

RELATIVE TUTTE POLYNOMIALS OF TENSOR PRODUCTS OF COLORED GRAPHS

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ABSTRACT. The tensor product (G_1, G_2) of a graph G_1 and a pointed graph G_2 (containing one distinguished edge) is obtained by identifying each edge of G_1 with the distinguished edge of a separate copy of G_2 , and then removing the identified edges. A formula to compute the Tutte polynomial of a tensor product of graphs was originally given by Brylawski. This formula was recently generalized to colored graphs and the generalized Tutte polynomial introduced by Bollobás and Riordan. In this paper we generalize the colored tensor product formula to relative Tutte polynomials of relative graphs, containing zero edges to which the usual deletion-contraction rules do not apply. As we have shown in a recent paper, relative Tutte polynomials may be used to compute the Jones polynomial of a virtual knot.

1. INTRODUCTION

The Tutte polynomial is one of the most important invariants in graph theory. It was first introduced and studied by Tutte for non-colored graphs, but has since been generalized to colored graphs [1] and to colored relative graphs in which some edges cannot be treated as regular colored edges in the computation of the Tutte polynomial [10]. The corresponding Tutte polynomial in the latter case is called the relative Tutte polynomial.

There are many situations in applied graph theory where an actual network is represented by a graph, whose edges turn out to denote subnetworks at closer inspection. A typical example is an electric circuit whose components are (identical) integrated circuits themselves. Theoretically, we may represent many such networks of subnetworks by using the *tensor product operation* of graphs. The tensor product operation associates a graph $G_1 \otimes G_2$ to a graph G_1 and a pointed graph G_2 containing one distinguished edge e . It is obtained by replacing each edge of G_1 with a copy of $G_2 \setminus e$, where e is used to mark the vertices of G_2 where we graft G_2 to the place of the removed edge of G_1 and is itself removed in the process. The Tutte polynomial of such a tensor product of graphs was first expressed by Brylawski [4]. He showed that the Tutte polynomial of a tensor product can be obtained from the Tutte polynomial G_1 and some ordinary and pointed Tutte polynomials associated to (G_2, e) through certain variable substitutions. The application of the initial tensor product is limited by the fact that its definition requires all edges to be replaced by the same graph. However, in the network setting, the components of an actual network may be integrated circuits of different kinds. Such a composite network cannot be obtained by replacing every connection with the same subnetwork (as required by the tensor product definition). Thus it is more practical and applicable to color the edges (links) of G_1 (a network) first and then replace only edges of a fixed color with the same graphs (subnetworks) G_2 ($G_2 \setminus e$ to be more precise). Repeating this operation would then allow replacing individual edges

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by different graphs. This new tensor product concept was introduced in [12] where it was shown that the results of Brylawski [4] on the Tutte polynomial of a tensor product of non-colored graphs can be generalized to the (generalized) Tutte polynomial of a (generalized) tensor product of colored graphs [12].

Another application of the colored Tutte polynomial is in the area of knot theory. It is well-known that the Jones polynomial of a link can be computed from the Kauffman bracket polynomial while the Kauffman bracket polynomial of a link can be computed from the (signed) Tutte polynomial of the face graph of a regular projection of the link. This was first shown for alternating links and the ordinary Tutte polynomial by Thistlethwaite [16], then generalized to arbitrary links and a signed Tutte polynomial by Kauffman [14]. This enables applications of the ordinary Tutte polynomials and their signed generalizations to classical knot theory such as those in [9, 11, 13]. For virtual knots the situation is a little more complicated. An appropriate generalization of the Kauffman bracket polynomial was developed by Kauffman himself [15]. However, until very recently, no appropriate generalization of the Tutte polynomial to face graphs of virtual links was known. In a series of papers, Chmutov, Pak and Voltz [6, 7, 8] developed a generalization of Thistlethwaite's theorem first to checkerboard-colorable [7] then to arbitrary [6, 8] virtual link diagrams. These express the Jones polynomial of a virtual link in terms of a signed generalization of the Bollobás-Riordan polynomial [2, 3] of a ribbon graph, obtained from the virtual link diagram. In [10], it is shown that a relative variant of the *other* generalization of the Tutte polynomial, also due to Bollobás and Riordan [1] may also be used to compute the Jones polynomial of a virtual link, this time directly from the face graph of the virtual link diagram. In a face graph of a virtual link diagram, edges corresponding to virtual crossings cannot be treated as a regular edge and are called *zero edges* in [10]. The Tutte polynomial of a colored graph with zero edges generalized in [10] is called a *relative Tutte polynomial*.

Given a colored graph G_1 and a pointed colored graph G_2 such that both may contain zero edges, their tensor product can be defined just as in the case of two colored graphs, so long as the edges in G_1 to be replaced by copies of G_2 are not zero edges and the distinguished edge marked in G_2 for the gluing purpose is not a zero edge either. The main goal of this paper is to formulate the relative Tutte polynomial of the tensor product of two colored graphs with zero edges, using only the Tutte polynomials obtained from G_1 and G_2 and certain substitution rules. As it turned out, we have to generalize the pointed polynomials used in the colored tensor product case, define new pointed polynomials and introduce a set of much more complicated substitution rules. Given the complexity level of the relative Tutte polynomial, this should not be a surprise. It is actually somewhat surprising that such a formulation still exists!

This paper is organized as follows. In Section 2 we review the relative Tutte polynomial and, in Section 3 we introduce the concept of the universal relative Tutte polynomial. Section 4 contains the definition of our pointed universal relative Tutte polynomials. These include the ones generalized from the pointed Tutte polynomials used previously and three new pointed Tutte polynomials. In Section 5 we discuss the contracting sets in a tensor product of colored graphs with zero edges. Section 6 contains our main result: the generalization of the tensor product formula to colored relative graphs. The concluding Section 7 contains a sample application of our main result and a few further remarks.

2. A REVIEW OF THE RELATIVE TUTTE POLYNOMIAL

In this section we review the notion of the relative Tutte polynomial of a colored graph G , with respect to a set of edges $\mathcal{H} \subset E(G)$ introduced in [10]. We observe that the results in [10] may be

easily generalized to the situation where the edges in \mathcal{H} do not all belong to the same color set. We also introduce the *universal relative Tutte polynomial* of a colored graph.

Definition 2.1. Let G be a graph with edge set $E(G)$ and let $\mathcal{H} \subseteq E(G)$. A subset \mathcal{C} of $E(G) \setminus \mathcal{H}$ is called a *contracting set* of G with respect to \mathcal{H} if \mathcal{C} contains no cycles and $E(G) \setminus (\mathcal{C} \cup \mathcal{H})$ contains no cocycles. Given a contracting set \mathcal{C} , the set $E(G) \setminus (\mathcal{C} \cup \mathcal{H})$ is called the *corresponding deleting set* and it is denoted by \mathcal{D} .

Recall that the cocycles of a graph are its minimal sets of edges whose removal increases the number of connected components. Sometimes we will refer to \mathcal{C} , \mathcal{D} and \mathcal{H} as *graphs*, by which we mean the subgraphs of G induced by the respective set of edges.

Definition 2.2. Let G be a graph and \mathcal{H} be a subset of $E(G)$. A *proper labeling* or *relative labeling* of the edges of G with respect to \mathcal{H} is a map $\phi : E(G) \rightarrow \mathbb{N}$ such that $\mathcal{H} = \{e \in E(G) : \phi(e) = 0\}$ and the restriction of ϕ to $E(G) \setminus \mathcal{H}$ is an injective map into \mathbb{Z}_+ . We say that e_1 is larger than e_2 if $\phi(e_1) > \phi(e_2)$. Let \mathcal{C} be a contracting set of G with respect to \mathcal{H} , then

- a) an edge $e \in \mathcal{C}$ is called *internally active* if $\mathcal{D} \cup \{e\}$ contains a cocycle D_0 in which e is the smallest edge, otherwise it is *internally inactive*.
- b) an edge $f \in \mathcal{D}$ is called *externally active* if $\mathcal{C} \cup \{f\}$ contains a cycle C_0 in which f is the smallest edge, otherwise it is *externally inactive*.

As noted in [10, Remark 3.12], activities of regular edges may be given in the following equivalent definition.

Definition 2.3. Let G be a graph and \mathcal{H} be a subset of $E(G)$ and that a proper labeling ϕ has been given. Let \mathcal{C} be a contracting set of G with respect to \mathcal{H} , then

- a) an edge $e \in \mathcal{C}$ is internally active if it becomes a bridge once all edges in \mathcal{D} larger than e are deleted, otherwise it is internally inactive;
- b) an edge $e \in \mathcal{D}$ is externally active if it becomes a loop after all edges in \mathcal{C} larger than f are contracted, otherwise it is externally inactive.

The above equivalent definition depends of the following description of contracting and deleting sets.

Lemma 2.4. Let G be a graph, let \mathcal{H} be a subset of $E(G)$, and let ϕ be a proper labeling. Let $\mathcal{C} \subseteq E(G) \setminus \mathcal{H}$ be a set of regular edges and let $\mathcal{D} = E(G) \setminus (\mathcal{C} \cup \mathcal{H})$. Then \mathcal{C} is a contracting set and \mathcal{D} is the corresponding deleting set if and only if the following holds for regular edge $e \in E(G) \setminus \mathcal{H}$ after contracting all edges in $f \in \mathcal{C}$ and all deleting edges $g \in \mathcal{D}$ satisfying $\phi(f) \geq \phi(e)$ and $\phi(g) \geq \phi(e)$:

- (1) If $e \in \mathcal{C}$ then e does not become a loop;
- (2) if $e \in \mathcal{D}$ then e does not become a bridge.

The proof is straightforward and left to the reader. As a consequence of Lemma 2.4 we may find each contracting set \mathcal{C} , together with the corresponding deleting set \mathcal{D} by going through the list of regular edges in the order of their labels and deciding to put each of them either into \mathcal{C} or into \mathcal{D} , contracting or deleting them accordingly, subject only to the rules that we are not allowed to contract a loop or delete a bridge.

The definition of a relative Tutte polynomial involves contracting all edges in \mathcal{C} and deleting all edges in \mathcal{D} . We perform these operations in decreasing order of the labels. The resulting graph $\mathcal{H}_{\mathcal{C}}$ contains only zero edges and will be replaced with a graph invariant $\psi(\mathcal{H}_{\mathcal{C}})$. The graph $\mathcal{H}_{\mathcal{C}}$ depends on the order of deletions and contractions determined by the proper labeling ϕ . However, the multiset of blocks of $\mathcal{H}_{\mathcal{C}}$ is independent of the order in which the deletions and contractions are performed, see [10, Lemma 3.14]. That's why we want the operator ψ to be a *block invariant* (see [10, Definition 3.13]), most of the times. For applications in knot theory a generalization of block invariants was introduced in [10]: maps on isomorphism classes on graphs that are invariant under *vertex pivots*. These operations are defined as sequences of *vertex splicings* and *vertex splittings*. A *vertex splicing* is an operation that merges two disjoint graphs by picking a vertex from each and identifying these selected vertices, thus creating a cutpoint. The opposite operation is *vertex splitting* that creates two disjoint graphs by replacing a cutpoint v with two copies v_1 and v_2 , and makes each block containing v contain exactly one of v_1 and v_2 .

Definition 2.5. Let G be a graph that has a cutpoint u . A *vertex pivot* is a sequence of vertex splittings and vertex splicings, of the following kind. First we split G by creating two copies of u and two disjoint graphs G_1 and G_2 . Then we take a vertex $v_1 \in V(G_1)$ from the connected component of u_1 and a vertex $v_2 \in V(G_2)$ in the connected component of u_2 and we merge G_1 and G_2 by identifying u_1 with u_2 .

As noted in [10, Section 4], $\mathcal{H}_{\mathcal{C}}$ will be the same up to performing a sequence of vertex pivots, independently of ϕ .

Let G be a graph and $\mathcal{H} \subseteq E(G)$. In [10] a coloring $c : E(G) \setminus \mathcal{H} \rightarrow \Lambda$ of the regular edges to a color set Λ was considered. However, the definitions and results stated in [10] may be generalized without any substantial change to the situation where we color all edges of G , including the zero edges, using a map $c : E(G) \rightarrow \Lambda$. Let us call a graph G , together with such a coloring $c : E(G) \rightarrow \Lambda$ a Λ -*colored graph*. We may fix a subset $\Lambda_0 \subseteq \Lambda$ and require all edges in \mathcal{H} to be with colors from Λ_0 . The subgraph \mathcal{H} is thus also a Λ_0 -colored graph.

Definition 2.6. We call two Λ_0 -colored graphs Γ and Γ' *vertex pivot equivalent* if Γ' is isomorphic to a graph obtained from Γ by performing a sequence of vertex pivot operations. We call an invariant ψ of Λ_0 -colored graphs a *vertex pivot invariant* if $\psi(\Gamma) = \psi(\Gamma')$ whenever Γ and Γ' are vertex pivot equivalent. The collection of vertex pivot equivalence classes of Λ_0 -colored graphs is denoted by $\text{VP}(\Lambda_0)$.

