HURWITZIAN CONTINUED FRACTIONS CONTAINING A REPEATED CONSTANT AND AN ARITHMETIC PROGRESSION

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Abstract. We prove an explicit formula for infinitely many convergents of Hurwitzian continued fractions that repeat several copies of the same constant and elements of one arithmetic progression, in a quasi-periodic fashion. The proof involves combinatorics and formal Laurent series. Using very little analysis we can express their limits in terms of (modified) Bessel functions and Fibonacci polynomials. The limit formula is a generalization of Lehmer's theorem that implies the continuous fraction expansions of e and $\tan(1)$, and it can also be derived from Lehmer's work using Fibonacci polynomial identities. We completely characterize those implementations of our limit formula for which the parameter of each Bessel function is the half of an odd integer, allowing them to be replaced with elementary functions.

Introduction

It is a remarkable property of infinite continued fractions that they often define a sequence whose limit is easier to describe than the individual entries. Even for $[1, 1, \ldots]$, the simplest example, the limit is easily found by solving a quadratic equation, it takes a bit longer to find a formula for the convergents, in terms of Fibonacci numbers.

The class of Hurwitzian continued fractions of the form

$$[\underbrace{\alpha, \ldots, \alpha}_{r}, \beta_0, \underbrace{\overline{\alpha, \ldots, \alpha, \beta_0 + \beta_1 \cdot n}}_{d}]_{n=1}^{\infty}$$

that we propose investigating seems to be no different in this regard. For the special case $r = 0$, D. H. Lehmer [9] proved a formula for the limit in terms of Bessel functions, that can be verified very easily, after having guessed the correct answer. Using another result from [9] and some Fibonacci polynomial identities, it is not hard to generalize Lehmer's formula to Hurwitzian continued fractions of the above form (see Remark 2.10). The resulting generalization has several famous special cases, the most famous ones being Euler's formula for e and the formula for $\tan(1)$. On the other hand, the only research regarding the convergents themselves seems to be the work [8] of D. N. Lehmer (D. H. Lehmer's father!), who proved congruences for their numerators and denominators by induction, in somewhat greater generality, but without stating the values of the convergents in an explicit fashion.

Our work contains such an explicit formula, stated in Section 2 and proved in Section 4. A variant of the Euler-Mindig formulas using shifted partial denominators yields a summation formula with many vanishing terms. This leads to a compact recurrence

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for the numerators of the $(nd+r-1)$st convergents that may be restated as a linear differential equation for a formal Laurent series. Classical textbooks on differential equations instruct us to solve the associated homogeneous equation first and then find the general solution by "variation of parameters". Unfortunately, in our case the "solution" of the homogeneous equation turns out to be a a two-way infinite formal sum. After discarding infinitely many terms to have a formal Laurent series, the "spirit" of the classical method still inspires a good guess for the form of the solution, where the transformed differential equation encodes a recurrence that is easily solved by inspection. Returning to the original Laurent series involves using two polynomial summation formulas that can be shown purely combinatorially, and seem to be interesting by their own right. These formulas are shown in Section 3. As outlined in Section 2, our explicit formulas for the $(nd+r-1)$st convergents allow us to calculate the limits using very little analysis. To find only the limits, this approach is a bit more tedious than the one outlined in Lehmer's work [9], but we gain a little more insight by also obtaining asymptotic formulas for the numerators and the denominators of the convergents.

Finally, Section 5 is a reflection on Komatsu's recent remark [7] stating that all known examples of Hurwitzian continued fractions seem to have a short quasi-period and involve (hyperbolic) trigonometric functions. For our class of Hurwitzian continued fractions, the limit may be expressed in terms of (modified) Bessel functions, which are known to have an elementary form, in terms of rational and (hyperbolic) trigonometric functions, when their parameter is the half of an odd integer. We describe all continued fractions in our class that yield such parameters, and find that, subject to this restriction, the quasi-period d can not be longer than 4.

Our work inspires several questions worth exploring in the future. Extending the validity of our formulas to all convergents of the same class seems to require only a little more work. As already indicated in Lehmer's paper [9], the calculation of the limits is easily extended to the class of Hurwitzian fractions whose quasi-periodic part contains several different constants and one arithmetic sequence. An explicit formula for the convergents should be obtained using some multivariate generalization of Fibonacci polynomials. The ideas used in our calculations of the convergents may also be useful in finding convergents of other Hurwitzian continued fractions. Finally, as indicated in Sections 3 and 4, there is combinatorics behind the formulas for the convergents. This combinatorics would be worth uncovering. A good starting point may be revisiting the weighted lattice-path model proposed in Flajolet's work [3], where convergents of generalized continued fractions arise as weights of infinite lattice paths of limited height.

1. Preliminaries

1.1. **Simple continued fractions.** A good reference on the basic facts of the subject is Perron's classic work [13, 14]. A *(generalized) finite continued fraction* is an expression of the form

$$a_0+\cfrac{b_1}{a_1+\cfrac{b_2}{a_2+\ddots\cfrac{b_{n-1}}{a_{n-1}+\cfrac{b_n}{a_n}}}}, \tag{1.1}$$

where the *initial term* a_0, the *partial denominators* $a_1, \ldots, a_n$ and the *partial numerators* $b_1, \ldots, b_n$ may be numbers or functions. An *infinite continued fraction* is obtained by letting n go to infinity. The arising questions of convergence have a reassuring answer for *(simple) continued fractions* where all b_i equal 1, the initial term a_0 is an integer, and the partial denominators a_i (for $i > 0$) are positive integers. In the present work we will only use simple continued fractions, and refer to them as continued fractions. Every rational number may be written uniquely as a finite continued fraction, subject to the restriction that the last partial denominator is at least 2 [13, §9, Satz 2.1]. Every infinite continued fraction converges to an irrational number and every irrational number may be uniquely written as a (necessarily infinite) continued fraction [13, §12, Satz 2.6]. As usual, for finite, respectively infinite, continued fraction we will use the shorthand notations $[a_0, a_1, \ldots, a_n]$ and $[a_0, a_1, \ldots]$, respectively. An infinite continued fraction $[a_0, a_1, \ldots]$ is the limit of its *convergents*, that is, of the finite continued fractions $[a_0, \ldots, a_n]$, obtained by reading the first n partial denominators. The convergents

$$[a_0, \ldots, a_n] = \frac{p_n}{q_n} \tag{1.2}$$

may be recursively computed from the initial conditions $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$ and $q_0 = 1$, and from the recurrences $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 1$, cf. [13, §2, Eq. (12), (13)]. The integers p_n and q_n are relative prime for all n [13, §9, Satz 2.1]. The *Euler-Mindig formulas*, derived for general continued fractions in [13, §3], allow to express the numerators p_n and the denominators q_n directly.

