the uniform bound on $\|u(t)\|_{L^2}^2$, it also follows that $\liminf_{t\to\infty} \|u(t)\|_{\dot{\mathbb{H}}^\beta} = 0$ for $0 < \beta \le 1$.

When $p = 2$, we have $\beta_c = \frac{1}{2}$ and consequently (3.2) holds, which implies (3.4).

For $p > 2$, (3.4) follows from a direct application of the Sobolev inequalities (4.8).

We will now focus on the case $1 < p < 2$. We will use the the vorticity $\omega = \nabla \times \mathbf{u}$. Note first that

$$\liminf_{t\to\infty} \|\omega\|_{L^2}^2 = \liminf_{t\to\infty} \|\nabla \mathbf{u}\|_{L^2}^2 = \liminf_{t\to\infty} \|\mathbf{u}\|_{\dot{\mathbb{H}}^1}^2 = 0.$$

From the vorticity equation, $\omega_t + u \cdot \nabla\omega - \Delta\omega = \omega\nabla u$, we have the uniform L^1 bound (see [9])

$$\|\omega(t)\|_{L^1} \le C(\|\mathbf{u}_0\|_{L^2}^2 + \|\omega_0\|_{L^1}). \tag{5.2}$$

The uniform L^1 bound and the interpolation inequality

$$\|\omega\|_{L^q} \le \|\omega\|_{L^1}^\theta \|\omega\|_{L^2}^{(1-\theta)}, \quad 1/q = \theta + (1-\theta)/2$$

implies

$$\liminf_{t\to\infty} \|\omega\|_{L^q} = 0, \quad 1 < q \le 2. \tag{5.3}$$

Recall that $\nabla \mathbf{u} = \nabla(\nabla \times (\Delta)^{-1}\omega)$. The operator $\nabla(\nabla \times (\Delta)^{-1})$ is a Fourier multiplier of homogeneous degree zero, and consequently, by the Calderon-Zygmund theorem, we have

$$\|\omega\|_{\dot{\mathbb{H}}_q^\zeta} = \|\Lambda^\zeta \omega\|_{L^q} \sim \|\Lambda^\zeta \nabla \mathbf{u}\|_{L^q} \sim \|\mathbf{u}\|_{\dot{\mathbb{H}}_q^{1+\zeta}}, \quad 1 < q < \infty, \quad \zeta \ge 0. \tag{5.4}$$

Consequently,

$$\liminf_{t\to\infty} \|\mathbf{u}\|_{\dot{\mathbb{H}}_q^1} = 0, \quad 1 < q < 2. \tag{5.5}$$

For $p = 3/2, \beta_c = 1$ and (3.2) follows directly from (5.5). For $3/2 < p < 2$ and β_c as above, applying (4.8), we have $\|\mathbf{u}\|_{\dot{\mathbb{H}}_p^{\beta_c}} \le \|\mathbf{u}\|_{\dot{\mathbb{H}}_{3/2}^1}$ and once again (3.2) follows from (5.5).

We now consider the case $1 < p < 3/2$. From (5.4) and (3.4) with $p = 2$ (which we already established), it follows that

$$\lim_{t\to\infty} \|\omega\|_{\dot{\mathbb{H}}^\zeta} \le \lim_{t\to\infty} \|\mathbf{u}\|_{\dot{\mathbb{H}}^{1+\zeta}} = 0, \quad \zeta \ge 0. \tag{5.6}$$

Recall now the generalization of the Gagliardo-Nirenberg inequalities for the fractional (homogeneous) Sobolev spaces (see [34]), namely,

$$\|\omega\|_{\dot{\mathbb{H}}_p^\alpha} \le C\|\omega\|_{\dot{\mathbb{H}}_{q_1}^{\alpha_1}}^\theta \|\omega\|_{L^{q_2}}^{1-\theta}, \quad \theta \in (0,1), \ \alpha = \theta\alpha_1, \quad \frac{1}{p} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}. \tag{5.7}$$

Combining (5.7) with (5.3) and (5.6) we have

$$\liminf_{t\to\infty} \|\omega\|_{\dot{\mathbb{H}}_p^\beta} = 0, \quad \beta > 0, \quad 1 < p < 2.$$

Consequently, by (5.4), we also have

$$\liminf_{t\to\infty} \|\mathbf{u}\|_{\dot{\mathbb{H}}_p^{1+\beta}} = 0, \quad 1 < p < 3/2, \quad \beta > 0.$$

Noting that for $1 < p < 3/2$, we have $\beta_c > 1$. By taking β such that $1 + \beta = \beta_c$, (3.2) follows.