For any contracting set \mathcal{C} of G with respect to \mathcal{H} , let $\mathcal{H}_{\mathcal{C}}$ be the graph obtained by deleting all edges in \mathcal{D} and contracting all edges in \mathcal{C} . Finally, we assign a proper labeling ϕ to the edges of G . We now define the relative Tutte polynomial of G with respect to \mathcal{H} and ψ as

$$(2.1) \quad T_{\mathcal{H}}^{\psi}(G) = \sum_{\mathcal{C}} \left(\prod_{e \in G \setminus \mathcal{H}} w(G, c, \phi, \mathcal{C}, e) \right) \psi(\mathcal{H}_{\mathcal{C}}) \in \mathcal{R}[x_{\lambda}, X_{\lambda}, y_{\lambda}, Y_{\lambda} : \lambda \in \Lambda],$$

where the summation is taken over all contracting sets \mathcal{C} and $w(G, c, \phi, \mathcal{C}, e)$ is the *weight* of the edge e with respect to the contracting set \mathcal{C} , which is defined as (assume that e has color λ):

$$(2.2) \quad w(G, c, \phi, \mathcal{C}, e) = \begin{cases} X_\lambda & \text{if } e \text{ is internally active;} \\ Y_\lambda & \text{if } e \text{ is externally active;} \\ x_\lambda & \text{if } e \text{ is internally inactive;} \\ y_\lambda & \text{if } e \text{ is externally inactive.} \end{cases}$$

To simplify our notation, we may use $T_{\mathcal{H}}(G)$ for $T_{\mathcal{H}}^\psi(G)$, with the understanding that some ψ has been chosen, unless there is a need to stress what ψ really is. Following [10], we then write

$$W(G, c, \phi, \mathcal{C}) = \prod_{e \in G \setminus \mathcal{H}} w(G, c, \phi, \mathcal{C}, e)$$

so that

$$(2.3) \quad T_{\mathcal{H}}(G, \phi) = \sum_{\mathcal{C}} W(G, c, \phi, \mathcal{C}) \psi(\mathcal{H}_{\mathcal{C}}).$$

One of our main results [10, Theorem 3.16] extends the famous result of Bollobás and Riordan [1, Theorem 2] on colored Tutte polynomials to colored relative Tutte polynomials. Its proof extends without any change to the situation when the set of zero edges is a Λ_0 -colored subgraph.

Theorem 2.7. *Assume I is an ideal of $\mathcal{R}[x_\lambda, X_\lambda, y_\lambda, Y_\lambda : \lambda \in \Lambda]$. Then the homomorphic image of $T_{\mathcal{H}}(G, \phi)$ in $\mathcal{R}[x_\lambda, X_\lambda, y_\lambda, Y_\lambda : \lambda \in \Lambda]/I$ is independent of ϕ (for any G and ψ) if and only if*

$$(2.4) \quad \det \begin{pmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{pmatrix} - \det \begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} \in I$$

and

$$(2.5) \quad \det \begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} - \det \begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix} \in I.$$

hold for all $\lambda, \mu \in \Lambda$.

Motivated by this result, we will assume $T_{\mathcal{H}}(G, \phi)$ is defined in the ring

$$\mathcal{T}(\mathcal{R}, \Lambda) := \mathcal{R}[x_\lambda, X_\lambda, y_\lambda, Y_\lambda : \lambda \in \Lambda]/I_1(\mathcal{R}, \Lambda)$$

where $I_1(\mathcal{R}, \Lambda)$ is the ideal of $\mathcal{R}[x_\lambda, X_\lambda, y_\lambda, Y_\lambda : \lambda \in \Lambda]$ generated by all polynomials of the form (2.4) and (2.5).

Definition 2.8. *We call the ring $\mathcal{T}(\mathcal{R}, \Lambda)$ the Tutte ring associated to the color set Λ and the ring of coefficients \mathcal{R} .*

The relative Tutte polynomial, considered as an element of the Tutte ring $\mathcal{T}(\mathcal{R}, \Lambda)$, becomes independent of the choice of the proper labeling ϕ and we may write $T_{\mathcal{H}}(G)$ for $T_{\mathcal{H}}(G, \phi)$. An immediate consequence of this fact is the following corollary, see [10, Corollary 3.17].

Corollary 2.9. *$T_{\mathcal{H}}(G)$ can be computed via the following recursive formula, valid for any regular edge e , i.e., any $e \notin \mathcal{H}$:*

$$(2.6) \quad T_{\mathcal{H}}(G) = \begin{cases} y_\lambda T_{\mathcal{H}}(G - e) + x_\lambda T_{\mathcal{H}}(G/e), & \text{if } e \text{ is neither a bridge nor a loop,} \\ X_\lambda T_{\mathcal{H}}(G/e), & \text{if } e \text{ is a bridge,} \\ Y_\lambda T_{\mathcal{H}}(G - e), & \text{if } e \text{ is a loop.} \end{cases}$$

In the above, $e \notin \mathcal{H}$ is a regular edge, $\lambda = c(e)$, $G - e$ is the graph obtained from G by deleting e and G/e is the graph obtained from G by contracting e .

Remark 2.10. In some situations, it is plausible to require that the set of colors used to color the regular edges be disjoint from the set Λ_0 used to color the zero edges, i.e. $c(E(G \setminus \mathcal{H})) \subseteq \Lambda \setminus \Lambda_0$, although we do not need this restriction in what is written above. For the sake of convenience and to avoid possible confusions, we will assume that $c(E(G \setminus \mathcal{H})) \subseteq \Lambda \setminus \Lambda_0$ in the rest of this paper.

3. THE UNIVERSAL RELATIVE TUTTE POLYNOMIAL

We now introduce the *universal relative Tutte polynomial* associated to a color set Λ_0 .

Definition 3.1. Let G be a Λ -colored graph and \mathcal{H} a Λ_0 -colored subset of its edges such that $\Lambda_0 \subseteq \Lambda$ and $c(E(G \setminus \mathcal{H})) \subseteq \Lambda \setminus \Lambda_0$. Let us introduce a distinct variable $z_{[\Gamma]}$ for each vertex pivot equivalence class $[\Gamma] \in \text{VP}(\Lambda_0)$. Let ψ_{Λ_0} be the vertex pivot invariant that assigns to each Λ_0 -colored graph Γ the variable $z_{[\Gamma]}$ in the polynomial ring $\mathcal{R}[z_{[\Gamma]} : [\Gamma] \in \text{VP}(\Lambda_0)]$. We call the relative Tutte polynomial

$$T_{\mathcal{H}}^{\psi_{\Lambda_0}}(G) \in \mathcal{R}[x_{\lambda}, X_{\lambda}, y_{\lambda}, Y_{\lambda}, z_{[\Gamma]} : \lambda \in \Lambda \setminus \Lambda_0, [\Gamma] \in \text{VP}(\Lambda_0)]/I_1(\mathcal{R}, \Lambda, \Lambda_0)$$

the *universal Λ_0 -colored relative Tutte polynomial of G with respect to \mathcal{H}* and denote it by $T_{\mathcal{H}}^{\Lambda_0}(G)$. Here $I_1(\mathcal{R}, \Lambda, \Lambda_0)$ is the ideal of $\mathcal{R}[x_{\lambda}, X_{\lambda}, y_{\lambda}, Y_{\lambda}, z_{[\Gamma]} : \lambda \in \Lambda \setminus \Lambda_0, [\Gamma] \in \text{VP}(\Lambda_0)]$ generated by all polynomials of the form (2.4) and (2.5) with $\lambda, \mu \in \Lambda \setminus \Lambda_0$. We call the ring

$$\mathcal{T}(\mathcal{R}, \Lambda, \Lambda_0) := \mathcal{R}[x_{\lambda}, X_{\lambda}, y_{\lambda}, Y_{\lambda}, z_{[\Gamma]} : \lambda \in \Lambda \setminus \Lambda_0, [\Gamma] \in \text{VP}(\Lambda_0)]/I_1(\mathcal{R}, \Lambda, \Lambda_0)$$

the *Λ_0 -pointed Tutte ring associated to the color set Λ and the ring of coefficients \mathcal{R}* .

The ideal $I_1(\mathcal{R}, \Lambda, \Lambda_0)$ in Definition 3.1 above is generated by polynomials not containing any of the variables $z_{[\Gamma]}$. Thus we have

$$(3.1) \quad \mathcal{T}(\mathcal{R}, \Lambda, \Lambda_0) = \mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0)[z_{[\Gamma]} : [\Gamma] \in \text{VP}(\Lambda_0)].$$

In other words, the Λ_0 -pointed Tutte ring $\mathcal{T}(\mathcal{R}, \Lambda, \Lambda_0)$ is a polynomial ring in which the Tutte ring $\mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0)$ is the ring of coefficients and $\{z_{[\Gamma]} : [\Gamma] \in \text{VP}(\Lambda_0)\}$ is the set of independent variables and the universal relative Tutte polynomial $T_{\mathcal{H}}^{\psi_{\Lambda_0}}(G)$ is a special element in the Tutte ring $\mathcal{T}(\mathcal{R}, \Lambda, \Lambda_0)$, namely one that is a $\mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0)$ -linear combination of the terms of the form $z_{[\Gamma]}$. This observation makes the substitution map, given in Theorem 3.2 below, well-defined. This theorem justifies the adjective *universal* in the name of the universal Λ_0 -pointed relative Tutte polynomial. We call it a theorem only because of its importance, its proof is straightforward.

Theorem 3.2. *Let G be a Λ -colored graph and \mathcal{H} a Λ_0 -colored subset of its edges where $\Lambda_0 \subseteq \Lambda$ and $c(E(G \setminus \mathcal{H})) \subseteq \Lambda \setminus \Lambda_0$. Let ψ be a vertex pivot invariant of Λ_0 -colored graphs with values in an integral domain \mathcal{R} . Then the homomorphism*

$$\mathcal{T}(\mathcal{R}, \Lambda, \Lambda_0) \rightarrow \mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0),$$

sending each element of $\mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0)$ into itself and sending each $z_{[\Gamma]}$ into $\psi([\Gamma])$, sends the universal Λ_0 -colored Tutte polynomial $T_{\mathcal{H}}^{\Lambda_0}(G)$ into the relative Tutte polynomial $T_{\mathcal{H}}^{\psi}(G)$.

Remark 3.3. $\mathcal{T}(\mathcal{R}, \Lambda, \Lambda_0)$ is a polynomial ring with infinitely many variables $z_{[\Gamma]}$. However, if we consider only colored graphs with at most N edges, where N is any positive integer, then it may be replaced with a polynomial ring with only finitely many variables.

4. POINTED UNIVERSAL RELATIVE TUTTE POLYNOMIALS

In analogy to the main results in [11, Theorem 5.1] and [12, Theorem 3], we want to obtain a formula for the universal relative Tutte polynomial of a λ -colored tensor product $G_1 \otimes_\lambda G_2$ of two Λ -colored graphs that have Λ_0 -colored subsets of zero edges \mathcal{H}_1 and \mathcal{H}_2 (we will assume $\Lambda_0 \subset \Lambda$ and $\lambda \in \Lambda \setminus \Lambda_0$). Similarly to the formulation in [11, 12], our formula will make use of pointed variants of the universal relative Tutte polynomial. Two of these variants will be generalizations of the polynomials $T_C(G, e)$ and $T_L(G, e)$ that were already introduced in [11, 12] and which are generalizations of pointed Tutte polynomials introduced by Brylawski [4, 5]. A third variant arises with the presence of zero edges. We will also have to introduce two further pointed Tutte polynomials which will assume the role played by $T(G/e)$ and by $T(G-e)$, respectively, in [11, 12]. As before, when considering the λ -colored tensor product $G_1 \otimes_\lambda G_2$, all pointed relative Tutte polynomials will be computed for the pointed graph G_2 .

In this section we assume that G is a pointed connected graph with a distinguished edge e which is neither a loop nor a bridge, and \mathcal{H} is a subset of $E(G)$ not containing e . We assume that G is a $\Lambda \cup \{\nu\}$ -colored graph where $\nu \notin \Lambda$, \mathcal{H} is Λ_0 -colored where $\Lambda_0 \subset \Lambda$. The distinguished edge e is marked by the unique color $\nu \notin \Lambda$ to avoid possible confusions. Denote the set $\Lambda \cup \{\nu\}$ by Λ' , $\Lambda_0 \cup \{\nu\}$ by Λ'_0 and $\mathcal{H} \cup \{e\}$ by \mathcal{H}' . The first three pointed Tutte polynomials to be introduced are homomorphic images of the universal Λ'_0 -colored relative Tutte polynomial $T_{\mathcal{H}'}^{\Lambda'_0}(G)$. When we calculate $T_{\mathcal{H}'}^{\Lambda'_0}(G)$, we consider the distinguished edge e as a zero edge. Again, let us stress that the color ν assigned to e is different from the colors of all other (regular or zero) edges.