Definition 1.1. *We call a set S of integers* even *if it is the disjoint union of intervals of even cardinality. Given two sets of integers S and T such that $S \subseteq T$, we say that T* evenly contains *S, denoted by $S \subseteq_e T$ or $T \supseteq_e S$, if $T \setminus S$ is an even set.*

Thus, for example $\{5\} \subseteq_e \{1, \ldots, 7\}$ since the difference $\{1, 2, 3, 4, 6, 7\}$ is the disjoint union of $\{1, \ldots, 4\}$ and $\{6, 7\}$, whereas $\{4\} \not\subseteq_e \{1, \ldots, 4\}$ as $\{1, 2, 3\}$ can not be written as the disjoint union of intervals of even cardinality. The Euler-Mindig formulas for (simple) continued fractions may be restated as

$$p_n = \sum_{S \subseteq_e \{0,\ldots,n\}} \prod_{i \in S} a_i \quad \text{and} \quad q_n = \sum_{S \subseteq_e \{1,\ldots,n\}} \prod_{i \in S} a_i. \tag{1.3}$$

The following observation is an immediate consequence of Eq. (1.3).

Lemma 1.2. *The denominator q_n associated to $[a_0, a_1, \ldots]$ is the same as the numerator p_{n-1} associated to $[a_1, a_2, \ldots]$.*

1.2. **Hurwitzian continued fractions.** A *Hurwitzian continued fraction* is a continued fraction of the form $[a_0, \ldots, a_h, \overline{\phi_0(\lambda), \ldots, \phi_{k-1}(\lambda)}]_{\lambda=0}^{\infty}$, where $\phi_0, \ldots, \phi_{k-1}$ are polynomial functions that send positive integers into positive integers. The definition given in [13, §32] is easily seen to be equivalent. In the most trivial examples of a Hurwitzian continued fraction all functions ϕ_j are constants, we then obtain a *periodic continued fraction*. The set of real numbers represented by a periodic continued fraction is exactly the set of quadratic irrationals [13, §20]. It should be noted that quadratic irrationals are sometimes excluded from the definition of Hurwitzian continued fractions in some recent papers [7], by requiring that at least one of the repeatedly used polynomial functions be non-constant. A real number is *Hurwitzian* if its continued fraction representation is a Hurwitzian continued fraction. A famous Hurwitzian number is

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots],$$

see [13, §34, Eq. (10)]. The main result on Hurwitzian numbers is Hurwitz theorem [13, §33] stating that for a Hurwitzian number $\xi_0 \in \mathbb{R}$, and any rational numbers a, b, c, d, satisfying $ad - bc \neq 0$, the number $(a\xi_0 + b)/(b\xi_0 + d)$ is also a Hurwitzian number. Hurwitz theorem provides also some estimate on the degrees of the polynomial functions appearing in the continued fraction representation of $(a\xi_0 + b)/(b\xi_0 + d)$, remains silent however on the issue how the length of the period may be affected by the transformation $\xi_0 \mapsto (a\xi_0 + b)/(b\xi_0 + d)$. A generalization of Hurwitz theorem may be found in [16].

Hurwitzian numbers may be computed from their continued fraction representation in some special cases, the expansion of the set of examples is subject of ongoing research. A frequently overlooked first attempt may be found in Perron's book [14, §48, Satz 6.3] which states (without proof) a formula for generalized continued fractions of the form (1.1) having the property that the partial numerators b_i are all equal, and that the numbers a_i form an arithmetic sequence. For simple continued fractions, Perron's result gives

$$[\beta_0, \beta_0 + \beta_1, \beta_0 + 2\beta_1, \beta_0 + 3\beta_1, \ldots] = \beta_1 \frac{\sum_{n=0}^{\infty} \frac{1}{\beta_1^{2n} n! \Gamma\left(\frac{\beta_0}{\beta_1} + n\right)}}{\sum_{n=0}^{\infty} \frac{1}{\beta_1^{2n} n! \Gamma\left(\frac{\beta_0}{\beta_1} + n + 1\right)}}. \tag{1.4}$$

The same class of Hurwitzian continued fractions was revisited by D. H. Lehmer [9], who proved the following formula.

$$[\beta_0, \beta_0 + \beta_1, \beta_0 + 2\beta_1, \beta_0 + 3\beta_1, \ldots] = I_{\beta_0/\beta_1 - 1}(2\beta_1^{-1})/I_{\beta_0/\beta_1}(2\beta_1^{-1}) \tag{1.5}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{\nu+2m}}{\Gamma(m+1)\Gamma(\nu+m+1)}. \tag{1.6}$$

D. H. Lehmer's result was generalized in recent papers to other Hurwitzian continued fractions [5, 7, 11]. Some analogous results were found for *Tasoev continued fractions* [5, 6, 11, 12], defined as continued fractions of the form $[a_0; \underbrace{\overline{a^k, \ldots, a^k}}_{m}]_{k=1}^{\infty}$. As pointed out by T. Komatsu [7], most of these recent results involve Hurwitzian and Tasoev continued fractions where the length of the quasi-period does not exceed 4, Komatsu's work [7] contains some sophisticated examples of longer quasi-periods.

1.3. **Bessel functions.** Besides the modified Bessel functions $I_\nu(x)$, our formulas will also involve the (original) Bessel functions of the first kind

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m}}{\Gamma(m+1)\Gamma(\nu+m+1)}. \tag{1.7}$$

There is an elementary expression for the functions $I_\nu(z)$ and $J_\nu(z)$ respectively, whenever ν is the half of an odd integer. Indeed, using $\Gamma(1/2)=\sqrt{\pi}$ and $\Gamma(z+1)=z\Gamma(z)$, it is easy to derive directly from the definitions (1.6) and (1.7) that we have

$$I_{-1/2}(z)=\sqrt{\frac{2}{\pi z}}\cosh(z),\quad I_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sinh(z), \tag{1.8}$$

$$J_{-1/2}(z)=\sqrt{\frac{2}{\pi z}}\cos(z),\quad \text{and}\quad J_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sin(z). \tag{1.9}$$

(see [10, List of formulæ: 44, 48, 182, 186]). The functions $I_\nu(z)$ and $J_\nu(z)$ satisfy very similar recurrence formulas:

$$I_{\nu+1}(x)=I_{\nu-1}(x)-\frac{2\nu}{x}I_\nu(x)\quad\text{and}\quad J_{\nu+1}(x)=-J_{\nu-1}(x)+\frac{2\nu}{x}J_\nu(x),$$

see [1, 9.1.27, 9.6.26]. These allow us to find explicit elementary expressions for $I_\nu(z)$ and $J_\nu(z)$, whenever ν is the half of an odd integer. In particular, for $\nu=3/2$ we obtain

$$I_{3/2}(x)=\sqrt{\frac{2}{\pi x}}\left(\cosh(x)-\frac{\sinh(x)}{x}\right)\quad\text{and}\quad J_{3/2}(x)=\sqrt{\frac{2}{\pi x}}\left(\frac{\sin(x)}{x}-\cos(x)\right), \tag{1.10}$$

see [10, List of formulæ: 45, 183].

1.4. **Fibonacci and Lucas polynomials.** The Fibonacci and Lucas polynomials are q-analogues of the usual Fibonacci and Lucas numbers.

Definition 1.3. *We define the* Fibonacci polynomials $F_n(q)$ *and the* Lucas polynomials $L_n(q)$ *by the initial conditions* $F_0(q)=0$, $F_1(q)=1$, $L_0(q)=2$ *and* $L_1(q)=q$; *and by the common recurrence* $P_n(q)=q\cdot P_{n-1}(q)+P_{n-2}(q)$ *for* $n\geq 2$, *where the letter* P *should be replaced by either* F *or* L *throughout the defining recurrence.*

Fibonacci and Lucas polynomials are widely studied, they even have their own Mathworld and Wikipedia entries. As a sample reference on Fibonacci polynomials see Yuang and Zhang [17], for further generalizations, see Cigler [2].