5.2. **Proof of Theorem 3.3.** For notational simplicity, we will write $\|\cdot\|_{Gv(t,0,p_0)} = \|\cdot\|_{Gv(t)}$. Note that in the notation of Theorem 2.1, we have $T_0 = \nabla, T_1 = \mathcal{R}, T_2 = I$. It is known that solution to (3.5) satisfies (see [10])

$$\lim_{t\to\infty} \|\eta\|_{L^p} = 0, 1 \leq p \leq \infty.$$

We will apply the local existence part of Theorem 2.1 with $d = 2, n = 2, \alpha_{T_0} = 1, \alpha_{T_1} = \alpha_{T_2} = 0, p_0 = \frac{2}{\kappa-1}, \beta_c = 0$. Let t_1 be such that $\|\eta(t_1)\|_{L^{p_0}} < \epsilon$ where ϵ is as in Theorem 2.1. Applying the global existence part of this theorem, we have

$$\sup_{t>t_1} \|\eta(t)\|_{Gv(t-t_1)} < \infty.$$

From (3.1), we obtain

$$\|\eta(t)\|_{\dot{\mathbb{H}}^\alpha_{p_0}} \leq \frac{C^\alpha \alpha^\alpha}{(t-t_1)^{\frac{\alpha}{\kappa}}} \leq \frac{C_1^\alpha \alpha^\alpha}{t^{\frac{\alpha}{\kappa}}} \text{ for all } t \geq t_1 + 1, \alpha > 0.$$

Also from the local existence part of the theorem, there exists $t_2 > 0, \beta > 0$ such that

$$\max\{\|\eta\|_{Gv(t)}, t^{\beta/\kappa}\|\eta\|_{\dot{\mathbb{H}}^\beta_{p_0}}\} \leq 2\|\eta_0\|_{L^{p_0}} \text{ for all } 0 < t \leq t_2. \tag{5.8}$$

Thus (3.6) holds for all $t \in (0, t_2] \cup [t_1 + 1, \infty)$.

To complete the proof, we will need to show (3.6) for $t \in [t_2, t_1 + 1]$. In fact, since t lies in the compact interval $[t_2, t_1 + 1]$, it will be enough to show an estimate of the form

$$\|\eta(t)\|_{\dot{\mathbb{H}}^\alpha_{p_0}} \leq C^\alpha \alpha^\alpha \tag{5.9}$$

for a constant C that does not depend on t or α. Note that due to (5.8), we have $\|\eta(t_2/2)\|_{\dot{\mathbb{H}}^\beta_{p_0}} < \infty$. Due to the global well-posedness for the sub-critical quasi-geostrophic equations, we have (see [47, 7])

$$M := \sup_{t_2 \leq t \leq t_1+1} \|\eta(t)\|_{\dot{\mathbb{H}}^\beta_{p_0}} < \infty.$$

Thus, applying the non-critical case of Theorem 2.1, there exists a time $0 < t_3 < t_2/2$ depending on M, such that

$$\|\eta(t)\|_{Gv(t_3)} \leq 2\|\eta(t-t_3)\|_{\dot{\mathbb{H}}^\beta_{p_0}} \leq 2M \text{ for all } t_2 \leq t \leq t_1 + 1.$$

Once again due to (3.1), this yields (5.9).

5.3. **Proof of Theorem 3.5.** As noted previously, it will suffice to prove (3.2). We begin with the the L^2 energy estimate. We multiply (3.7) by u and integrate over $\mathbb{R}$. Using integration by parts, we get

$$\|u(t)\|_{L^2}^2 + \int_0^t \|u(s)\|_{\dot{\mathbb{H}}^1}^2 \, ds \leq \|u_0\|_{L^2}^2.$$

As in the previous subsection, this implies $\liminf_{t\to\infty} \|u(t)\|_{\dot{\mathbb{H}}^1} = 0$. Consequently, due to the uniform bound on $\|u\|_{L^2}^2$, it also follows that $\liminf_{t\to\infty} \|u(t)\|_{\dot{\mathbb{H}}^\beta} = 0$ for all $0 < \beta \leq 1$.

If $n \geq 4$, $\beta_c \in (0, 1)$ and so the decay now follows.