We want to classify the pairs $(\mathcal{C}, \mathcal{D})$ of contracting sets and corresponding deleting sets with respect to \mathcal{H}' into three classes, depending on their relation to the distinguished edge e , as follows:

Definition 4.1. *Let G be a pointed graph with distinguished edge e and set of zero edges $\mathcal{H}' = \mathcal{H} \cup \{e\}$. Let \mathcal{C} be a contracting set of G with respect to \mathcal{H}' and let \mathcal{D} be the corresponding deleting set.*

- (i) *We say that $(\mathcal{C}, \mathcal{D})$ has type \mathcal{C} if $\mathcal{C} \cup \{e\}$ contains a cycle;*
- (ii) *We say that $(\mathcal{C}, \mathcal{D})$ has type \mathcal{D} if $\mathcal{D} \cup \{e\}$ contains a cocycle;*
- (iii) *We say that $(\mathcal{C}, \mathcal{D})$ has type zero if it has neither type \mathcal{C} nor type \mathcal{D} .*

To simplify our terminology we will also say that a contracting set \mathcal{C} , or a deleting set \mathcal{D} has type \mathcal{C} , \mathcal{D} or zero, if the unique pair $(\mathcal{C}, \mathcal{D})$ formed with the corresponding deleting or contracting set has the same type.

The choice of letters to denote the types may seem counter-intuitive in this section, the motivation will become clear in Section 5. Notice that an equivalent condition for $(\mathcal{C}, \mathcal{D})$ to be of type zero is that $\mathcal{C} \cup \mathcal{H}'$ contains a cycle but $\mathcal{C} \cup \{e\}$ does not. Furthermore, $(\mathcal{C}, \mathcal{D})$ cannot have type \mathcal{C} and type \mathcal{D} simultaneously: if e closes a cycle with \mathcal{C} in G then after removing all edges of \mathcal{D} and the edge e from G , the endpoints of e are still connected via a path containing the edges in \mathcal{C} , hence deleting $\mathcal{D} \cup \{e\}$ will not increase the number of connected components in G . Thus another equivalent description of the three types may be stated as follows.

Proposition 4.2. *$(\mathcal{C}, \mathcal{D})$ has type \mathcal{C} , \mathcal{D} , or zero, respectively, if and only if after contracting the edges of \mathcal{C} and deleting the edges of \mathcal{D} in G , the edge e becomes a loop, bridge, or neither loop nor bridge, respectively. In the type zero case, after contracting the edges of \mathcal{C} and deleting the edges of \mathcal{D} in G , there is a path consisting of zero edges only (in \mathcal{H}) connecting the endpoints of e .*

The next statements depend on, and also characterize the type of $(\mathcal{C}, \mathcal{D})$.

Proposition 4.3. *If $(\mathcal{C}, \mathcal{D})$ is of type \mathcal{C} then \mathcal{C} is a contracting set with respect to \mathcal{H} but $\mathcal{C} \cup \{e\}$ is not a contracting set.*

Proof. Clearly $\mathcal{C} \cup \{e\}$ is not a contracting set since \mathcal{C} contains a path connecting the endpoints of e , and adding e to this path creates a cycle. As seen in the proof of Proposition 5.1, the set \mathcal{C} does not contain any cycle. We only need to check that $\mathcal{D} \cup \{e\}$ contains no cocycle in G . This may be performed in perfect analogy to the corresponding part in the proof of Proposition 5.1, the only difference being that the path γ' introduced in that proof may now be replaced by the path in \mathcal{C} connecting the endpoints of e . \square

Proposition 4.4. *If $(\mathcal{C}, \mathcal{D})$ is of type \mathcal{D} then $\mathcal{C} \cup \{e\}$ is a contracting set with respect to \mathcal{H} but \mathcal{C} is not a contracting set.*

Proof. Deleting all edges of $\mathcal{D} \cup \{e\}$ from G disconnects the endpoints of e . Thus the set $\mathcal{D} \cup \{e\}$ is not a deleting set with respect to \mathcal{H} and there is no path in \mathcal{C} connecting the end points of e . Equivalently, \mathcal{C} is not a contracting set and $\mathcal{C} \cup \{e\}$ contains no cycle. The proof of the fact that \mathcal{D} contains no cocycle of G is identical to the corresponding part of the proof of Proposition 5.1. \square

Proposition 4.5. *If $(\mathcal{C}, \mathcal{D})$ is of type zero then both \mathcal{C} and $\mathcal{C} \cup \{e\}$ are contracting sets with respect to \mathcal{H} .*

Proof. Since $(\mathcal{C}, \mathcal{D})$ is of type zero, $\mathcal{C} \cup \{e\}$ (hence \mathcal{C}) contains no cycle, but there is a path consisting of edges of $\mathcal{C} \cup \mathcal{H}$ connecting the end vertices of e . The set \mathcal{D} contains no cocycle by Proposition 5.1. Since there is a path consisting of edges of $\mathcal{C} \cup \mathcal{H}$ connecting the end vertices of e , adding e to \mathcal{D} does not create a cocycle in G_2^f . \square

Remark 4.6. Since the premises in Propositions 4.3, 4.4 and 4.5 mutually exclude each other, the conclusions provide a *characterization* of the types of $(\mathcal{C}, \mathcal{D})$: it has type \mathcal{C} exactly when \mathcal{C} is a contracting set with respect to \mathcal{H} but $\mathcal{C} \cup \{e\}$ is not, type \mathcal{D} exactly when $\mathcal{C} \cup \{e\}$ is a contracting set with respect to \mathcal{H} but \mathcal{C} is not, and it type zero exactly when both \mathcal{C} and $\mathcal{C} \cup \{e\}$ are contracting sets with respect to \mathcal{H} .

To define our pointed Tutte polynomials we introduce five endomorphisms of the Λ_0 -pointed Tutte ring $\mathcal{T}(\mathcal{R}, \Lambda', \Lambda'_0)$, which is a polynomial ring by (3.1). The restriction of each of these endomorphisms to $\mathcal{T}(\mathcal{R}, \Lambda)$ will be the identity map, thus they can be given by prescribing their effect on the variables $\{z_{[\Gamma]} : [\Gamma] \in \text{VP}(\Lambda'_0)\}$. The first three maps, π_C , π_L and π_0 , leave $z_{[\Gamma]}$ unchanged for select types of graphs Γ and they send all the other $z_{[\Gamma]}$ into zero:

$$\begin{aligned} \pi_C(z_{[\Gamma]}) &= \begin{cases} z_{[\Gamma]} & \text{if } \Gamma \text{ has exactly one edge } f \text{ of color } \nu \text{ and } f \text{ is a bridge (coloop);} \\ 0 & \text{otherwise.} \end{cases} \\ \pi_L(z_{[\Gamma]}) &= \begin{cases} z_{[\Gamma]} & \text{if } \Gamma \text{ has exactly one edge } f \text{ of color } \nu \text{ and } f \text{ is a loop;} \\ 0 & \text{otherwise.} \end{cases} \\ \pi_0(z_{[\Gamma]}) &= \begin{cases} z_{[\Gamma]} & \text{if } \Gamma \text{ has exactly one edge } f \text{ of color } \nu \text{ and } f \text{ is neither a loop nor a bridge;} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The last two maps, π_+ and π_- perform a contraction or deletion on some graphs Γ , and send all other $z_{[\Gamma]}$ into zero:

$$\begin{aligned} \pi_+(z_{[\Gamma]}) &= \begin{cases} z_{[\Gamma/f]} & \text{if } \Gamma \text{ has exactly one edge } f \text{ of color } \nu \text{ and } f \text{ is not a loop;} \\ 0 & \text{otherwise.} \end{cases} \\ \pi_-(z_{[\Gamma]}) &= \begin{cases} z_{[\Gamma-f]} & \text{if } \Gamma \text{ has exactly one edge } f \text{ of color } \nu \text{ and } f \text{ is not a bridge;} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Definition 4.7. We define the pointed universal Λ_0 -colored relative Tutte polynomials $T_{\mathcal{H},C}^{\Lambda'_0}(G, e)$, $T_{\mathcal{H},L}^{\Lambda'_0}(G, e)$ and $T_{\mathcal{H},0}^{\Lambda'_0}(G, e)$, respectively, as the image of $T_{\mathcal{H}'}^{\Lambda'_0}(G)$ under the endomorphism $\pi_+ \circ \pi_C$, $\pi_- \circ \pi_L$ and π_0 , respectively.

Notice that in the special case that $\mathcal{H} = \emptyset$, the definitions of $T_{\mathcal{H},C}^{\Lambda'_0}(G, e)$ and $T_{\mathcal{H},L}^{\Lambda'_0}(G, e)$ yield $T_C(G, e) \cdot z_{[\bullet]}$ and $T_L(G, e)z_{[\bullet]}$, respectively. Here $T_C(G, e)$ and $T_L(G, e)$ are the polynomials defined in [11, 12] and \bullet is the graph containing a single vertex. Thus, in situations where there is no confusion about the sets Λ , Λ_0 and \mathcal{H} , we will simply use $T_C(G, e)$, $T_L(G, e)$ and $T_0(G, e)$ as the abbreviations for $T_{\mathcal{H},C}^{\Lambda'_0}(G, e)$, $T_{\mathcal{H},L}^{\Lambda'_0}(G, e)$ and $T_{\mathcal{H},0}^{\Lambda'_0}(G, e)$ respectively. As a consequence of Theorem 2.7, the pointed universal Λ_0 -colored relative Tutte polynomials defined above may be computed by summing weights of contracting sets of G with respect to \mathcal{H}' . The weights will be assigned using a proper labeling with respect to \mathcal{H}' , but the outcome will be independent of the labeling. The following lemmas are direct consequences of the definitions of T_C , T_L and T_0 .

Lemma 4.8. A contracting set \mathcal{C} of G with respect to \mathcal{H}' contributes a zero term to $T_C(G, e)$ unless it has type \mathcal{D} .

Lemma 4.9. A contracting set \mathcal{C} of G with respect to \mathcal{H}' contributes a zero term to $T_L(G, e)$ unless it has type \mathcal{E} .

Lemma 4.10. A contracting set \mathcal{C} of G with respect to \mathcal{H}' contributes a zero term to $T_0(G, e)$ unless it has type zero.

Next we define the pointed universal relative Tutte polynomials which will assume the roles played by $T(G/e)$ and by $T(G - e)$ respectively in [11, 12].

Definition 4.11. We define the pointed universal relative Tutte polynomials $T_{\mathcal{H},/}^{\Lambda'_0}(G, e)$ and $T_{\mathcal{H},-}^{\Lambda'_0}(G, e)$, respectively, as $T_{\mathcal{H}}^{\Lambda_0}(G/e) - \pi_+T_{\mathcal{H},0}^{\Lambda'_0}(G, e)$ and $T_{\mathcal{H}}^{\Lambda_0}(G - e) - \pi_-T_{\mathcal{H},0}^{\Lambda'_0}(G, e)$, respectively.

Remark 4.12. Notice that although the type zero contracting sets of $G - e$ are exactly those type zero contracting sets that make non-zero contributions in $\pi_-T_{\mathcal{H},0}^{\Lambda'_0}(G, e)$, they may not make the same contributions in $T_{\mathcal{H}}^{\Lambda_0}(G - e)$ and $\pi_-T_{\mathcal{H},0}^{\Lambda'_0}(G, e)$. The reason is that in $T_{\mathcal{H}}^{\Lambda_0}(G - e)$, the edge e is removed first while in $\pi_+T_{\mathcal{H},0}^{\Lambda'_0}(G, e)$ the edge e is removed last. This means that the contributions of the type zero contracting sets to $T_{\mathcal{H}}^{\Lambda_0}(G - e)$ may not cancel with the contributions of the corresponding type zero contracting sets to $\pi_+T_{\mathcal{H},0}^{\Lambda'_0}(G, e)$. In general, $T_{\mathcal{H},/}^{\Lambda'_0}(G, e)$ may even contain negative terms. The graph on the left side of Figure 1 shows such an example. We will leave it to our reader to verify that $T_{\mathcal{H}}^{\Lambda_0}(G - e) = X_\mu z_{[\Gamma_b]}$ and $\pi_-T_{\mathcal{H},0}^{\Lambda'_0}(G, e) = x_\mu z_{[\Gamma_b]}$, here Γ_b is the graph that consists of a single zero edge that is a bridge. Thus $T_{\mathcal{H},-}^{\Lambda'_0}(G, e) = (X_\mu - x_\mu)z_{[\Gamma_b]}$. The situation for $T_{\mathcal{H},/}^{\Lambda'_0}(G, e)$ is similar. For the

graph G shown on the right side of Figure 1, we have $T_{\mathcal{H}}^{\Lambda_0}(G/e) = Y_\mu z_{[\Gamma_b]}$ while $\pi/T_{\mathcal{H},0}^{\Lambda'_0}(G, e) = y_\mu z_{[\Gamma_b]}$, where Γ_b is the graph that consists of a single zero loop edge. Thus $T_{\mathcal{H},/}^{\Lambda'_0}(G, e) = (Y_\mu - y_\mu)z_{[\Gamma_b]}$.