The coefficients of the Fibonacci and Lucas polynomials are listed as sequences A102426 and A034807 in the Online Encyclopedia of Integer Sequences [15]. In a recent work of Foata and Han [4] the coefficient of q^{n+1-m} in $F_{n+1}(q)$ appears as the number of *t-compositions* of n into m parts. Here we record another combinatorial interpretation. The coefficient of q^{n+1-m} in $F_{n+1}(q)$ is the number of m-element even sets contained in $\{1,\ldots,n\}$:

$$F_{n+1}(q)=\sum_{\emptyset\subseteq_e S\subseteq\{1,\ldots,n\}} q^{n-|S|}=\sum_{S\subseteq_e\{1,\ldots,n\}} q^{|S|}. \tag{1.11}$$

Equation (1.11) may be shown by induction on n, using the defining recurrence. Just like for the usual Fibonacci and Lucas numbers, a closed formula for $F_n(q)$ and $L_n(q)$ may be obtained after solving the characteristic equation. We have

$$F_n(q) = \frac{1}{\sqrt{q^2+4}}(\rho_1^n-\rho_2^n)\quad\text{and} \tag{1.12}$$

$$L_n(q) = \rho_1^n+\rho_2^n \tag{1.13}$$

where

$$\rho_1 = \frac{q+\sqrt{q^2+4}}{2} \quad \text{and} \quad \rho_2 = \frac{q-\sqrt{q^2+4}}{2}. \tag{1.14}$$

2. Our main result

All our results will be about Hurwitzian continued fractions of the form

$$\xi(\alpha,\beta_0,\beta_1,d,r) := [\underbrace{\alpha,\ldots,\alpha}_{r},\beta_0,\overline{\underbrace{\alpha,\ldots,\alpha,\beta_0+\beta_1\cdot n}_{d}}]_{n=1}^{\infty}$$

where α, β_0, β_1 and d are positive integers and and r is a nonnegative integer. In order to state them we need to introduce two *magic numbers* associated to such a continued fraction.

Definition 2.1. *The* magic sum *associated to* $\xi(\alpha,\beta_0,\beta_1,d,r)$ *is the sum*

$$\sigma(\alpha,\beta_0,\beta_1,d,r) = \frac{\beta_0-\alpha}{\beta_1} + \frac{L_d(\alpha)}{\beta_1 F_d(\alpha)},$$

the magic quotient *associated to* $\xi(\alpha,\beta_0,\beta_1,d,r)$ *is the quotient*

$$\rho(\alpha,\beta_0,\beta_1,d,r) = \frac{(-1)^{d-1}}{\beta_1^2\cdot F_d(\alpha)^2}.$$

Whenever this does not lead to confusion, we will omit the parameters $(\alpha,\beta_0,\beta_1,d,r)$ and denote the magic numbers simply by σ and ρ, respectively. Note that σ does not depend on r and that ρ depends only on α, β_1 and d. Our main result is the following.

Theorem 2.2. *The* $(nd+r-1)$*st convergent of* $\xi(\alpha,\beta_0,\beta_1,d,r)$ *may be written as* p_{nd+r-1}/q_{nd+r-1} *where* p_{nd+r-1} *is given by*

$$\begin{aligned}\frac{p_{nd+r-1}}{F_d(\alpha)^n\beta_1^n} &= F_{r+1}(\alpha)\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(n-k)!}{k!}\binom{n+\sigma-1-k}{n-2k}\rho^k + (-1)^{d-r}F_{d-r-1}(\alpha)\cdot\\ &\quad\cdot F_d(\alpha)\beta_1\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(n-k-1)!}{k!}\binom{n+\sigma-1-k}{n-2k-1}\rho^{k+1},\end{aligned}$$

and q_{nd+r-1} *is given by*

$$\begin{aligned}\frac{q_{nd+r-1}}{F_d(\alpha)^n\beta_1^n} &= F_{r}(\alpha)\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(n-k)!}{k!}\binom{n+\sigma-1-k}{n-2k}\rho^k + (-1)^{d+1-r}F_{d-r}(\alpha)\cdot\\ &\quad\cdot F_d(\alpha)\beta_1\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(n-k-1)!}{k!}\binom{n+\sigma-1-k}{n-2k-1}\rho^{k+1}.\end{aligned}$$

We postpone the proof of Theorem 2.2 till Section 4. In this section we only show how to find an asymptotic formula for the numerator and the denominator, which yields essentially the same limit formula that can also be derived from Lehmer's work [9]. This is the only part of our paper where the notion of limits and convergence from analysis will be used, the proof of Theorem 2.2 will only involve purely algebraic and combinatorial manipulations. We begin with estimating the magic sum σ.

Lemma 2.3. *The magic sum σ is always positive.*

Proof. Since β_0 and β_1 are positive, it suffices to prove that $L_d(\alpha) \geq \alpha F_d(\alpha)$ holds for all positive integer α and all nonnegative integer d. This may be shown by induction on d, using the common recurrence of the Fibonacci and Lucas polynomials. □

Corollary 2.4. *The falling factorial $(\sigma+n-1)_n = (\sigma+n-1)\cdots(\sigma)$ is always positive.*

As a consequence, we may divide $p_{nd+r-1}/(F_d(\alpha)^n\beta_1^n)$, as well as $q_{nd+r-1}/(F_d(\alpha)^n\beta_1^n)$ by $(\sigma+n-1)_n$ and ask whether these quotients converge as n goes to infinity. It will turn out that they do.

Lemma 2.5. *Let σ be any positive real number and ρ be any real number. Then*

$$\lim_{n\to\infty} \frac{1}{(\sigma+n-1)_n} \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(n-k)!}{k!} \binom{n+\sigma-1-k}{n-2k} \rho^k = \sum_{m=0}^{\infty} \frac{\rho^m}{m!(\sigma+m-1)_m}.$$

Proof. Note first that the series on the right hand side is absolute convergent for any ρ as each of its term has smaller absolute value than the corresponding term of $e^{\rho/\sigma} = \sum_{m=0}^{\infty} \rho^m\sigma^{-m}/m!$, an absolute convergent series. For any $\varepsilon > 0$ there is an M such that

$$\sum_{m=M+1}^{\infty} \left|\frac{\rho^m}{m!(\sigma+m-1)_m}\right| < \frac{\varepsilon}{3}. \tag{2.1}$$

Let us consider now the expression of n on the left hand side. It may be rewritten as

$$\sum_{k=0}^{\lfloor n/2\rfloor} \frac{\rho^k}{k!(\sigma+k-1)_k} \cdot \delta_{n,k} \quad \text{where} \quad \delta_{n,k} = \frac{(n-k)_k}{(\sigma+n-1)_k}.$$