For $n = 3$, the critical space is L^2. Therefore, we need to show $\liminf_{t\to\infty} \|\mathbf{u}(t)\|_{L^2} = 0$. To do this, it will be enough to obtain a time independent bound for $\|u(t)\|_{\dot{\mathbb{H}}^{-1}}$. To this end, we multiply (3.7) by $\Lambda^{-2}u$ and integrate by parts over $\mathbb{R}$ to get

$$\begin{aligned}\frac{d}{dt}\|\Lambda^{-1}u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 &= \int_{\mathbb{R}} \Lambda^{-2}u(t)\partial_x(u(t)^3)dx \\ &\le \|\Lambda^{-1}u(t)\|_{L^2}\|u(t)\|_{L^2}\|u(t)\|_{L^\infty}^2 \\ &\le C\Big[\|\Lambda^{-1}u(t)\|_{L^2}^2\|u(t)\|_{L^\infty}^4\Big] + \frac{1}{2}\|u(t)\|_{L^2}^2,\end{aligned} \tag{5.10}$$

where we have used the fact that the Hilbert transform $\partial_x\Lambda^{-1}$ is a bounded operator on L^2. In one dimension, we have the inequality [20]:

$$\|u(t)\|_{L^\infty} \le C\|u(t)\|_{L^2}^{\frac{1}{2}}\|\nabla u(t)\|_{L^2}^{\frac{1}{2}}. \tag{5.11}$$

Using (5.10), (5.11) and Gronwall's inequality, we obtain

$$\begin{aligned}\|\Lambda^{-1}u(t)\|_{L^2}^2 &\le C\|\Lambda^{-1}u_0\|_{L^2}^2 \exp\int_0^t \|u(s)\|_{L^\infty}^4 ds \\ &\le C\|\Lambda^{-1}u_0\|_{L^2}^2 \exp\int_0^t \|u(s)\|_{L^2}^2\|\nabla u(s)\|_{L^2}^2 ds \\ &\le C\|\Lambda^{-1}u_0\|_{L^2}^2 \exp\Big[\|u_0\|_{L^2}^2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds\Big] \\ &\le C\|\Lambda^{-1}u_0\|_{L^2}^2 \exp\Big[\|u_0\|_{L^2}^4\Big].\end{aligned}$$

This finishes the proof.

5.4. **Proof of Theorem 3.7.** We only need to show (3.2). Multiplying (3.9) by u and integrating by parts, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \le -3\beta\int_{R^d} |u|^2|\nabla u|^2 dx.$$

Noting $\beta > 0$, as before, this immediately yields

$$\|u(t)\|_{L^2}^2 + \int_0^t \|\Delta u(s)\|_{L^2}^2 ds \le \|u_0\|_{L^2}^2.$$

This yields an time independent uniform bound for $\|u(t)\|_{L^2}$ and also that $\liminf_{t\to\infty} \|\Delta u\|_{L^2} = 0$. Interpolation immediately yields (3.2) in case $d \ge 3$. For cases $d = 1, 2$, multiplying (3.9) by $\Lambda^{-2}u$ (recall $\Lambda^2 = -\Delta$) and integrating by parts, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-1}u\|_{L^2}^2 + \|\Lambda u\|_{L^2}^2 \le -3\beta\int_{R^d} |u|^4 dx.$$

This yields

$$\|u\|_{\dot{\mathbb{H}}^{-1}} \le \|u_0\|_{\dot{\mathbb{H}}^{-1}}, \quad \liminf_{t\to\infty} \|u\|_{\dot{\mathbb{H}}^1} = 0.$$

Noting that for $d = 1$ and $d = 2$, we have $\beta_c = -1/2$ and $\beta_c = 0$ respectively, we have (3.2).

5.5. **Proof of Theorem 3.8.** Multiplying the equation by u and integrating by parts, we have the energy inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2+\|\Delta u\|_{L^2}^2 \le \sum_{j=1}^{2N-2} j|a_j|\int_{R^d}|\nabla u|^2|u|^{j-1}dx - (2N-1)a_{2N-1}\int_{R^d}|\nabla u|^2 u^{2(N-1)}dx.$$

Let

$$I_1=\sum_{j=1}^{2N-2} j|a_j|\int_{u:|u|\le 1}|\nabla u|^2|u|^{j-1}dx, \quad I_2=\sum_{j=1}^{2N-2} j|a_j|\int_{u:|u|>1}|\nabla u|^2|u|^{j-1}dx$$

and note that applying (3.10) and Poincare inequality, we get

$$I_1\le \delta\int_{R^d}|\nabla u|^2dx\le \delta\lambda_0\|u\|_{L^2}^2, \quad I_2\le\delta\int_{R^d}|\nabla u|^2u^{2(N-1)}dx.$$