FIGURE 1. Simple examples of graphs G with the property that $T_{\mathcal{H},/}^{\Lambda'_0}(G, e)$ (left) or $T_{\mathcal{H},-}^{\Lambda'_0}(G, e)$ (right) contains negative terms.

Of course, in the case when $\mathcal{H} = \emptyset$, we have $T_{\mathcal{H},/}^{\Lambda'_0}(G, e) = T(G/e) \cdot z_{[\bullet]}$ and $T_{\mathcal{H},-}^{\Lambda'_0}(G, e) = T(G-e) \cdot z_{[\bullet]}$. Again, in situations where there will be no confusion about the sets \mathcal{H} , Λ and Λ_0 , we will use $T_/(G, e)$ and $T_-(G, e)$ as the abbreviations for $T_{\mathcal{H},/}^{\Lambda'_0}(G, e)$ and $T_{\mathcal{H},-}^{\Lambda'_0}(G, e)$ respectively.

We conclude this section with a generalization of [12, Theorem 2] and of its consequences. We will need this result to justify why the *regular Brylawski homomorphism*, to be introduced in Section 6, is well-defined.

Theorem 4.13. *The pointed universal Λ_0 -colored relative Tutte polynomials $T_C(G, e)$, $T_L(G, e)$, $T_/(G, e)$, and $T_-(G, e)$ satisfy the following two identities for any $\mu \in \Lambda$:*

$$(4.1) \quad x_\mu (T_/(G, e) - T_C(G, e)) = (Y_\mu - y_\mu)T_L(G, e),$$

$$(4.2) \quad y_\mu (T_-(G, e) - T_L(G, e)) = (X_\mu - x_\mu)T_C(G, e).$$

Proof. (4.1) is proved in a way that is analogous to the establishment of equation (5) in the proof of [12, Theorem 2]. Because of the presence of the zero edges, the proof is harder here and we choose to provide a detailed proof for our reader. By the definition of $T_/(G, e)$, what we need to prove is

$$(4.3) \quad x_\mu (T(G/e) - \pi/T_0(G, e) - T_C(G, e)) = (Y_\mu - y_\mu)T_L(G, e).$$

Notice that in order to compute each of the three polynomials on the left side of (4.3), we only need to consider type \mathcal{D} contracting sets with respect to \mathcal{H}' . For the calculation of $T_C(G, e)$ this observation is stated in Lemma 4.8. To calculate $T(G/e)$ and $\pi/T_0(G, e)$ we need to sum over contracting sets \mathcal{C} of G with respect to \mathcal{H}' that have the property that contracting all edges of \mathcal{C} does not turn e into a loop. By the converse of Proposition 4.4, stated in Remark 4.6, these are exactly the type \mathcal{D} contracting sets.

Let \mathcal{C} be a type \mathcal{D} contracting set and let $f \in \mathcal{D}$ be a regular edge in the corresponding deleting set. Let us call the pair (\mathcal{C}, f) a *special pair* if $\mathcal{C} \cup \{e, f\}$ contains a cycle $C(\mathcal{C}, f)$ containing e and f has the smallest label in $C(\mathcal{C}, f) \setminus \{e\}$. Observe that the cycle $C(\mathcal{C}, f)$ is unique since $\mathcal{C} \cup \{e\}$ contains no cycle (equivalently, \mathcal{C} is a contracting set of G/e with respect to \mathcal{H}), thus $\mathcal{C} \cup \{f\}$ contains at most one cycle. Furthermore f is externally active in G/e exactly if it belongs to a special pair (\mathcal{C}, f) .

Let us consider a special pair (\mathcal{C}, f) . As seen in [10, Lemma 3.7], $\mathcal{C}' = \mathcal{C} \cup \{f\}$ is a contracting set of G that does not contain e . Thus \mathcal{C}' is also a contracting set of $G - e$ and, by the external activity

of f with respect to \mathcal{C} in G/e , the edge f is the element on the cycle $C(\mathcal{C}, f)$ with the smallest label, so it will be internally active in $G - e$ but internally inactive in the computation of $T_L(G, e)$ (since in the latter case e is deleted last). By the converse of Proposition 4.3, stated in Remark 4.6, \mathcal{C}' has type \mathcal{C} .

Conversely, let \mathcal{C}' be a type \mathcal{C} contracting set of G with respect to \mathcal{H}' . By Proposition 4.4, the set \mathcal{C}' is a contracting set of $G - e$ with respect to \mathcal{H} and e closes a cycle (denoted by $C(\mathcal{C}', e)$) with some edges from \mathcal{C}' . Again, the cycle $C(\mathcal{C}', e)$ is unique. Let f be the element in the set $\mathcal{C}' \cap C(\mathcal{C}', e)$ with the smallest label. We may use [10, Lemma 3.7] again to see that $\mathcal{C}' \setminus \{f\} \cup \{e\}$ is a contracting set of G with respect to \mathcal{H} containing e and it is easy to check that $\mathcal{C} := \mathcal{C}' \setminus \{f\}$ forms a special pair (\mathcal{C}, f) with f . We thus obtain a bijection between the type \mathcal{C} contracting sets \mathcal{C}' of $G \setminus e$ and the special pairs (\mathcal{C}, f) of G .

Let \mathcal{C} be a type \mathcal{D} contracting set of G with respect to \mathcal{H}' such that there is no edge f in the corresponding deleting set with the property that (\mathcal{C}, f) is a special pair. By Proposition 4.4, the set \mathcal{C} is also a contracting set of G/e with respect to \mathcal{H} . Also, contracting e first or last does not affect the activities of the edges in \mathcal{C} and \mathcal{D} . If we contract all edges in \mathcal{C} and delete all edges in \mathcal{D} first, e will not be a loop in the resulting graph $\Gamma(\mathcal{C}, e)$. If e is a bridge in $\Gamma(\mathcal{C}, e)$, then \mathcal{C} makes a contribution in $T_C(G, e)$ (after e is contracted in $\Gamma(\mathcal{C}, e)$). If e is not a bridge in $\Gamma(\mathcal{C}, e)$, then it forms a cycle with some zero edges in $\Gamma(\mathcal{C}, e)$. In this case \mathcal{C} makes a contribution to $\pi/T_0(G, e)$ after e is contracted in $\Gamma(\mathcal{C}, e)$. So, in the case that there are no special pairs (\mathcal{C}, f) , then either all edges make the same contributions to $T(G/e)$ and $T_C(G, e)$ (if e is a bridge in $\Gamma(\mathcal{C}, e)$), or all edges make the same contributions to $T(G/e)$ and $\pi/T_0(G, e)$ (if e is a not bridge in $\Gamma(\mathcal{C}, e)$). Thus for all contracting sets \mathcal{C} that do not form any special pairs with edges from their corresponding deleting sets \mathcal{D} , their total contribution to $T(G/e) - T_{0,/}(G, e) - T_C(G, e)$ is zero.

Now assume that \mathcal{C} is a type \mathcal{D} contracting set of G with respect to \mathcal{H}' that forms special pairs with some edges from \mathcal{D} . Without loss of generality, assume that $f_1, \dots, f_k \in \mathcal{D}$ are all the edges that form special pairs with \mathcal{C} and that they have been listed in the increasing order according to their labels. Furthermore, let us assume that the color of f_i is μ_i . Each f_i is externally active in G/e hence their total contribution to $T(G/e)$ is $\prod_{i=1}^k Y_{\mu_i}$. However, in the computation of $T_C(G, e)$ or $\pi/T_0(G, e)$, e is to be contracted last (hence it has the smallest label) so the total contribution of f_1, f_2, \dots, f_k to $T_C(G, e)$ or to $T_{0,/}(G, e)$ is $\prod_{i=1}^k y_{\mu_i}$. All other edges have the same contributions to both polynomials, since their activities are the same, whether e is contracted first or last. Thus the combined contribution of \mathcal{C} and \mathcal{D} to the left hand side of (4.3) is $x_\mu \left(\prod_{i=1}^k Y_{\mu_i} - \prod_{i=1}^k y_{\mu_i} \right) z_{[\Gamma(\mathcal{C})]}$ times the product of the weights of the edges that are different from $\{f_1, \dots, f_k\}$, where $\Gamma(\mathcal{C})$ is the graph of zero edges obtained after contracting the edges in $\mathcal{C} \cup \{e\}$ and deleting the edges in the corresponding deleting set \mathcal{D} . The term $(\prod_{i=1}^k y_{\mu_i}) z_{[\Gamma(\mathcal{C})]}$ appears either in $T_C(G, e)$ or in $\pi/T_0(G, e)$, but it makes no difference in our argument. By [12, Lemma 3],

$$x_\mu \left(\prod_{i=1}^k Y_{\mu_i} - \prod_{i=1}^k y_{\mu_i} \right) = (Y_\mu - y_\mu) \sum_{i=1}^k x_{\mu_i} \prod_{j=1}^{i-1} Y_{\mu_j} \prod_{j=i+1}^k y_{\mu_j}.$$

Thus it is sufficient to prove the following:

- (1) The graph $\Gamma(\mathcal{C}_i) - e$, obtained by contracting edges in \mathcal{C}_i and deleting e and the edges in \mathcal{D} , is (vertex pivot equivalent to) $\Gamma(\mathcal{C}, e)/e$.

- (2) The product of the weights of the edges f_1, \dots, f_k in the contribution of $\mathcal{C}_i = \mathcal{C} \cup \{f_i\}$ to $T_L(G, e)$ is $x_{\mu_i} \prod_{j=1}^{i-1} Y_{\mu_j} \prod_{j=i+1}^k y_{\mu_j}$.
- (3) The weight of any edge $f \notin \{e, f_1, \dots, f_k\}$ is the same in the contribution of \mathcal{C} to $T(G/e)$ as in the contribution of any $\mathcal{C}_i = \mathcal{C} \cup \{f_i\}$ to $T_L(G, e)$.

The first statement above is true since f_i and e belong to the same cycle in which the other edges are all from \mathcal{C} . Thus deleting any one of the edges in the cycle and contracting the rest has the same effect. The vertices of all the edges involved become one single vertex.

Since $f_i \in \mathcal{C}_i$ and it belongs to $C(\mathcal{C}_i, e)$, it contributes a factor of x_{μ_i} to $T_L(G, e)$ by the exception rule (since e is deleted last, f_i can never become a bridge). Since $C(\mathcal{C}_i, e)$ and $C(\mathcal{C}_j, e)$ both contain e , one can show that for any $i \neq j$, there exist a unique cycle $C(\mathcal{C}_i, \mathcal{C}_j)$, consisting of f_i, f_j and edges from $C(\mathcal{C}_i, e) \cup C(\mathcal{C}_j, e)$, but not e (see [12, Lemma 2]). Furthermore, if $i < j$, then the label of f_i is smaller than those of the other edges in this cycle. Since $f_j \in \mathcal{D}_i$ and $j > i$, f_j has a larger label (recall that the labels of f_1, f_2, \dots, f_k are in increasing order by our choice), it is externally inactive. But if $j < i$, then f_j has the smallest label among the edges in $C(\mathcal{C}_i, \mathcal{C}_j)$ so it is externally active. Thus the product of the weights of the edges f_1, \dots, f_k in the contribution of $\mathcal{C}_i = \{f_i\} \cup \mathcal{C} \setminus \{e\}$ to $T_L(G, e)$ is $x_{\mu_i} \prod_{j=1}^{i-1} Y_{\mu_j} \prod_{j=i+1}^k y_{\mu_j}$. This proves the second statement above.

To prove the third statement, observe that a regular edge $f \notin \{e, f_1, \dots, f_k\}$ either belongs to \mathcal{C} , in which case it would belong to all contracting sets \mathcal{C}_i , or it belongs to \mathcal{D} , in which case it belongs to \mathcal{D}_i for each i , where \mathcal{D}_i is the deleting set corresponding to \mathcal{C}_i . Consider first the case $f \in \mathcal{C}$, i.e., $f \in \mathcal{C}_i$ for all i . For each i we have either $f \in C(\mathcal{C}_i, e)$ or $f \notin C(\mathcal{C}_i, e)$. If $f \in C(\mathcal{C}_i, e)$ then f is internally inactive with respect to \mathcal{C} since f has a label larger than that of f_i . It is also internally inactive with respect to \mathcal{C}_i in the computation of $T_L(G, e)$ since e is considered as an edge in the corresponding deleting set and it has the smallest label (among all edges). If $f \notin C(\mathcal{C}_i, e)$ then its activity is the same with respect to \mathcal{C} or with respect to \mathcal{C}_i , since its activity is determined by comparing its label with the labels of edges in the deleting set that are on cycles containing f , yet e and f_i are not on such cycles. Thus a regular edge $f \in \mathcal{C} \setminus \{e, f_1, \dots, f_k\}$ has the same weight in the contribution of \mathcal{C} to $T(G/e)$ and in the contribution of \mathcal{C}_i to $T_L(G, e)$. In the second case, $f \in \mathcal{D}$, hence $f \in \mathcal{D}_i$ holds for all i . Here \mathcal{D} , respectively \mathcal{D}_i is the deleting set corresponding to \mathcal{C} , respectively \mathcal{C}_i . In this case f either does not close any cycle with edges from $\mathcal{C} \cup \{e\}$ that contains e , or it closes such a cycle but it does not have the smallest label compared to other edges from \mathcal{C} on this cycle (since it is not one of the f_j 's). If f does not close any cycle with edges from $\mathcal{C} \cup \{e\}$ that contains e , then f will not close any cycle with edges from \mathcal{C}_i that also contains f_i , hence the determination of its activity does not involve e or f_i , and it has the same activity with respect to \mathcal{C} and with respect to \mathcal{C}_i . Assume finally f closes a cycle $C(\mathcal{C}, f)$ with some edges from $\mathcal{C} \cup \{e\}$ and e is on this cycle but f does not have the smallest label among the edges on this cycle. Then f is externally inactive with respect to \mathcal{C} . In this case f also closes a (unique) cycle with edges from \mathcal{C}_i that contains f_i . Denote this cycle $C(\mathcal{C}_i, f)$. Let $g \in C(\mathcal{C}, f)$ be an edge with label smaller than that of f . Then one can show that either $g \in C(\mathcal{C}_i, f)$ or $g \in C(\mathcal{C}, f_i)$. If $g \in C(\mathcal{C}_i, f)$, then f is externally inactive since g has a smaller label. If $g \in C(\mathcal{C}, f_i)$, then the label of f_i is smaller than that of f since f_i has the smallest label among the edges of $C(\mathcal{C}, f_i)$ (which contains g). So f is again externally inactive. To summarize, in all cases, f has the same weight in (the contribution of \mathcal{C} to) $T(G/e)$ and in (the contribution of \mathcal{C}_i to) $T_L(G, e)$.