The factor $\delta_{n,k}$ may be estimated as follows:

$$\left(\frac{n-2k+1}{\sigma+n-k}\right)^k \leq \delta_{n,k} = \frac{(n-k)\cdots(n-2k+1)}{(\sigma+n-1)\cdots(\sigma+n-k)} \leq \left(\frac{n-k}{\sigma+n-1}\right)^k.$$

Our upper bound for $\delta_{n,k}$ is 1 for $k=0$ and it is less than 1 for $k>0$, since $n+\sigma-1 > n-1 > n-k$. For fixed k, our lower bound converges to 1 as n goes to infinity. Now, for all $n > M$ we may write

$$\sum_{k=0}^{\lfloor n/2\rfloor} \frac{\rho^k}{k!(\sigma+k-1)_k} \cdot \delta_{n,k} = \sum_{k=0}^{M} \frac{\rho^k}{k!(\sigma+k-1)_k} \cdot \delta_{n,k} + \sum_{k=M+1}^{\lfloor n/2\rfloor} \frac{\rho^k}{k!(\sigma+k-1)_k} \cdot \delta_{n,k}$$

By our choice of M, and by $|\delta_{n,k}| \leq 1$, the second sum on the right hand side has absolute value less than $\varepsilon/3$, for all $n > M$. As n goes to infinity, each of $\delta_{n,0}, \ldots, \delta_{n,M}$ converges to 1, so there is an n_0 such that for all $n > n_0$ we have

$$\left|\sum_{k=0}^{M} \frac{\rho^k}{k!(\sigma+k-1)_k} \cdot \delta_{n,k} - \sum_{k=0}^{M} \frac{\rho^k}{k!(\sigma+k-1)_k}\right| < \frac{\varepsilon}{3},$$

implying

$$\left|\sum_{k=0}^{\lfloor n/2\rfloor} \frac{\rho^k}{k!(\sigma+k-1)_k} \cdot \delta_{n,k} - \sum_{k=0}^{M} \frac{\rho^k}{k!(\sigma+k-1)_k}\right| < \frac{2\varepsilon}{3}, \tag{2.2}$$

Combining (2.1) and (2.2) we obtain

$$\left|\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\rho^k}{k!(\sigma+k-1)_k}\cdot\delta_{n,k}-\sum_{k=0}^{\infty}\frac{\rho^k}{k!(\sigma+k-1)_k}\right|<\varepsilon\quad\text{for all } n\geq n_0.$$

□

Lemma 2.6. *Let σ be any positive real number and ρ be any real number. Then*

$$\lim_{n\to\infty}\frac{1}{(\sigma+n-1)_n}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(n-k-1)!}{k!}\binom{n+\sigma-1-k}{n-2k-1}\rho^{k+1}=\sum_{m=0}^{\infty}\frac{\rho^{m+1}}{m!(\sigma+m)_{m+1}}$$

Proof. The proof is very similar to the proof of Lemma 2.5, thus we omit the details. We only note that this time we have

$$\frac{1}{(\sigma+n-1)_n}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(n-k-1)!}{k!}\binom{n+\sigma-1-k}{n-2k-1}\rho^{k+1}=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{\rho^{k+1}}{k!(\sigma+k)_{k+1}}\cdot\delta'_{n,k}$$

where

$$\delta'_{n,k}=\frac{(n-k-1)_k}{(\sigma+n-1)_k}.$$

Again $|\delta'_{n,k}|\leq 1$ for all n and k and, for any fixed k, the limit of $\delta'_{n,k}$ is 1 as n goes to infinity. □

Combining Lemmas 2.5 and 2.6 with Theorem 2.2 we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{p_{nd+r-1}}{F_d(\alpha)^n\beta_1^n(\sigma+n-1)_n} &= F_{r+1}(\alpha)\sum_{m=0}^{\infty}\frac{\rho^m}{m!(\sigma+m-1)_m}\\ &\quad+(-1)^{d-r}F_{d-r-1}(\alpha)F_d(\alpha)\beta_1\sum_{m=0}^{\infty}\frac{\rho^{m+1}}{m!(\sigma+m)_{m+1}}\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}\lim_{n\to\infty}\frac{q_{nd+r-1}}{F_d(\alpha)^n\beta_1^n(\sigma+n-1)_n} &= F_{r}(\alpha)\sum_{m=0}^{\infty}\frac{\rho^m}{m!(\sigma+m-1)_m}\\ &\quad+(-1)^{d-r+1}F_{d-r}(\alpha)F_d(\alpha)\beta_1\sum_{m=0}^{\infty}\frac{\rho^{m+1}}{m!(\sigma+m)_{m+1}}.\end{aligned}\tag{2.4}$$

Taking the quotients of the right hand sides of (2.3) and (2.4) we obtain the following formula for $\xi=\xi(\alpha,\beta_0,\beta_1,d,r)$:

$$\xi=\frac{F_{r+1}(\alpha)\displaystyle\sum_{m=0}^{\infty}\frac{\rho^m}{m!(\sigma+m-1)_m}+(-1)^{d-r}F_{d-r-1}(\alpha)F_d(\alpha)\beta_1\sum_{m=0}^{\infty}\frac{\rho^{m+1}}{m!(\sigma+m)_{m+1}}}{F_{r}(\alpha)\displaystyle\sum_{m=0}^{\infty}\frac{\rho^m}{m!(\sigma+m-1)_m}+(-1)^{d-r+1}F_{d-r}(\alpha)F_d(\alpha)\beta_1\sum_{m=0}^{\infty}\frac{\rho^{m+1}}{m!(\sigma+m)_{m+1}}}.\tag{2.5}$$

This is the formula that could have been discovered in a "parallel universe" where interest in Hurwitzian continued fractions were to arise a long time before defining Bessel functions. We may restate Eq. (2.5) in terms of Bessel functions using the following two obvious statements.

Lemma 2.7. *For any positive ρ and σ we have*

$$\sum_{m=0}^{\infty}\frac{\rho^m}{m!(\sigma+m-1)_m}=\frac{I_{\sigma-1}(2\sqrt{\rho})\Gamma(\sigma)}{\rho^{(\sigma-1)/2}} \quad \text{and} \quad \sum_{m=0}^{\infty}\frac{\rho^{m+1}}{m!(\sigma+m)_{m+1}}=\frac{I_{\sigma}(2\sqrt{\rho})\Gamma(\sigma)}{\rho^{(\sigma-2)/2}}.$$

Here $I_\nu(z)$ is the modified Bessel function defined in (1.6).

Lemma 2.8. *For any negative ρ and positive σ we have*

$$\sum_{m=0}^{\infty}\frac{\rho^m}{m!(\sigma+m-1)_m}=\frac{J_{\sigma-1}(2\sqrt{-\rho})\Gamma(\sigma)}{(-\rho)^{(\sigma-1)/2}} \quad \text{and}$$

$$\sum_{m=0}^{\infty}\frac{\rho^{m+1}}{m!(\sigma+m)_{m+1}}=-\frac{J_{\sigma}(2\sqrt{-\rho})\Gamma(\sigma)}{(-\rho)^{(\sigma-2)/2}}.$$

Here $J_\nu(z)$ is the Bessel function given by (1.7).

Using Lemmas 2.8 and 2.7 above we may rephrase Eq. (2.5) as follows.