Provided δ is sufficiently small, from the energy inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2+\gamma\|\Delta u\|_{L^2}^2\le 0,$$

where $\gamma>0$ is an adequate constant. As before, from here we obtain

$$\|u\|_{L^2}^2\le\|u_0\|_{L^2}^2, \quad \limsup_{t\to\infty}\|\Delta u\|_{L^2}=0.$$

By interpolation, this implies that $\liminf_{t\to\infty}\|u\|_{\dot{\mathbb{H}}^\beta}=0$ for all $\beta<2$. By Poincare inequality, this in turn implies $\liminf_{t\to\infty}\|u\|_{L^2}=0$ as well. Consequently, $\liminf_{t\to\infty}\|u\|_{\mathbb{H}^\beta}=0$ for all $\beta<2$. This finishes the proof.

6. Appendix

We now present the Littlewood-Paley theory and its application to the Kato-Ponce inequality. For more details of the Littlewood-Paley theory, see [8, 11].

6.1. **Littlewood-Paley theory.** We take a couple of smooth functions (χ,φ) supported on $\{\xi;|\xi|\le 1\}$ with values in $[0,1]$ such that for all $\xi\in R^d$,

$$\chi(\xi)+\sum_{j=0}^{\infty}\psi(2^{-j}\xi)=1,$$

where $\psi(\xi)=\varphi(\frac{\xi}{2})-\varphi(\xi)$. We denote $\psi(2^{-j}\xi)$ by $\psi_j(\xi)$. The homogeneous dyadic blocks and lower frequency cut-off functions are defined by

$$\triangle_j u=2^{jd}\int_{R^d}h(2^jy)u(x-y)dy, \quad S_ju=2^{jd}\int_{R^d}\tilde{h}(2^jy)u(x-y)dy,$$

where $h=\mathcal{F}^{-1}\psi$ and $\tilde{h}=\mathcal{F}^{-1}\chi$. We note that $u=\sum_{j\in\mathbb{Z}}\triangle_j u$ in $\mathcal{S}_h'$, where $\mathcal{S}_h'$ is the space of tempered distributions u such that $\lim_{j\to-\infty}S_ju=0$ in $\mathcal{S}'$. This is called the Littlewood-Paley decomposition.

This decomposition allows us to characterize a large range of functions spaces in a unified way. In this section, we only consider the homogeneous Triebel-Lizorkin spaces ([19]):

$$\|f\|_{\dot{F}^\alpha_{p,q}}=\left\|\Big(\sum_{j\in\mathbb{Z}}2^{2j\alpha}|\triangle_j f|^2\Big)^{\frac{1}{2}}\right\|_{L^p}$$

In particular, for $q = 2$,

$$\|f\|_{\dot{\mathbb{H}}^\alpha_p} \simeq \Big\| \Big(\sum_{j\in\mathbb{Z}} 2^{2j\alpha} |\triangle_j f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p}. \tag{6.1}$$

The concept of paraproduct is to deal with the interaction of two functions in terms of low or high frequency parts. For u, v two tempered distributions,

$$uv = T_u v + T_v u + R(u,v), \quad \text{where}$$

$$T_u v = \sum_{i\le j-2} \triangle_i u \triangle_j v = \sum_{j\in\mathbb{Z}} S_{j-1} u \triangle_j v, \quad S_j u = \sum_{l\le j-1} \triangle_l u,$$

$$R(u,v) = \sum_{|j-j'|\le 1} \triangle_j u \triangle_{j'} v.$$

Then, up to finitely many terms,

$$\triangle_j (T_u v) = S_{j-1} u \triangle_j v, \quad \triangle_j (R(u,v)) = \sum_{k\ge j-2} \triangle_k u \triangle_k v.$$

6.2. **Proof of the Kato-Ponce inequality ([28]).** We only prove the homogeneous part: for p, p_i, and q_i such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$, $1 \le p < \infty$, $p_i, q_i \ne 1$, we have the following estimation:

$$\|\Lambda^s(fg)\|_{L^p} \le C\Big[\|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|\Lambda^s g\|_{L^{q_2}} \Big]. \tag{6.2}$$

Proof. We decompose fg as follows:

$$\begin{aligned} fg &= \sum_k \Big(\sum_{j\le k-2} \triangle_j f \Big) \triangle_k g + \sum_k \Big(\sum_{j\le k-2} \triangle_j g \Big) \triangle_k f + \sum_{|j-k|\le 1} \triangle_j f \triangle_k g \\ &= \sum_k S_{k-1} f \triangle_k g + \sum_k \triangle_k S_{k-1} g \triangle_k f + \sum_{|j-k|\le 1} \triangle_j f \triangle_k g \\ &= (a) + (b) + (c). \end{aligned} \tag{6.3}$$