Equation (4.2) is a direct generalization of equation (6) in [12, Theorem 2]. In [12] we invoked matroid duality to derive this equation from the preceding one. We want to avoid doing so this time since the presence of zero edges makes questions of duality less clear, and since, in an effort to state our results in a language that is more directly applicable in knot theory, we avoided stating the matroid

theoretic generalizations. Fortunately there is another easy way to show (4.2) after having shown (4.1): it suffices to prove the validity of the *sum* of the two equations, which is equivalent to

$$x_\mu T_/(G, e) + y_\mu T_-(G, e) = X_\mu T_C(G, e) + Y_\mu T_L(G, e).$$

After adding $x_\mu(T_{\mathcal{H}}^{\Lambda_0}(G/e) - T_/(G, e)) + y_\mu(T_{\mathcal{H}}^{\Lambda_0}(G - e) - T_-(G, e))$ to both sides, we obtain the equivalent equation

$$\begin{aligned} & x_\mu T_{\mathcal{H}}^{\Lambda_0}(G/e) + y_\mu T_{\mathcal{H}}^{\Lambda_0}(G - e) \\ &= X_\mu T_C(G, e) + x_\mu(T_{\mathcal{H}}^{\Lambda_0}(G/e) - T_/(G, e)) \\ &+ Y_\mu T_L(G, e) + y_\mu(T_{\mathcal{H}}^{\Lambda_0}(G - e) - T_-(G, e)) \\ (4.4) \quad &= X_\mu T_C(G, e) + x_\mu \pi_+/T_0(G, e) + Y_\mu T_L(G, e) + y_\mu \pi_-T_0(G, e). \end{aligned}$$

Let G' be the colored graph that is identical to G , except that the edge e is colored with color μ instead of ν . Let us consider $T_{\mathcal{H}}^{\Lambda_0}(G')$, whose definition is labeling independent. We will show that both sides of (4.4) equal to $T_{\mathcal{H}}^{\Lambda_0}(G')$. First, by Corollary 2.9, $T_{\mathcal{H}}^{\Lambda_0}(G') = x_\mu T_{\mathcal{H}}^{\Lambda_0}(G/e) + y_\mu T_{\mathcal{H}}^{\Lambda_0}(G - e)$ if we contract and delete e first, since e is neither a bridge nor a loop. That is, the left side of (4.4) is equal to $T_{\mathcal{H}}^{\Lambda_0}(G')$. Next, let us now select any proper labeling such that the label of e is the smallest, so e will be the last edge to be contracted and/or deleted for each given contracting set \mathcal{C} . For each type \mathcal{C} contracting set \mathcal{C} with respect to \mathcal{H}' , e becomes a loop after all edges of \mathcal{C} have been contracted, hence it will contribute a Y_μ term at the end. By Lemma 4.9, the collection of all such contracting sets are exactly those that make non-zero contributions to $T_L(G, e)$, thus the combined contributions of all such contracting sets yield $Y_\mu T_C(G, e)$. Similarly, the combined contributions of all type \mathcal{D} contracting sets \mathcal{C} of G with respect to \mathcal{H}' yield exactly $X_\mu T_L(G, e)$ by Lemma 4.8. Finally, for each type zero contracting set \mathcal{C} , the edge e becomes neither a bridge nor a loop, after all edges in \mathcal{C} have been contracted and all edges in \mathcal{D} have been deleted. In this case \mathcal{C} contributes a term to $T_0(G, e)$. If, in the last step, e is contracted, we obtain a term x_μ and \mathcal{C} makes a contribution to $\pi_+/T_0(G, e)$ by the definition of $\pi_+/T_0(G, e)$. Similarly, if e is deleted in the last step, we get the expected y_μ term and \mathcal{C} makes a non-zero contribution to $\pi_-T_0(G, e)$. Combining the above, we see that the right side of (4.4) is also equal to $T_{\mathcal{H}}^{\Lambda_0}(G')$, hence establishing the equality of (4.4). \square

In analogy to equations (8) and (9) in [12], equations (4.1) and (4.2) may be restated as

$$(4.5) \quad \det \begin{pmatrix} T_L(G, e) & T_C(G, e) \\ x_\lambda & y_\lambda \end{pmatrix} = \det \begin{pmatrix} T_L(G, e) & T_/(G, e) \\ x_\lambda & Y_\lambda \end{pmatrix}$$

and

$$(4.6) \quad \det \begin{pmatrix} T_L(G, e) & T_C(G, e) \\ x_\lambda & y_\lambda \end{pmatrix} = \det \begin{pmatrix} T_-(G, e) & T_C(G, e) \\ X_\lambda & y_\lambda \end{pmatrix}.$$

Remark 4.14. The analogue of Theorem 2.7 in [11] (and in [12]) is used to prove that the definition of the pointed Tutte polynomials $T_C(G, e)$ and $T_L(G, e)$ is independent of the labeling, see [12, Corollary 2]. This time we do not prove labeling-independence of our pointed relative Tutte polynomials, since it is obvious from the definition. Note that this also applies to the special case when $\mathcal{H} = \emptyset$. Thus the labeling independence of the polynomials $T_C(G, e)$ and $T_L(G, e)$ defined in [11, 12] is also a consequence of the labeling independence of the relative Tutte polynomial shown in [10].

5. CONTRACTING SETS IN A TENSOR PRODUCT OF GRAPHS HAVING ZERO EDGES

A crucial idea behind proving the main results [11, Theorem 5.1] and [12, Theorem 3], providing a formula for the Tutte polynomial of a tensor product $G_1 \otimes_\lambda G_2$ of a colored connected graph G_1 with a pointed colored connected graph G_2 (with distinguished edge e) was to understand the composite structure of a spanning tree of $G_1 \otimes_\lambda G_2$ in terms of considering an induced spanning tree of G_1 and a collection of spanning trees of $G_2 - e$ and G_2/e . In this section we generalize this description to understanding the composite structure of a contracting set and of the corresponding deleting set in a tensor product $G_1 \otimes_\lambda G_2$ where both graphs may have zero edges.

Let Λ be a color set, $\nu \notin \Lambda$ is a distinguished color and $\Lambda' = \Lambda \cup \{\nu\}$. From now on we assume that G_1 is a Λ -colored graph, together with a set of zero edges $\mathcal{H}_1 \subset E(G_1)$ which form a Λ_0 -colored subgraph for some $\Lambda_0 \subset \Lambda$. Assume that the color $\lambda \in \Lambda \setminus \Lambda_0$ appears in $E(G_1)$ as a color of regular edges only. Let G_2 be a pointed Λ' -colored graph with a distinguished edge e that is neither a loop nor a bridge, together with a set of zero edges $\mathcal{H}_2 \subseteq E(G_2) \setminus \{e\}$, which form a Λ_0 -colored subgraph. To simplify our arguments, we will assume that no edge of G_2 has color λ , e is colored with ν and that no other edges in $E(G_2)$ are colored with ν .

As in [11] and [12] we define the λ -colored tensor product $G_1 \otimes_\lambda G_2$ as the graph obtained as follows. We associate a distinct copy G_2^f of G_2 to each edge f of color λ in G_1 by identifying the edge f with the copy e_f of e in G_2^f , and then removing the identified edges e_f and f . In particular, if f is a loop, then we will identify the endpoints of e_f in G_2^f (and remove e_f). The resulting graph will contain the edges of \mathcal{H}_1 and several copies of the edges of \mathcal{H}_2 . We define the set \mathcal{H} of zero edges of $G_1 \otimes_\lambda G_2$ as the set of all edges belonging to \mathcal{H}_1 or any copy of \mathcal{H}_2 .

Let us fix a contracting set \mathcal{C} of $G_1 \otimes_\lambda G_2$ with respect to \mathcal{H} and let \mathcal{D} be the corresponding deleting set. Let $f \in E(G_1)$ be of color λ and let G_2^f be the copy of G_2 associated to f with e_f being the corresponding distinguished edge of G_2^f . First we would like to make the following fundamental observation on the intersection of \mathcal{C} and \mathcal{D} with $E(G_2^f)$.

Proposition 5.1. *Let $\mathcal{C}_f = \mathcal{C} \cap E(G_2^f)$, $\mathcal{D}_f = \mathcal{D} \cap E(G_2^f)$ and $\mathcal{H}_f = \mathcal{H} \cap E(G_2^f)$. Then \mathcal{C}_f is a contracting set of G_2^f with respect to $\mathcal{H}_f \cup \{e_f\}$ and \mathcal{D}_f is the corresponding deleting set.*

Proof. Since \mathcal{C}_f is a subset of \mathcal{C} , it clearly does not contain any cycle. If \mathcal{D}_f contains a cocycle, then after deleting the edges of \mathcal{D}_f , there exist two vertices v_1 and v_2 that are not connected by a path in $G_2^f - \mathcal{D}_f$. Since $\mathcal{D}_f \subset \mathcal{D}$ and \mathcal{D} does not contain any cocycle in $G_1 \otimes_\lambda G_2$, there must be a path γ from v_1 to v_2 in $G_1 \otimes_\lambda G_2$. Since the only vertices where a path of $G_1 \otimes_\lambda G_2$ can leave or enter G_2^f are the endpoints of e_f , the part of γ that lies outside G_2^f must form a path γ' connecting the endpoints of e_f . Replacing γ' with e_f results in a path in $G_2^f - \mathcal{D}_f$ connecting v_1 and v_2 , a contradiction to our assumption that v_1 and v_2 that are not connected by a path in $G_2^f - \mathcal{D}_f$. \square

As a consequence of Proposition 5.1 we may use Definition 4.1 to classify the pairs $(\mathcal{C}_f, \mathcal{D}_f)$ into type \mathcal{C} , type \mathcal{D} and type zero. Using this classification we define an *induced partition* $(\mathcal{C}_1, \mathcal{D}_1, \widehat{\mathcal{H}}_1)$ of $E(G_1)$ as follows:

- (i) $f \in \mathcal{C}_1$ if the color of f is not λ and $f \in \mathcal{C}$, or the color of f is λ and $(\mathcal{C}_f, \mathcal{D}_f)$ has type \mathcal{C} ;

- (ii) $f \in \mathcal{D}_1$ if the color of f is not λ and $f \in \mathcal{D}$, or the color of f is λ and $(\mathcal{C}_f, \mathcal{D}_f)$ has type \mathcal{D} ;
- (iii) $f \in \widehat{\mathcal{H}}_1$ if the color of f is not λ and $f \in \mathcal{H}_1$, or the color of f is λ and $(\mathcal{C}_f, \mathcal{D}_f)$ has type zero.