Theorem 2.9. *Let α, β_0, β_1 and d be positive integers, and let r be nonnegative integers. Then the Hurwitzian continued fraction*

$$\xi(\alpha,\beta_0,\beta_1,d,r)=[\underbrace{\alpha,\ldots,\alpha}_{r},\beta_0,\overline{\underbrace{\alpha,\ldots,\alpha,\beta_0+\beta_1\cdot n}_{d}}]_{n=1}^{\infty}$$

is given by

$$\xi(\alpha,\beta_0,\beta_1,d,r)=\frac{F_{r+1}(\alpha)I_{\sigma-1}(2\sqrt{\rho})+(-1)^{r+1}F_{d-r-1}(\alpha)I_{\sigma}(2\sqrt{\rho})}{F_r(\alpha)I_{\sigma-1}(2\sqrt{\rho})+(-1)^{r}F_{d-r}(\alpha)I_{\sigma}(2\sqrt{\rho})}$$

if d is odd, and it is given by

$$\xi(\alpha,\beta_0,\beta_1,d,r)=\frac{F_{r+1}(\alpha)J_{\sigma-1}(2\sqrt{-\rho})+(-1)^{r+1}F_{d-r-1}(\alpha)J_{\sigma}(2\sqrt{-\rho})}{F_r(\alpha)J_{\sigma-1}(2\sqrt{-\rho})+(-1)^{r}F_{d-r}(\alpha)J_{\sigma}(2\sqrt{-\rho})},$$

if d is even. Here $I_\nu(z)$ and $J_\nu(z)$, respectively, denotes the is the modified respectively original Bessel function defined in (1.6) and (1.7), respectively.

Proof. We work out only the case of odd d in detail, the case of even d is completely analogous. Direct substitution of Lemma 2.7 into (2.5) yields

$$\xi(\alpha,\beta_0,\beta_1,d,r)=\frac{F_{r+1}(\alpha)\dfrac{I_{\sigma-1}(2\sqrt{\rho})}{\rho^{(\sigma-1)/2}}+(-1)^{d-r}F_{d-r-1}(\alpha)F_d(\alpha)\beta_1\dfrac{I_{\sigma}(2\sqrt{\rho})}{\rho^{(\sigma-2)/2}}}{F_r(\alpha)\dfrac{I_{\sigma-1}(2\sqrt{\rho})}{\rho^{(\sigma-1)/2}}+(-1)^{d-r+1}F_{d-r}(\alpha)F_d(\alpha)\beta_1\dfrac{I_{\sigma}(2\sqrt{\rho})}{\rho^{(\sigma-2)/2}}}.$$

After multiplying the numerator and the denominator by $\rho^{(\sigma-1)/2}$ we get

$$\xi(\alpha,\beta_0,\beta_1,d,r)=\frac{F_{r+1}(\alpha)I_{\sigma-1}(2\sqrt{\rho})+(-1)^{d-r}\sqrt{\rho}F_{d-r-1}(\alpha)F_d(\alpha)\beta_1I_{\sigma}(2\sqrt{\rho})}{F_r(\alpha)I_{\sigma-1}(2\sqrt{\rho})+(-1)^{d-r+1}\sqrt{\rho}F_{d-r}(\alpha)F_d(\alpha)\beta_1I_{\sigma}(2\sqrt{\rho})}.$$

The first equality in Theorem 2.9 follows from $\sqrt{\rho}=\beta_1^{-1}F_d(\alpha)^{-1}$ and from the fact that $(-1)^d=(-1)$ in this case. A similar reasoning for even d yields

$$\xi(\alpha,\beta_0,\beta_1,d,r)=\frac{F_{r+1}(\alpha)J_{\sigma-1}(2\sqrt{-\rho})-(-1)^{d-r}F_{d-r-1}(\alpha)J_\sigma(2\sqrt{-\rho})}{F_r(\alpha)J_{\sigma-1}(2\sqrt{-\rho})-(-1)^{d-r+1}F_{d-r}(\alpha)J_\sigma(2\sqrt{-\rho})},$$

a final simplification may be made by observing that $(-1)^d=1$ in this case. □

Remark 2.10. Theorem 2.9 may also be derived from Lehmer's work [9] directly, using a few, easily verifiable facts about Fibonacci and Lucas polynomials. Again, we outline the proof for odd d only, the case of even d being completely analogous. Consider first the case when $d=1$ and $r=0$. In this case we get $\sigma=\beta_0/\beta_1$, regardless of of α. Theorem 2.9 takes the form $\xi(\alpha,\beta_0,\beta_1,1,0)=I_{\sigma-1}(2\sqrt{\rho})/I_\sigma(2\sqrt{\rho})$, which, by $\rho=\beta_1^{-2}$, is exactly Lehmer's formula (1.5). Consider next the case when d is an arbitrary odd integer and $r=0$. In this case, we may use Lehmer's [9, Theorem 4] with all constants being equal to α. Using the fact that

$$[\underbrace{\alpha,\ldots,\alpha}_{d}]=\frac{F_{d+1}(\alpha)}{F_d(\alpha)},$$

the fractions A/B, A'/B' and A''/B'' appearing in [9, Theorem 4] are easily seen to correspond to $((\beta_0-\alpha)F_d(\alpha)+F_{d+1}(\alpha))/F_d(\alpha)$, $((\beta_0-\alpha)F_{d-1}(\alpha)+F_d(\alpha))/F_{d-1}(\alpha)$ and $F_d(\alpha)/F_{d-1}(\alpha)$, respectively, in our notation. Lehmer's $(bB+B'+B'')/(aB)$ corresponds to our

$$\frac{\beta_0F_d(\alpha)+2F_{d-1}(\alpha)}{\beta_1F_d(\alpha)}=\frac{(\beta_0-\alpha)F_d(\alpha)+(\alpha F_d(\alpha)+2F_{d-1}(\alpha))}{\beta_1F_d(\alpha)}=\sigma,$$

since $\alpha F_d(\alpha)+2F_{d-1}(\alpha)=L_d(\alpha)$ holds for all d. Lehmer's [9, Theorem 4] gives

$$\xi(\alpha,\beta_0,\beta_1,d,0)=\frac{1}{F_d(\alpha)}\left(-F_{d-1}(\alpha)+\frac{I_{\sigma-1}(2(\beta_1F_d(\alpha))^{-1})}{I_\sigma(2(\beta_1F_dF_d(\alpha))^{-1})}\right)$$

which is the same as the formula implied by Theorem 2.9, after noting that we have $\rho=(\beta_1F_d(\alpha))^{-2}$. Finally, for arbitrary r we may substitute $\eta:=\xi(\alpha,\beta_0,\beta_1,d,0)$ into the formula

$$[\underbrace{\alpha,\ldots,\alpha}_{r},\eta]=\frac{F_{r+1}(\alpha)\eta+F_r(\alpha)}{F_r(\alpha)\eta+F_{r-1}(\alpha)}$$

and obtain

$$\xi(\alpha,\beta_0,\beta_1,d,r)=\frac{F_{r+1}(\alpha)\left(\frac{1}{F_d(\alpha)}\left(-F_{d-1}(\alpha)+\frac{I_{\sigma-1}(2(\beta_1F_d(\alpha))^{-1})}{I_\sigma(2(\beta_1F_dF_d(\alpha))^{-1})}\right)+F_r(\alpha)\right)}{F_r(\alpha)\left(\frac{1}{F_d(\alpha)}\left(-F_{d-1}(\alpha)+\frac{I_{\sigma-1}(2(\beta_1F_d(\alpha))^{-1})}{I_\sigma(2(\beta_1F_dF_d(\alpha))^{-1})}\right)+F_{r-1}(\alpha)\right)}.$$