We apply Λ^s to (6.3) and estimate three terms separately. We begin with (a).

$$\begin{aligned} \|\Lambda^s(a)\|_{L^p} &\le C \Big\| \Big[\sum_k \Big| \Lambda^s \big(\sum_l S_{l-1} f \triangle_l g \big) \Big|^2 \Big]^{\frac{1}{2}} \Big\|_{L^p} \\ &\le C \Big\| \Big[\sum_k \big| 2^{ks} S_{k-1} f \triangle_k g \big|^2 \Big]^{\frac{1}{2}} \Big\|_{L^p}. \end{aligned} \tag{6.4}$$

Since for any k,

$$|S_{k-1} f(x)| \le C\mathcal{M} f(x) = \sup_{r>0} \frac{1}{r^d} \int_{B(x,r)} |f(y)| dy,$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator, we can estimate the right-hand side of (6.4) by

$$\begin{aligned} \|\Lambda^s(a)\|_{L^p} &\le C \Big\| \mathcal{M} f \Big[\sum_k \big| 2^{ks} \triangle_k g \big|^2 \Big]^{\frac{1}{2}} \Big\|_{L^p} \\ &\le C \|\mathcal{M} f\|_{L^{p_1}} \Big\| \Big[\sum_k \big| 2^{ks} \triangle_k g \big|^2 \Big]^{\frac{1}{2}} \Big\|_{L^{q_1}} \le C \|f\|_{L^{p_1}} \|\Lambda^s g\|_{L^{q_1}}, \end{aligned} \tag{6.5}$$

where we use the fact that $\mathcal{M}$ maps L^p to L^p for all $p > 1$. By using the same method,

$$\|\Lambda^s(b)\|_{L^p} \le C\|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}. \tag{6.6}$$

We finally estimate $\Lambda^s(c)$.

$$\begin{aligned}
\|\Lambda^s(c)\|_{L^p} &\le C\Big\|\Big[\sum_k \Big|\Lambda^s\big(\sum_{|l-l'|\le 1} \triangle_l f \triangle_{l'} g\big)\Big|^2\Big]^{\frac12}\Big\|_{L^p} \\
&\le C\Big\|\Big[\sum_k \Big|2^{ks}\sum_{l\ge k-2} \triangle_l f \triangle_l g\Big|^2\Big]^{\frac12}\Big\|_{L^p} \\
&= \Big\|\Big[\Big\{2^{ks}\Big(\sum_{l\ge k-2}\big(2^{-ls}\big)^2\Big)^{\frac12} \times \Big(\sum_l \big(\triangle_k(\triangle_l f 2^{ls}\triangle_l g)\big)^2\Big)^{\frac12}\Big\}^2\Big]^{\frac12}\Big\|_{L^p} \\
&\le C\Big\|\Big[\sum_k\sum_l \Big(\triangle_k\big(\triangle_l f 2^{ls}\triangle_l g\big)\Big)^2\Big]^{\frac12}\Big\|_{L^p}.
\end{aligned} \tag{6.7}$$

We now need to use the extension of the Littlewood-Paley operator [31]: if $\mathbb{L} : L^p \to L^p l^2$ is a Littlewood-Paley operator, then $\mathcal{L} : L^p l^2 \to L^p l^2 l^2$ is the extension of $\mathbb{L}$ such that

$$\Big\|\mathcal{L}\Big\|_{L^p l^2 \to L^p l^2 l^2} \le C\Big\|\mathbb{L}\Big\|_{L^p \to L^p l2}.$$

Using this relation, we can replace the last term in (6.7) by

$$\begin{aligned}
\|\Lambda^s(c)\|_{L^p} &\le C\Big\|\Big[\sum_l \Big(\triangle_l f 2^{ls}\triangle_l g\Big)^2\Big]^{\frac12}\Big\|_{L^p} \le C\Big\|\Big[\mathcal{M} f\sum_l \Big(2^{ls}\triangle_l g\Big)^2\Big]^{\frac12}\Big\|_{L^p} \\
&\le C\|\mathcal{M} f\|_{L^{p_2}}\Big\|\Big[\sum_l \Big(2^{ls}\triangle_l g\Big)^2\Big]^{\frac12}\Big\|_{L^q_2} \le C\|f\|_{L^{p_2}}\|\Lambda^s g\|_{L^{q_2}}.
\end{aligned} \tag{6.8}$$

By (6.5), (6.6), and (6.8), we obtain (6.2). □

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