Proposition 5.2. *Let \mathcal{C} be a contracting set of $G_1 \otimes_\lambda G_2$ with respect to \mathcal{H} and let \mathcal{D} be the corresponding deleting set. Let $(\mathcal{C}_1, \mathcal{D}_1, \widehat{\mathcal{H}}_1)$ be the induced partition of $E(G_1)$. Then \mathcal{C}_1 is a contracting set of G_1 with respect to $\widehat{\mathcal{H}}_1$, and \mathcal{D}_1 is the corresponding deleting set.*

Proof. Assume, by way of contradiction, that \mathcal{C}_1 contains a cycle $C = \{f_1, \dots, f_k\}$. After replacing each f_i of color λ with a path in \mathcal{C} connecting the endpoints of the distinguished edge e in the associated type \mathcal{C} copy of G_2 we obtain a cycle in \mathcal{C} , a contradiction. Therefore \mathcal{C}_1 cannot contain any cycle. We obtain a similar contradiction if we assume that \mathcal{D}_1 contains a cocycle $\{f_1, \dots, f_k\}$, after replacing each f_i of color λ with a minimal set of edges belonging to \mathcal{D} in the associated type \mathcal{D} copy of G_2 . \square

We conclude this section with the converse of Proposition 5.2. Consider a colored graph G_1 and a pointed colored graph G_2 subject to the assumptions made at the beginning of this section. Let \mathcal{H}_λ be a subset of the λ colored edges of G_1 and let $\widehat{\mathcal{H}}_1 = \mathcal{H}_1 \cup \mathcal{H}_\lambda$. Let \mathcal{C}_1 be a contracting set of G_1 with respect to $\widehat{\mathcal{H}}_1$ and let \mathcal{D}_1 be the corresponding deleting set. For each edge $f \in E(G_1)$ of color λ , let G_2^f be the copy of G_2 associated to f in $G_1 \otimes_\lambda G_2$ with e_f being the distinguished edge. Let us select a contracting set \mathcal{C}_f (and the corresponding deleting set \mathcal{D}_f) of G_2^f respect to $\mathcal{H}_f \cup \{e_f\}$ in the following way: if $f \in \mathcal{C}_1$, we select a pair $(\mathcal{C}_f, \mathcal{D}_f)$ of type \mathcal{C} , if $f \in \mathcal{D}_1$, we select a pair $(\mathcal{C}_f, \mathcal{D}_f)$ of type \mathcal{D} and if $f \in \mathcal{H}_\lambda$, we select a pair $(\mathcal{C}_f, \mathcal{D}_f)$ of type zero. Let \mathcal{C} be the union of all edges in \mathcal{C}_1 whose color is not λ and of all the sets \mathcal{C}_f and let \mathcal{D} be the union of all edges in \mathcal{D}_1 not whose color is not λ and of all the sets \mathcal{D}_f .

Theorem 5.3. *The edge set \mathcal{C} defined above is a contracting set with respect to \mathcal{H} and \mathcal{D} is the corresponding deleting set.*

Proof. Clearly, $\mathcal{D} = E(G_1 \otimes_\lambda G_2) \setminus (\mathcal{C} \cup \mathcal{H})$, so we only need to prove that \mathcal{C} contains no cycle and \mathcal{D} contains no cocycle.

Assume, by way of contradiction, that \mathcal{C} contains a cycle C . This cycle cannot be contained entirely in a copy G_2^f of G_2 since no set \mathcal{C}_f contains a cycle. Thus \mathcal{C}_f must be either the empty set or a path γ_f connecting the endpoints of e_f . In the latter case $f \in E(G_1)$ must belong to \mathcal{C}_1 since $\gamma_f \cup \{e_f\}$ forms a cycle hence is of type \mathcal{C} . After replacing each such path γ_f with the edge $f \in \mathcal{C}_1$ we obtain a cycle contained in \mathcal{C}_1 , in contradiction with \mathcal{C}_1 being a contracting set.

To show that \mathcal{D} contains no cocycle it suffices to show that for every edge $g \in \mathcal{D}$ there is a walk contained in $\mathcal{C} \cup \mathcal{H}$ connecting the endpoints of g . We will have two cases, depending on whether g belongs to $E(G_1)$ or it belongs to a copy G_2^f of G_2 . Consider first the case $g \in E(G_1)$. Then $g \in \mathcal{D}_1$, and there is a path $\gamma_1(g)$ contained in $\mathcal{C}_1 \cup \widehat{\mathcal{H}}_1$ connecting the endpoints of g in G_1 . If the color of an edge h in $\gamma_1(g)$ is not λ then this edge also belongs to $E(G_1 \otimes_\lambda G_2)$. If the color of $h \in \gamma_1(g)$ is λ then $(\mathcal{C}_h, \mathcal{D}_h)$ has type \mathcal{C} or zero and we may replace h with a path $\gamma(h)$ connecting the endpoints of h in $\mathcal{C} \cup \mathcal{H}$. Thus we obtain a walk in $\mathcal{C} \cup \mathcal{H}$ that connects the endpoints of g . Consider finally the case when g belongs to a copy G_2^f of G_2 . Then $g \in \mathcal{D}_f$, and there is a path $\gamma_f(g)$ in $\mathcal{C}_f \cup \mathcal{H}_f \cup \{e_f\}$ connecting the endpoints of g . If $\gamma_f(g)$ does not contain e_f then all of its edges belong to $\mathcal{C} \cup \mathcal{H}$ and we are done. Thus we may assume $\gamma_f(g)$ contains e_f . If f belongs to $\mathcal{C}_1 \cup \widehat{\mathcal{H}}_1$ then, in analogy to the previous case, we may replace e_f with a path $\gamma_f(e_f)$ connecting its endpoints in $\mathcal{C}_f \cup \mathcal{H}_f$ and obtain a walk contained in $\mathcal{C} \cup \mathcal{H}$ connecting the endpoints of g . We are left with the case when f belongs to

\mathcal{D}_1 . Repeating the argument of the case $g \in E(G_1)$ for f , there is a walk connecting the endpoints of f in $\mathcal{C}_1 \cup \mathcal{D}_1$ which may be transformed into a path $\gamma(f)$ connecting the endpoints of f in $\mathcal{C} \cup \mathcal{H}$. We may replace e_f in $\gamma_f(g)$ with $\gamma(f)$ and obtain a walk connecting the endpoints of g in $\mathcal{C} \cup \mathcal{H}$. \square

6. THE TENSOR PRODUCT FORMULA

This section contains the main result of our paper. The issue here is to find a way to compute the relative Tutte polynomial of $G_1 \otimes_\lambda G_2$ in terms of the relative Tutte polynomial of G_1 and the pointed relative Tutte polynomials of G_2 via some suitable variable substitutions. In the case that there are no zero edges involved, this is done by keeping all variables of color $\mu \neq \lambda$ in $T(G_1)$ unchanged, and using the substitutions $X_\lambda \mapsto T(G_1 - e)$, $x_\lambda \mapsto T_L(G_1, e)$, $Y_\lambda \mapsto T(G_1/e)$ and $y_\lambda \mapsto T_C(G_1, e)$, see [11, Theorem 5.1] and [12, Theorem 3]. The new obstacle we face here (when there are zero edges present) is that a choice of contracting set in $G_1 \otimes_\lambda G_2$ may turn some G_2 copies into the zero types by the results we have established in the previous sections. That is, a λ colored edge in G_1 may not always be treated as a regular edge (which is either in a contracting set or a deleting set). To simplify our arguments, in this section we also assume that G_1 is a connected graph and that G_2 is a connected graph in which the pointed edge e is neither a loop nor a coloop. To distinguish an edge of color λ , treated as zero edge, from the regular edge of the same color, we will change its color to a new color $\lambda_0 \notin \Lambda$, as defined more formally in the following definition.

Definition 6.1. *Let G be any Λ -colored graph together with a set of zero edges \mathcal{H} , let $\lambda \in \Lambda$ be a color used to color regular edges only and let S be any subset of the set $E_\lambda \subseteq E(G) \setminus \mathcal{H}$ of λ -colored edges. Let $\lambda_0 \notin \Lambda$ be a new color. We define the graph G_S as the $\Lambda \cup \{\lambda_0\}$ -colored graph obtained from G by changing the color of each edge belonging to S to λ_0 .*

In analogy to [11, Theorem 5.1] and [12, Theorem 3], we will express $T_{\mathcal{H}}^{\Lambda_0}(G_1 \otimes_\lambda G_2)$ as a function of graph polynomials associated to G_2 and all the graphs of the form $G_{1,S}$, respectively. Since we are dealing with more than just the graphs G_1 and G_2 (as is the case when there are no zero edges involved), the procedure is much more complex, we will break down our process into a sequence of homomorphisms and $\mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0)$ -linear maps. The first homomorphism applied is a direct generalization of the substitutions used in [11, Theorem 5.1] and [12, Theorem 3], which generalize a transformation introduced by Brylawski [4, 5].

Definition 6.2. *Let G be a pointed Λ' -colored graph, together with a Λ_0 -colored subgraph \mathcal{H} of zero edges (recall that $\Lambda' = \Lambda \cup \{\nu\}$ and ν is the unique color for the pointed edge e). We define the regular Brylawski map $\beta_{\lambda,G} \in \text{End}(\mathcal{T}(\mathcal{R}, \Lambda, \Lambda_0))$, as the endomorphism sending each variable x_μ , X_μ , y_μ , Y_μ such that $\mu \neq \lambda$ into itself and sending X_λ into $T_-(G, e)$, x_λ into $T_L(G, e)$, Y_λ into $T_/(G, e)$ and y_λ into $T_C(G, e)$. We will use β_λ for $\beta_{\lambda,G}$ when the graph G is clear in the context of the problem.*

In analogy to [12, Lemma 4], the fact that the regular Brylawski map is well-defined is a direct consequence of equations (4.5) and (4.6). Furthermore, for each variable $z_{[\Gamma]}$ that appears in the polynomials $T_-(G, e)$, $T_L(G, e)$, $T_/(G, e)$ and $T_C(G, e)$, the corresponding graph Γ has no edge of color ν , thus β_λ indeed takes the ring $\mathcal{T}(\mathcal{R}, \Lambda, \Lambda_0)$ into itself. However, when applying β_λ to a relative Tutte polynomial of a graph with $k \geq 2$ λ colored edges, the result is a polynomial containing terms of the form $z_{[\Gamma_1]} \cdot z_{[\Gamma_2]} \cdots z_{[\Gamma_k]}$, which is not a universal relative Tutte polynomial (recall that a universal relative Tutte polynomial is a linear combination of the terms of the form $z_{[\Gamma]}$). The need to change this into a universal relative Tutte polynomial leads to the next definition.

Definition 6.3. *The regular splicing map $\sigma : \mathcal{T}(\mathcal{R}, \Lambda, \Lambda_0) \rightarrow \mathcal{T}(\mathcal{R}, \Lambda, \Lambda_0)$ is the $\mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0)$ -linear map induced by $\sigma(z_{[\Gamma_1]} \cdots z_{[\Gamma_k]}) = z_{[\Sigma([\Gamma_1], \dots, [\Gamma_k])]}$. Here $\Sigma([\Gamma_1], \dots, [\Gamma_k])$ is the connected graph obtained by the repeated splicing of all connected components of the graph $\Gamma_1 \uplus \cdots \uplus \Gamma_k$ (here \uplus stands for disjoint union).*

When we perform the vertex splicing operation on a pair of connected graphs, the resulting graph is connected and unique up to vertex pivot equivalence. It follows by induction on the number of connected components m of $\Gamma_1 \uplus \cdots \uplus \Gamma_k$ that, repeating the vertex splicing operation $m - 1$ times in such a way that each splicing operation merges vertices from different connected components, results in a connected graph which is unique up to vertex pivot equivalence. Thus σ is well defined.

For each $S \subseteq E_\lambda(G_1)$, the polynomial $\sigma\beta_\lambda(T_{\mathcal{H}_1 \cup S}^{\Lambda_0 \cup \{\lambda_0\}}(G_{1S}))$ (where $\beta_\lambda = \beta_{\lambda, G_2}$) is a $\mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0)$ -linear combination of the variables $\{z_{[\Gamma]} : [\Gamma] \in \text{VP}(\Lambda_0 \cup \lambda_0)\}$. In other words,

$$\sigma\beta_\lambda(T_{\mathcal{H}_1 \cup S}^{\Lambda_0 \cup \{\lambda_0\}}(G_{1S})) \in \bigoplus_{[\Gamma] \in \text{VP}(\Lambda_0 \cup \lambda_0)} \mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0) z_{[\Gamma]}.$$

We will use this module as the domain of the *zero Brylawski map*, to be defined below. This map is analogous to the maps introduced by Brylawski [4, 5] (and generalized in [11, Theorem 5.1] and [12, Theorem 3]) only in the sense that they are associated to replacing edges in (a recolored variant of) G_1 with copies of G_2 . Let G_2 be a graph with a distinguished (pointed) edge colored with the unique color ν . Recall that $T_0(G_2, e)$ is a $\mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0)$ -linear combination of terms of the form $z_{[\Gamma]}$ where each Γ contains exactly one edge of color ν . Thus we have $T_0(G_2, e) = \sum_{1 \leq j \leq n} p_j \cdot z_{[\Gamma_j]}$ where $p_j \in \mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0)$ and Γ_j contains exactly one ν -colored edge for each j . Let $\mathbf{P} = \{p_1, p_2, \dots, p_n\}$. For any graph Γ consisting of only zero edges and k λ_0 -colored edges, let us number its λ_0 -colored edges by $1, 2, \dots, k$ in an arbitrary way. For any choice of $q_1 = p_{j_1} \in \mathbf{P}$, $q_2 = p_{j_2} \in \mathbf{P}$, \dots , $q_k = p_{j_k} \in \mathbf{P}$, let Γ_{q_1, \dots, q_k} be the graph obtained by identifying the i -th λ_0 -colored edge in Γ with the ν -colored edge in Γ_{j_i} first, then removing the identified edge, for each $1 \leq i \leq k$ (so Γ_{q_1, \dots, q_k} is obtained through a total of k 2-sum operations on Γ).