(Recall that σ does not depend on r.) Equivalently,

$$\xi=\frac{F_{r+1}(\alpha)I_{\sigma-1}(2(\beta_1F_d(\alpha))^{-1})+\left(F_r(\alpha)F_d(\alpha)-F_{r+1}(\alpha)F_{d-1}(\alpha)\right)I_\sigma(2(\beta_1F_dF_d(\alpha))^{-1})}{F_r(\alpha)I_{\sigma-1}(2(\beta_1F_d(\alpha))^{-1})+\left(F_{r-1}(\alpha)F_d(\alpha)-F_r(\alpha)F_{d-1}(\alpha)\right)I_\sigma(2(\beta_1F_dF_d(\alpha))^{-1})}.$$

Theorem 2.9 now follows after observing that $F_r(\alpha)F_d(\alpha)-F_{r+1}(\alpha)F_{d-1}(\alpha)$ and and $F_{r-1}(\alpha)F_d(\alpha)-F_r(\alpha)F_{d-1}(\alpha)$, respectively, may be replaced by $(-1)^{r+1}F_{d-r-1}(\alpha)$ and $(-1)^rF_{d-r}(\alpha)$ respectively.

3. Two useful lemmas

In this section we provide a combinatorial proof for two polynomial identities which seem to be interesting by their own right. They will play a crucial role in Section 4 where we prove Theorem 2.2. Our lemmas will be summation formulas for the polynomials

$$R_n(x,y) := \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\binom{n+y}{n-k}(n-k)!^2x^k \tag{3.1}$$

and

$$S_n(x,y) := \frac{1}{n!}\sum_{k=0}^{n-1}\binom{n}{k}\binom{n+y}{n-k-1}(n-k)!(n-k-1)!x^{k+1} \tag{3.2}$$

Lemma 3.1. *The polynomials $R_n(x,y)$ satisfy*

$$\sum_{m=0}^{n}\frac{(-x)^{n-m}}{(n-m)!}R_m(x,y) = \frac{1}{n!}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n+y-k}{n-2k}(n-k)!^2x^k \quad \textit{for all } n \geq 0.$$

Proof. We consider both sides of the equation as polynomials in the variable y with coefficients from the field $\mathbb{Q}(x)$. Since a nonzero polynomial has only finitely many roots, it suffices to show that the two sides equal for any nonnegative integer value of y.

For $y \in \mathbb{N}$, the left hand side may then be rewritten as

$$\frac{1}{n!}\sum_{m=0}^{n}\binom{n}{n-m}(-x)^{n-m}\sum_{k=0}^{m}\binom{m}{k}\binom{m+y}{y+k}(m-k)!^2x^k.$$

This is $1/n!$ times the total weight of all quadruplets $(\pi_1,\pi_2,\gamma_1,\gamma_2)$ subject to the following conditions:

1. π_1 is a permutation of $\{1,\ldots,n\}$, π_2 is a permutation of $\{1,\ldots,n+y\}$, the functions $\gamma_1 : \{1,\ldots,n\} \to \{0,1,2\}$ and $\gamma_2 : \{1,\ldots,n+y\} \to \{0,1,2\}$ are colorings;
2. for $i \in \{1,2\}$ an element j satisfying $\gamma_i(j) = 1$ must be a fixed point of π_i;
3. for any j, $\gamma_1(j) = 2$ is equivalent to $\gamma_2(j) = 2$ and $\gamma_1(j) = \gamma_2(j) = 2$ implies that j is a common fixed point of π_1 and π_2;
4. the colorings γ_1 and γ_2 satisfy $|\{j \in \{1,\ldots,n+y\} : \gamma_2(j) > 0\}| = |\{j \in \{1,\ldots,n\} : \gamma_1(j) > 0\}| + y$;

We define the weight of $(\pi_1,\pi_2,\gamma_1,\gamma_2)$ as

$$x^{|\{j\in\{1,\ldots,n\}\,:\,\gamma_1(j)=1\}|}\cdot(-x)^{|\{j\in\{1,\ldots,n\}\,:\,\gamma_1(j)=2\}|}.$$

Indeed, after setting $n-m$ as the number of elements j satisfying $\gamma_1(j) = \gamma_2(j) = 2$, there are $\binom{n}{n-m}$ ways to select them. By rule (3) these are common fixed points of π_1 and π_2 (and thus elements of $\{1,\ldots,n\}$). Next we set k as the number of elements $j \in \{1,\ldots,n\}$ satisfying $\gamma_1(j) = 1$, and select these elements, in $\binom{m}{k}$ ways. The remaining elements of $\{1,\ldots,n\}$ satisfy $\gamma_1(j) = 0$. At this point we have $|\{j \in \{1,\ldots,n\} : \gamma_1(j) > 0\}| = n-m+k$. Thus, by rule (4), the set $\{j \in \{1,\ldots,n+y\} : \gamma_2(j) > 0\}$ must have $n-m+k+y$ elements. Exactly $n-m$ of these elements satisfy $\gamma_2(j) = 2$, therefore the number of elements $j \in \{1,\ldots,n+y\}$ satisfying $\gamma_2(j) = 1$ must be $y+k$.

There are $\binom{m+y}{y+k}$ ways to select them. So far we have selected $n-m+k$ fixed points of π_1 and $n-m+y+k$ fixed points of π_2. We can complete prescribing the action of π_1 and π_2 in $(m-k)!^2$ ways.

The same total weight may also be found by fixing the pair (π_1, π_2) first, and summing over all *allowable pairs* of colorings (γ_1, γ_2). We call the pair (γ_1, γ_2) allowable, if the quadruplet $(\pi_1, \pi_2, \gamma_1, \gamma_2)$ satisfies the conditions listed above. Suppose j_0 is a common fixed point of π_1 and π_2. Observe that the contribution of all allowable pairs (γ_1, γ_2) satisfying $\gamma_1(j_0) = \gamma_2(j_0) = 2$ cancels the contribution of all allowable pairs satisfying $\gamma_1(j_0) = \gamma_2(j_0) = 1$. Indeed, let (γ_1, γ_2) an allowable pair satisfying $\gamma_1(j_0) = \gamma_2(j_0) > 0$. Then the pair (γ_1', γ_2') defined by

$$\gamma_i'(j) = \begin{cases} \gamma_i(j) & \text{if } j \neq j_0 \\ 3 - \gamma_i(j) & \text{if } j = j_0 \end{cases}$$

is also allowable and also satisfies $\gamma_1'(j_0) = \gamma_2'(j_0) > 0$. (Here $j \in \{1, \ldots, n\}$ for γ_1 and γ_1' and $j \in \{1, \ldots, n+y\}$ for γ_2 and γ_2'.) The map $(\gamma_1, \gamma_2) \mapsto (\gamma_1', \gamma_2')$ is an involution that matches canceling terms: the only difference between the respective contribution is a factor of x or $-x$ associated to j_0. Therefore we may restrict our attention to the total weight of quadruplets $(\pi_1, \pi_2, \gamma_1, \gamma_2)$ satisfying the following additional criteria:

(5) no j satisfies $\gamma_1(j) = \gamma_2(j) = 2$, in particular, the colorings γ_1 and γ_2 map into the set $\{0, 1\}$;

(6) no j satisfies $\gamma_1(j) = \gamma_2(j) = 1$, in other words, the sets $\{j \in \{1, \ldots, n\} : \gamma_1(j) = 1\}$ and $\{j \in \{1, \ldots, n+y\} : \gamma_2(j) = 1\}$ are disjoint.

