Consistency Results in the Theory of Continuous Functions and Selective Separability

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Abstract

We study of the notion of selective separability (SS), which was introduced by Marion Scheepers and its connection with the game-theoritic strengthening, strategically selective separable spaces (SS^+). It is known that every set of countable π-weight is selectively separable and if X is selectively separable, then all dense subsets of X are selectively separable. We know that some dense countable subsets of $2^{\mathfrak{c}}$ are selectively separable and some are not. It is also known that $C_p(X)$ is selectively separable if and only if it is separable and has countable fan tightness. Here we prove that separable Fréchet spaces are selectively separable. It is also shown that consistently the product of two separable Fréchet spaces might not be selectively separable. Also we show that adding a Sacks real can destroy the property of being selectively separable.

We introduce a notion stronger than selective separability and named it strategically selectively separable or SS^+ and considered the properties in countable dense subsets of uncountable powers. It is shown that there is an SS space which fail to be SS^+. The motivation for studying SS^+ is that it is a property possessed by all separable subsets of $C_p(X)$ for each σ-compact space X. We prove that the winning strategy for countable SS^+ spaces can be chosen to be Markov.

We introduce the notion of being compactlike of a collection of open sets in a topological space and with the help of this notion we prove that there are two countable SS^+ spaces such that the union fails to be SS^+, which contrasts the known result about the union of SS spaces. We also prove that the product of two countable SS^+ spaces is again countable SS^+.

We prove a very interesting result which consistently contrasts our previous result, that the proper forcing axiom, PFA, implies that the product of two countable Fréchet spaces is SS. Also we show that consistently with the negation of CH that all separable Fréchet spaces have π-weight at most ω_1.

We also worked on an open question posed by Ohta and Yamasaki in open problems in topology which is, whether every C^*-embedded subset of a first countable is C-embedded. It is known that a counterexample can be derived from the assumption $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ and that if the Product Measure Extension Theorem (PMEA) holds then the answer is affirmative in some cases . We show that in the model obtained by adding κ-many random reals, where κ is a supercompact cardinal, every C^*-embedded subset of a first countable space (even with character smaller than κ) is C-embedded. The result was derived from the interesting fact that, if two ground model sets are completely separated after adding a random real, then they were completely separated originally.

The dissertation is divided as follows. The first chapter contains the topological properties of selectively separable spaces. The second chapter contains all the results we obtained about SS^+ spaces. The third chapter is devoted to the theorems involving CH and forcing extensions. The final chapter contains the results we obtained in the random real model about the C-embedding and C^*-embedding properties.

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