Definition 6.4. *We define the zero Brylawski map β_{0, G_2} as the $\mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0)$ -linear map from $\bigoplus_{[\Gamma] \in \text{VP}(\Lambda_0 \cup \lambda_0)} \mathcal{T}(\mathcal{R}, \Lambda \setminus \Lambda_0) z_{[\Gamma]}$ to $\mathcal{T}(\mathcal{R}, \Lambda, \Lambda_0)$ induced by the mapping that sends each $z_{[\Gamma]}$ to the symmetric sum*

$$\sum_{q_1 \in \mathbf{P}, \dots, q_k \in \mathbf{P}} q_1 q_2 \cdots q_k \cdot z_{[\Gamma_{q_1, \dots, q_k}]}.$$

Notice that although this definition presupposes labeling the λ_0 -colored edges of Γ in some order, the end result is independent of the choice of this labeling, since the summation is symmetric hence is invariant under the permutations of the factors. Again we will use the short hand notation β_0 for β_{0, G_2} when G_2 is clear from the context of the problem.

Theorem 6.5. *The universal Λ_0 -colored relative Tutte polynomial $T_{\mathcal{H}}^{\lambda_0}(G_1 \otimes_\lambda G_2)$ is given by*

$$T_{\mathcal{H}}^{\lambda_0}(G_1 \otimes_\lambda G_2) = \sum_{S \subseteq E_\lambda(G_1)} \Phi(T_{\mathcal{H}_1 \cup S}^{\Lambda_0 \cup \{\lambda_0\}}(G_{1S})),$$

where $\Phi = \beta_0 \circ \sigma \circ \beta_\lambda$.

Proof. We generalize the proofs of [11, Theorem 5.1] and [12, Theorem 3], using the description of the contracting and deleting sets of $G_1 \otimes_\lambda G_2$ given in Section 5. The polynomial $T_{\mathcal{H}}^{\lambda_0}(G_1 \otimes_\lambda G_2)$ is the

total weight of all contracting sets \mathcal{C} of the graph $G_1 \otimes_\lambda G_2$, calculated using any proper labeling of the edges of $G_1 \otimes_\lambda G_2$ with respect to \mathcal{H} . Let us select our proper labeling in three steps as follows:

- (i) Label all edges in \mathcal{H} with zero.
- (ii) Label all regular edges (including the λ -colored edges) of G_1 with pairwise distinct positive integers that are multiples of $|E(G_2)|$.
- (iii) If the label of a λ -colored edge $f \in E_\lambda(G_1)$ is $k \cdot |E(G_2)|$ for some $k > 0$ then number the regular edges in the copy of G_2 replacing f using the elements of the set $\{(k-1)|E(G_2)| + 1, (k-1)|E(G_2)| + 2, \dots, k|E(G_2)|\}$.

We obtain a proper labeling of $G_1 \otimes_\lambda G_2$ with respect to \mathcal{H} that has the following property: if we list the regular edges of $G_1 \otimes_\lambda G_2$ in increasing order of labels, the regular edges belonging to a copy of G_2 associated to the same $f \in E_\lambda(G_1)$ form a sublist of consecutive elements. By the description given in Section 5, there is a one to one correspondence between the contracting sets \mathcal{C} of $G_1 \otimes_\lambda G_2$ and the contracting sets generated by the following three-step procedure:

- (1a) Select a subset S of $E_\lambda(G_1)$ and define $\widehat{\mathcal{H}}_1 := \mathcal{H}_1 \cup S$.
- (1b) Select a contracting/deleting set pair $\mathcal{C}_1, \mathcal{D}_1$ of G_1 with respect to $\widehat{\mathcal{H}}_1$.
- (2) For each copy G_2^f of G_2 , associated to an edge $f \in E_\lambda(G_1)$, partition the set of regular edges $E(G_2^f) \setminus (\mathcal{H}_f \cup \{f\})$ of the copy into a contracting set \mathcal{C}_f and a deleting set \mathcal{D}_f such that $(\mathcal{C}_f, \mathcal{D}_f)$ has type \mathcal{C} (or type \mathcal{D} , or type zero, respectively) exactly when $f \in \mathcal{C}_1$ (or $f \in \mathcal{D}_1$, or $f \in \widehat{\mathcal{H}}_1$, respectively).

We then define the contracting set \mathcal{C} as the union of the sets \mathcal{C}_f and of $\mathcal{C}_1 \setminus E_\lambda(G_1)$. The corresponding deleting set is the union of the sets \mathcal{D}_f and of $\mathcal{D}_1 \setminus E_\lambda(G_1)$. Note that there is no other restriction on the choices made in the three steps of the above procedure than the ones stated. If we group the weights of the contracting sets \mathcal{C} according to the choices in the above procedure, the choice made in step (1a) corresponds to summing over all subsets S of $E_\lambda(G_1)$. After fixing S , summing over all possible choices \mathcal{C}_1 in step (1b) calls for summing over the same contracting sets that are used to compute the relative Tutte polynomial $T_{\mathcal{H}_1 \cup S}^{\Lambda_0 \cup \{\lambda_0\}}(G_{1S})$. For a fixed contracting set \mathcal{C}_1 of G_{1S} with respect to $\widehat{\mathcal{H}}_1 = \mathcal{H}_1 \cup S$, step (2) calls for summing over all contracting sets \mathcal{C}_f of the same type, where the type depends on f belonging to $\mathcal{C}_1, \mathcal{D}_1$ or $\widehat{\mathcal{H}}_1$. A key observation in understanding the rest of the proof below is that we may replace steps (1a) and (1b) above with the following step.

- (1) In decreasing order of their labels, put each $f \in E_\lambda(G_1)$ into $\mathcal{C}_1, \mathcal{D}_1$ or S , subject to the following restrictions: an edge $f \in E_\lambda(G_1)$ cannot be put into \mathcal{C}_1 if it becomes a loop in G_1 after contracting all higher labeled edges of \mathcal{C}_1 , and it cannot be put into \mathcal{D}_1 if it becomes a bridge in G_1 after deleting all higher labeled edges of \mathcal{D}_1 .

The restrictions are necessary and sufficient to guarantee that the resulting \mathcal{C}_1 is a contracting set and the resulting \mathcal{D}_1 is the corresponding deleting set with respect to $\widehat{\mathcal{H}}_1 = \mathcal{H}_1 \cup S$. Furthermore, as noted in the alternative Definition 2.3, the external or internal activity of an edge in \mathcal{C}_1 or \mathcal{D}_1 is determined by whether the edge in question becomes a loop or bridge after contracting all higher labeled edges of \mathcal{C}_1 and deleting all higher labeled edges of \mathcal{D}_1 . Since the labeling on the edges of $G_1 \otimes_\lambda G_2$ is obtained by replacing each λ -colored edge f by a consecutive run of edges of G_2^f , and due to the special dependence of the pair $(\mathcal{C}, \mathcal{D})$ on the pair $(\mathcal{C}_1, \mathcal{D}_1)$, the above observation regarding activities may be extended to $G_1 \otimes_\lambda G_2$ in the following way. Consider any $f \in E_\lambda(G_1)$. Contract all

edges of \mathcal{C} and delete all edges of \mathcal{D} whose label is higher than the label of any edge in the copy G_2^f of G_2 . After performing these operations, the endpoints of f get identified if and only if f becomes a loop after contracting all higher labeled edges of \mathcal{C}_1 in G_1 . Similarly, the endpoints of f are only connected by paths in the copy G_2^f if and only if f becomes a bridge after deleting all higher labeled edges of \mathcal{D}_1 in G_1 . The first observation is true because contracting any λ -colored edge $g \in \mathcal{C}_1$ in G_1 (with higher label) identifies the endpoints of g in G_1 and the same effect is achieved by contracting all edges in \mathcal{C}_g in $G_1 \otimes_\lambda G_2$ as $(\mathcal{C}_g, \mathcal{D}_g)$ has type \mathcal{C} . Similarly, deleting any λ -colored edge $g \in \mathcal{D}_1$ in G_1 (with higher label) removes the possibility of going from one endpoint of g to the other endpoint, without visiting any other vertex of G_1 , and the same effect is achieved by deleting all edges in \mathcal{D}_g in $G_1 \otimes_\lambda G_2$, as $(\mathcal{C}_g, \mathcal{D}_g)$ has type \mathcal{D} .

We now consider three cases depending on whether an edge $f \in E_\lambda(G_1)$ becomes a loop, a bridge, or neither after contracting all higher labeled edges of \mathcal{C}_1 and deleting all higher labeled edges of \mathcal{D}_1 in step (1) above.

Case 1. $f \in E_\lambda(G_1)$ becomes neither a loop nor a bridge after contracting all higher labeled edges of \mathcal{C}_1 and deleting all higher labeled edges of \mathcal{D}_1 . In this case, f may be put into either of \mathcal{C}_1 , \mathcal{D}_1 , or S . Furthermore, if we use the contraction/deletion formula (2.6) (in the decreasing order of the labels of the edges) to compute $T(G_1 \otimes_\lambda G_2)$ then, after contracting all edges in \mathcal{C} and deleting all edges in \mathcal{D} whose label is higher than the label of the edges in G_2^f , the endpoints of f are still distinct, and there is a path outside the copy G_2^f of G_2 connecting the endpoints of f .

Subcase 1(a). $f \in \mathcal{C}_1$. Since f is inactive, it contributes a term x_λ to $T_{\mathcal{H}_1 \cup S}^{\Lambda_0 \cup \{\lambda_0\}}(G_{1S})$. The total weight of all type \mathcal{C} contracting sets of G_2^f is precisely $T_L(G_2, e)$ (Lemma 4.9).

Subcase 1(b). $f \in \mathcal{D}_1$. Since f is inactive, it contributes a term y_λ to $T_{\mathcal{H}_1 \cup S}^{\Lambda_0 \cup \{\lambda_0\}}(G_{1S})$. The total weight of all type \mathcal{D} contracting sets of G_2^f is precisely $T_C(G_2, e)$ (Lemma 4.8).

Subcase 1(c). $f \in S$. and is “inactive” in the sense that if we follow the contraction/deletion formula (in the decreasing order of the labels of the edges) The total weight of of all type zero contracting sets of G_2^f is precisely $T_0(G_2, e)$ (Lemma 4.10).

Notice that in all three cases above, replacing f by any terminal graph resulted from a contracting set of the corresponding type (through the splicing or the 2-sum operations defined in the mappings β_λ , σ and β_0) will not affect the activities of the remaining edges. Thus, 1(a) and 1(b) prove the validity of the substitutions $x_\lambda \rightarrow T_L(G_2, e)$ and $y_\lambda \rightarrow T_C(G_2, e)$, while 1(c) shows the validity of replacing a λ_0 colored edge in the terminal graph of G_{1S} by a copy of $T_0(G_2, e)$.

Case 2. $f \in E_\lambda(G_1)$ becomes a bridge after contracting all higher labeled edges of \mathcal{C}_1 and deleting all higher labeled edges of \mathcal{D}_1 . In this case, f may be put into either of \mathcal{C}_1 or S . Furthermore, if we use the contraction/deletion formula (2.6) (in the decreasing order of the labels of the edges) to compute $T(G_1 \otimes_\lambda G_2)$ then, after contracting all edges in \mathcal{C} and deleting all edges in \mathcal{D} whose label is higher than the label of the edges in G_2^f , the endpoints of f are still distinct, but there is no path outside the copy G_2^f of G_2 connecting the endpoints of f . If we choose to put f into \mathcal{C}_1 , it becomes internally active in G_{1S} thus contributing a term X_λ to $T_{\mathcal{H}_1 \cup S}^{\Lambda_0 \cup \{\lambda_0\}}(G_{1S})$.

In this case, the combined contribution of all type \mathcal{C} and type zero contracting sets of G_2^f is $T_{\mathcal{H}}^{\Lambda_0}(G_2^f - e_f) = T_{\mathcal{H}}^{\Lambda_0}(G_2^f - e_f) - \pi_- T_0(G_2^f, e_f) + \pi_- T_0(G_2^f, e_f) = T_-(G_2, e) + \pi_- T_0(G_2, e)$. Since we need to replace X_λ by a polynomial that is label independent, we will simply substitute $T_-(G_2, e)$ into X_λ . On the other hand, if $f \in S$, then we will still replace the corresponding λ_0 colored edge in the terminal graph of G_{1S} by a copy of $T_0(G_2^f, e_f)$. In this particular case, the terminal graphs obtained by using the type zero contracting sets in $\pi_- T_0(G_2^f, e_f)$ through splicing and using the corresponding type zero contracting sets in $T_0(G_2^f, e_f)$ through 2-sum operation are in fact vertex pivot equivalent (since the end points of f are cut vertices) and their total contributions are the same. Thus the combined contribution of $T_-(G_2, e)$ (for $f \in \mathcal{C}_1$) and $T_0(G_2, 0)$ (for $f \in S$) is equal to $T_{\mathcal{H}}^{\Lambda_0}(G_2^f - e_f)$, which is the correct contribution of G_2^f in this case. The key point of this substitution rule is that the λ_0 colored edges are now treated equally in terms of substitution and we have found a right substitution for X_λ that is label independent.