When computing the total weight of such quadruplets, we may first select k as the number of elements $j \in \{1, \ldots, n\}$ satisfying $\gamma_1(j) = 1$ and select them in $\binom{n}{k}$ ways. By rule (4) there must be $y + k$ elements $j \in \{1, \ldots, n+y\}$ satisfying $\gamma_2(j) = 1$ and, by rule (6), this set is disjoint of the previously selected k-element set. Thus there are $\binom{n+y-k}{y+k}$ ways to select them. So far we have selected k fixed points of π_1 and $y+k$ fixed points of π_2. There are $(n-k)!^2$ ways to complete prescribing π_1 and π_2, the weight of the quadruplet is x^k. We obtain a total contribution of $\sum_{k=0}^{n} \binom{n}{k}\binom{n+y-k}{y+k}(n-k)!^2 x^k$ which is exactly $n!$ times the right and side. □

Lemma 3.2. *The polynomials $S_n(x, y)$ satisfy*

$$\sum_{m=0}^{n} \frac{(-x)^{n-m}}{(n-m)!} S_m(x, y) = \frac{1}{n!} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{k} \binom{n+y-k}{n-2k-1} (n-k)!(n-k-1)! x^{k+1}$$

for all $n \geq 0$.

Proof. A proof may be obtained by performing slight modifications to the proof of Lemma 3.1, which we outline below. The left hand side equals

$$\frac{x}{n!} \sum_{m=0}^{n} \binom{n}{n-m} (-x)^{n-m} \sum_{k=0}^{m} \binom{m}{k} \binom{m+y}{y+k+1} (m-k)!(m-k-1)! x^k,$$

which may be considered as $x/n!$ times the total weight of quadruplets $(\pi_1, \pi_2, \gamma_1, \gamma_2)$, where the only change to the definition is that, instead of (4), now we require

(4') the colorings γ_1 and γ_2 satisfy $|\{j \in \{1,\ldots,n+y\} : \gamma_2(j) > 0\}| = |\{j \in \{1,\ldots,n\} : \gamma_1(j) > 0\}| + y + 1$.

Thus we will have to select $y+k+1$ elements j (instead of $y+k$) satisfying $\pi_2(j) = 1$, in $\binom{m+y}{y+k+1}$. In the last stage, we will have selected $n-m+y+k+1$ fixed points of π_2, thus we will only have $(m-k-1)!$ ways to complete the selection of π_2.

The same involution as before shows that we may again restrict our attention to those quadruplets which satisfy the additional conditions (5) and (6). Let us set k again as the number of elements $j \in \{1,\ldots,n\}$ satisfying $\gamma_1(j) = 1$. The only adjustment we need to make to the reasoning is to observe that now we need to have $y+k+1$ elements satisfying $|\gamma_2(j) = 1|$ which may be selected in $\binom{n+y-k}{y+k+1}$ ways, instead of $\binom{n+y-k}{y+k}$ ways. Finally, we may complete the selection of π_2 in $(n-k-1)!$ ways, instead of $(n-k)!$. The resulting total weight is exactly $x/n!$ times the right hand side of our stated equality. $\square$

For the sake of use in Section 4 let us note that $R_n(x,y)$ and $S_n(x,y)$ may also be written in the following shorter form, using falling factorials:

$$R_n(x,y) = \sum_{k=0}^{n} \frac{x^k (y+n)_{n-k}}{k!} \quad \text{and} \quad S_n(x,y) = \sum_{k=0}^{n-1} \frac{x^{k+1}(y+n)_{n-k-1}}{k!} \tag{3.3}$$

A similar simplification yields that Lemma 3.1 is equivalent to

$$\sum_{m=0}^{n} \frac{(-x)^{n-m}}{(n-m)!} R_m(x,y) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(n-k)!}{k!} \binom{n+y-k}{n-2k} x^k \quad \text{for all } n \geq 0, \tag{3.4}$$

and that Lemma 3.2 has the compact form

$$\sum_{m=0}^{n} \frac{(-x)^{n-m}}{(n-m)!} S_m(x,y) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(n-k-1)!}{k!} \binom{n+y-k}{n-2k-1} x^{k+1} \quad \text{for all } n \geq 0. \tag{3.5}$$

4. Calculating the convergents

In this section we calculate $(nd+r-1)$st convergent of $\xi(\alpha,\beta_0,\beta_1,d,r)$ directly, from the Euler-Mindig formulas (1.3). We begin by observing that, for an arbitrary continued fraction $[a_0,a_1,\ldots]$ and an arbitrary positive integer α, the numerator p_n in (1.3) may be rewritten as

$$p_n = \sum_{S \subseteq_e \{0,\ldots,n\}} \prod_{i \in S} (a_i - \alpha + \alpha) = \sum_{S \subseteq_e \{0,\ldots,n\}} \sum_{T \subseteq S} \prod_{i \in T} (a_i - \alpha) \alpha^{|S \setminus T|}.$$

Changing the order of summation gives

$$p_n = \sum_{T \subseteq \{0,\ldots,n\}} \prod_{i \in T} (a_i - \alpha) \sum_{T \subseteq S \subseteq_e \{0,\ldots,n\}} \alpha^{|S|}. \tag{4.1}$$

For $\xi(\alpha,\beta_0,\beta_1,d,r)$ we have $a_i = \alpha$ unless i is congruent to r modulo d. Thus, to compute p_{nd+r-1} using (4.1), we only need to sum over subsets T whose elements are all congruent to r modulo d. This observation yields the following recurrence:

$$p_{nd+r-1} = F_{nd+r+1}(\alpha) + \sum_{k=0}^{n-1} p_{kd+r-1} \cdot (\beta_0 + \beta_1 \cdot k - \alpha) \cdot F_{(n-k)d}(\alpha). \tag{4.2}$$

By Eq. (1.11), the term $F_{nd+r+1}(\alpha)$ above is the contribution of $T=\emptyset$, whereas the term $p_{kd+r-1}\cdot(\beta_0+\beta_1\cdot k-\alpha)\cdot F_{(n-k)d+1}(\alpha)$ is the total contribution of all sets T whose largest element is $kd+r$. Substituting $n=0$ into (4.2) yields the initial condition

$$p_{r-1}=F_{r+1}(\alpha)$$

which is obviously true for $r>0$, and it is also valid when $r=0$ after setting $p_{-1}=1$, as usual. Consider the formal Laurent series

$$y(t):=\sum_{n=0}^{\infty} p_{nd+r-1}\cdot t^{\beta_0+\beta_1\cdot n-\alpha}\in\mathbb{Q}((t)).$$