Case 3. $f \in E_\lambda(G_1)$ becomes a loop after contracting all higher labeled edges of \mathcal{C}_1 and deleting all higher labeled edges of \mathcal{D}_1 . In this case, f may be put into either of \mathcal{D}_1 or S . Furthermore, if we use the contraction/deletion formula (2.6) (in the decreasing order of the labels of the edges) to compute $T(G_1 \otimes_\lambda G_2)$ then, after contracting all edges in \mathcal{C} and deleting all edges in \mathcal{D} whose label is higher than the label of the edges in G_2^f , the endpoints of f become identical. If we choose to put f into \mathcal{C}_1 , it becomes externally active in G_{1S} thus contributing a term Y_λ to $T_{\mathcal{H}_1 \cup S}^{\Lambda_0 \cup \{\lambda_0\}}(G_{1S})$.

In this case the combined contribution of all type \mathcal{D} and type zero contracting sets of G_2^f is $T_{\mathcal{H}}^{\Lambda_0}(G_2^f/e_f) = T_{\mathcal{H}}^{\Lambda_0}(G_2^f/e_f) - \pi_/T_0(G_2^f, e_f) + \pi_/T_0(G_2^f, e_f) = T_/(G_2, e) + \pi_/T_0(G_2, e)$. As we did in 2(a), we will simply substitute Y_λ by $T_/(G_2, e)$. On the other hand, if $f \in S$, then we will still substitute the corresponding λ_0 colored edge in the terminal graph of G_{1S} by a copy of $T_0(G_2^f, e_f)$. Again, the terminal graphs obtained by using the type zero contracting sets in $\pi_/T_0(G_2^f, e_f)$ through splicing and using the corresponding type zero contracting sets in $T_0(G_2^f, e_f)$ through 2-sum operation are also vertex pivot equivalent (the end points of f are identified and is also a cut vertex in $G_1 \otimes_\lambda G_2$) and their total contributions are the same. Thus this substitution rule is also valid. \square

7. EXAMPLES AND ENDING REMARKS

Let us end this paper by a couple of examples and remarks.

Remark 7.1. In the case that G_1 contains zero edges but G_2 does not, there are no type zero contracting sets in G_2 and we have $T_-(G_2, e) = T(G_2 - e)$, $T_/(G_2, e) = T(G_2/e)$. In this case the substitution rule obtained in this paper is the same as the one given [12]. That is, the main result in [12] can be extended to the relative Tutte polynomial of $G_1 \otimes_\lambda G_2$ without having to modifying the definitions of the pointed Tutte polynomials of G_2 and the substitution formula.

Remark 7.2. Another extreme (and trivial) example is when G_2 consists of only two edges which are not loop edges: one is the special edge e and the other a zero edge. In this case any graph G_1 with zero edges can be obtained by color the zero edges by λ and then take the tensor product $G_1 \otimes_\lambda G_2$. In this case all pointed Tutte polynomials are zero except $T_0(G_2, e)$. Consequently, the only non-trivial substitution (as expected) happens only when the set S contains all λ -colored edges.

Next, let us use a relatively simple example to illustrate the application of Theorem 6.5. Figure 2 shows the graphs G_1 and G_2 , as well as their corresponding tensor product $G_1 \otimes_\lambda G_2$. We will assume

that the regular edges in G_2 are labeled in such a way that the top edge has the highest label and the bottom edge has the lowest label, see the numbers 1 through 3 in Figure 2.

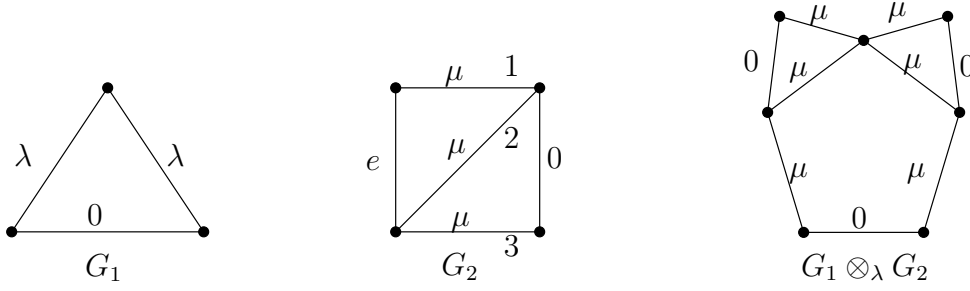


FIGURE 2. The graphs G_1 , G_2 and $G_1 \otimes_\lambda G_2$.

For the two regular edges in G_1 that are λ -colored, there are three cases: none of them is in $\widehat{\mathcal{H}}_1$, one of them is in $\widehat{\mathcal{H}}_1$ (and there are two symmetric cases here) and both are in $\widehat{\mathcal{H}}_1$. Using Corollary 2.9, we get (the details are left to our reader)

$$(7.1) \quad \sum_{S \subseteq E_\lambda(G_1)} T_{\widehat{\mathcal{H}}_1 \cup S}^{\Lambda_0 \cup \{\lambda_0\}}(G_{1S}) = x_\lambda^2 z [\text{circle with dot}] + y_\lambda (x_\lambda + X_\lambda) z [\text{circle with dot}] + 2x_\lambda z [\text{circle with dot}] + 2y_\lambda z [\text{circle with dot}] + z [\text{circle with dot}],$$

where the thicker edges in the graphs are of color λ_0 and the rest are zero edges. Next we will compute the pointed relative Tutte polynomials $T_C(G_2, e)$, $T_L(G_2, e)$, $T_0(G_2, e)$, $T_\gamma(G_2, e)$ and $T_-(G_2, e)$. To help our reader, we list all possible contracting sets and their contributions in Table 1. In the case of a type zero contracting set, the edge e in the terminal graph is marked by a thickened and dashed line. Summing up the weights listed in the table, we obtain

\mathcal{C}	Type of \mathcal{C}	Contributions
$\{1\}$	\mathcal{D}	$X_\mu y_\mu^2 z [\text{circle with dot}]$ to $T_C, T(G_2/e)$
$\{2\}$	\mathcal{D}	$x_\mu y_\mu^2 z [\text{circle with dot}]$ to $T_C, x_\mu y_\mu Y_\mu z [\text{circle with dot}]$ to $T(G_2/e)$
$\{3\}$	\mathcal{D}	$x_\mu y_\mu^2 z [\text{circle with dot}]$ to $T_C, T(G_2/e)$
$\{1, 2\}$	\mathcal{C}	$x_\mu^2 y_\mu z [\text{circle with dot}]$ to $T_L, X_\mu^2 y_\mu z [\text{circle with dot}]$ to $T(G_2 - e)$
$\{1, 3\}$	zero	$x_\mu^2 y_\mu z [\text{circle with dot}]$ to $T_0, x_\mu^2 y_\mu z [\text{circle with dot}]$ to $T(G_2/e), x_\mu y_\mu X_\mu z [\text{circle with dot}]$ to $T(G_2 - e)$
$\{2, 3\}$	\mathcal{D}	$x_\mu^2 y_\mu z [\text{circle with dot}]$ to $T_C, x_\mu^2 Y_\mu z [\text{circle with dot}]$ to $T(G_2/e)$
$\{1, 2, 3\}$	\mathcal{C}	$x_\mu^3 z [\text{circle with dot}]$ to $T_L, X_\mu x_\mu^2 z [\text{circle with dot}]$ to $T(G_2 - e)$

TABLE 1. Contributions of the contracting sets to the pointed relative Tutte polynomials associated to G_2 .

$$\begin{aligned} T(G_2 - e) &= x_\mu^2 X_\mu z [\text{circle with dot}] + y_\mu X_\mu (x_\mu + X_\mu) z [\text{circle with dot}], \\ T(G_2/e) &= x_\mu^2 (Y_\mu + y_\mu) z [\text{circle with dot}] + y_\mu (x_\mu y_\mu + x_\mu Y_\mu + y_\mu X_\mu) z [\text{circle with dot}], \\ T_0(G_2, e) &= x_\mu^2 y_\mu z [\text{circle with dot}] \end{aligned}$$

and

$$\begin{aligned}
T_C(G_2, e) &= x_\mu^2 y_\mu z[\text{⊙}] + y_\mu^2 (2x_\mu + X_\mu) z[\text{⦿}] \\
T_J(G_2, e) &= T(G_2/e) - \pi/T_0(G_2, e) \\
&= x_\mu^2 Y_\mu z[\text{⊙}] + (X_\mu y_\mu^2 + x_\mu y_\mu Y_\mu + x_\mu y_\mu^2) z[\text{⦿}], \\
T_L(G_2, e) &= x_\mu^3 z[\text{⊙}] + x_\mu^2 y_\mu z[\text{⦿}], \\
T_-(G_2, e) &= T(G_2 - e) - \pi_- T_0(G_2, e) \\
&= x_\mu^2 X_\mu z[\text{⊙}] + (x_\mu X_\mu + X_\mu^2 - x_\mu^2) y_\mu z[\text{⦿}].
\end{aligned}$$

We can now apply the mapping $\Phi = \beta_0 \circ \sigma \circ \beta_\lambda$ to $\sum_{S \subseteq E_\lambda(G_1)} T_{\mathcal{H}_1 \cup S}^{\Lambda_0 \cup \{\lambda_0\}}(G_{1S})$ in an almost term by term fashion. The final answer contains 7 different vertex pivot equivalent classes of graphs (with only zero edges). For example, to compute $\Phi(2x_\lambda z[\text{⦿}])$, we would replace x_λ by $T_L(G_2, e)$, apply the regular splicing map σ , and perform a two sum operation on the resulting graph with Γ_2^3 . This leads to

$$\Phi(2x_\lambda z[\text{⦿}]) = 2x_\mu^5 y_\mu z[\text{⊙}] + 2x_\mu^4 y_\mu^2 z[\text{⦿}].$$

Similarly,

$$\begin{aligned}
\Phi(x_\lambda^2 z[\text{⊙}]) &= x_\mu^6 z[\text{⊙}] + 2x_\mu^5 y_\mu z[\text{⦿}] + x_\mu^4 y_\mu^2 z[\text{⦿}], \\
\Phi(y_\lambda(x_\lambda + X_\lambda) z[\text{⦿}]) &= x_\mu^4 y_\mu (x_\mu + X_\mu) z[\text{⦿}] + x_\mu^2 y_\mu^2 (2x_\mu^2 + 4X_\mu x_\mu + 2X_\mu^2) z[\text{⦿}] \\
&\quad + X_\mu y_\mu^3 (2x_\mu^2 + 3x_\mu X_\mu + X_\mu^2) z[\text{⦿}], \\
\Phi(2y_\lambda z[\text{⦿}]) &= 2x_\mu^4 y_\mu^2 z[\text{⦿}] + 2x_\mu^2 y_\mu^3 (2x_\mu + X_\mu) z[\text{⦿}], \\
\Phi(z[\text{⦿}]) &= x_\mu^4 y_\mu^2 z[\text{⦿}].
\end{aligned}$$

We leave the verification of the details to our reader. Summing up the previous equations yields

$$\begin{aligned}
T_{\mathcal{H}}^\Lambda(G_1 \otimes_\lambda G_2) &= x_\mu^6 z[\text{⊙}] + x_\mu^4 y_\mu (3x_\mu + X_\mu) z[\text{⦿}] + x_\mu^2 y_\mu^2 (5x_\mu^2 + 4x_\mu X_\mu + 2X_\mu^2) z[\text{⦿}] \\
&\quad + y_\mu^3 (4x_\mu^3 + 4x_\mu^2 X_\mu + 3x_\mu X_\mu^2 + X_\mu^3) z[\text{⦿}] + 2x_\mu^5 y_\mu z[\text{⊙}] + 2x_\mu^4 y_\mu^2 z[\text{⦿}] + x_\mu^4 y_\mu^2 z[\text{⦿}].
\end{aligned}$$

As an exercise, we encourage our reader to verify this result by direct contraction/deletion computation using the recursive formula (2.6), keeping in mind Definition 2.3.

We end our paper with the following remark. In [10] we showed that the relative Tutte polynomial can be used to compute the Jones polynomial of a virtual knot, with a formulation very similar to the original work of Kauffman [14]. In the case of classical knot theory, our generalized formulation of the Tutte polynomial for a tensor product of colored graphs [11, 12] enables one to derive a fast computation of the Jones polynomial of a knot obtained through repeated tangle replacement operations [9]. Thus, the implication of our main result in virtual knot theory is that similar approaches are also possible for virtual knots obtained through repeated tangle replacement operations, so long as the tangle replacement does not occur at a virtual crossing (which corresponds to a zero edge in our graphs). A precise formulation and detailed analysis is, however, much more involved and is beyond the scope of this paper and shall be addressed by the authors in a future work.

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