For $y(t)$, Eq. (4.2) yields

$$y(t):=\sum_{n=0}^{\infty} F_{nd+r+1}(\alpha)t^{\beta_1 n}\cdot t^{\beta_0-\alpha}+t\cdot\sum_{n=1}^{\infty}F_{nd}(\alpha)t^{\beta_1 n}\cdot y'(t), \tag{4.3}$$

where $y'(t)$ is the formal derivative of $y(t)$ with respect to t. To write (4.3) in a more explicit form, observe that, by Eq. (1.12), we have

$$\sum_{n=0}^{\infty} F_{nd+r+1}(\alpha)t^n=\sum_{n=0}^{\infty}\frac{\rho_1^{nd+r+1}-\rho_2^{nd+r+1}}{\sqrt{\alpha^2+4}}t^n=\frac{1}{\sqrt{\alpha^2+4}}\left(\frac{\rho_1^{r+1}}{1-\rho_1^d t}-\frac{\rho_2^{r+1}}{1-\rho_2^d t}\right).$$

Using the fact that $\rho_1\rho_2=-1$, the above equation may be rewritten as

$$\begin{aligned}\sum_{n=0}^{\infty} F_{nd+r+1}(\alpha)t^n &= \frac{1}{\sqrt{\alpha^2+4}}\frac{\rho_1^{r+1}-\rho_2^{r+1}-(\rho_1^{r+1}\rho_2^d-\rho_2^{r+1}\rho_1^d)t}{1-(\rho_1^d+\rho_2^d)t+(-1)^d t^2}\\ &= \frac{1}{\sqrt{\alpha^2+4}}\frac{\rho_1^{r+1}-\rho_2^{r+1}+(-1)^{r+1}(\rho_1^{d-r-1}-\rho_2^{d-r-1})t}{1-(\rho_1^d+\rho_2^d)t+(-1)^d t^2}.\end{aligned}$$

By Eqs. (1.11) and (1.13) we obtain

$$\sum_{n=0}^{\infty} F_{nd+r+1}(\alpha)t^n=\frac{F_{r+1}(\alpha)+(-1)^{r+1}F_{d-r-1}(\alpha)t}{1-L_d(\alpha)t+(-1)^d t^2}. \tag{4.4}$$

Note that substituting $r=d-1$ in Eq. (4.4) yields

$$\sum_{n=1}^{\infty} F_{nd}(\alpha)t^n=t\cdot\sum_{n=0}^{\infty}F_{nd+d}(\alpha)t^n=\frac{F_d(\alpha)\cdot t}{1-L_d(\alpha)t+(-1)^d t^2},$$

since $F_0(\alpha)=0$. Using these last two equations, we may rewrite (4.3) as

$$y(t)=\frac{F_{r+1}(\alpha)+(-1)^{r+1}F_{d-r-1}(\alpha)t^{\beta_1}}{1-L_d(\alpha)t^{\beta_1}+(-1)^d t^{2\beta_1}}\cdot t^{\beta_0-\alpha}+\frac{F_d(\alpha)\cdot t^{\beta_1+1}}{1-L_d(\alpha)t^{\beta_1}+(-1)^d t^{2\beta_1}}\cdot y'(t).$$

Rearranging to express the derivative of $y(t)$ yields

$$\begin{aligned}y'(t) &= \frac{y(t)\cdot(t^{-\beta_1-1}-L_d(\alpha)t^{-1}+(-1)^d t^{\beta_1-1})}{F_d(\alpha)}\\ &\quad -\frac{(F_{r+1}(\alpha)+(-1)^{r+1}F_{d-r-1}(\alpha)t^{\beta_1})\cdot t^{\beta_0-\alpha}}{F_d(\alpha)\cdot t^{\beta_1+1}}.\end{aligned} \tag{4.5}$$

Inspired by the way we solve ordinary differential equations in analysis, we will *guess* the solution of (4.5) by "solving" first the corresponding homogeneous equation and then replace the arbitrary constant by a formal Laurent series. The next few lines will not make sense, they just indicate how one may come up with a good guess for $y(t)$. A reader who does not like "obscure reasoning," should skip ahead to (4.6) and accept that there is no "rational explanation" as to why introducing the formal Laurent series $z(t)$ is a good idea.

The homogeneous equation

$$y_H'(t) = \frac{y_H(t)\cdot(t^{-\beta_1-1} - L_d(\alpha)t^{-1} + (-1)^d t^{\beta_1-1})}{F_d(\alpha)}$$

"may be rewritten as"

$$\frac{d}{dt}\ln(y_H(t)) = \frac{t^{-\beta_1-1} - L_d(\alpha)t^{-1} + (-1)^d t^{\beta_1-1}}{F_d(\alpha)},$$

"yielding"

$$y_H(t) = C\cdot t^{-L_d(\alpha)/F_d(\alpha)} \exp\left(\frac{-t^{-\beta_1} + (-1)^d t^{\beta_1}}{\beta_1\cdot F_d(\alpha)}\right).$$

where C is an arbitrary constant. This "solution" to the "homogeneous equation" suggests looking for a solution to Eq. (4.5) of the form

$$y(t) = z(t)\cdot t^{-L_d(\alpha)/F_d(\alpha)} \exp\left(\frac{-t^{-\beta_1} + (-1)^d t^{\beta_1}}{\beta_1\cdot F_d(\alpha)}\right).$$

Equivalently, we would want to set

$$z(t) := y(t)\cdot t^{L_d(\alpha)/F_d(\alpha)} \exp\left(\frac{t^{-\beta_1} + (-1)^{d-1} t^{\beta_1}}{\beta_1\cdot F_d(\alpha)}\right).$$

Alas, the resulting formal expression would contain arbitrary large positive and as well as arbitrary small negative powers of t, it does not resemble a Laurent series at all. Hoping that a slight change would not upset our calculations irreparably, we *define* $z(t)$ by setting

$$z(t) := y(t)\cdot t^{L_d(\alpha)/F_d(\alpha)} \exp\left(\frac{(-1)^{d-1} t^{\beta_1}}{\beta_1\cdot F_d(\alpha)}\right). \tag{4.6}$$

Note that $z(t)$ is a formal Laurent series in the variable $t^{1/F_d(\alpha)}$, an infinite formal sum of the form

$$z(t) = \sum_{n=0}^{\infty} s_n t^{\beta_1\cdot n+\beta_0-\alpha+L_d(\alpha)/F_d(\alpha)} \in \mathbb{Q}((t^{1/F_d(\alpha)})).$$

Taking the derivative on both sides of (4.6) yields

$$\begin{aligned}
z'(t) &= y'(t)\cdot t^{\frac{L_d(\alpha)}{F_d(\alpha)}} \exp\left(\frac{(-1)^{d-1}t^{\beta_1}}{\beta_1\cdot F_d(\alpha)}\right) \\
&\quad + y(t)\cdot t^{\frac{L_d(\alpha)}{F_d(\alpha)}} \exp\left(\frac{(-1)^{d-1}t^{\beta_1}}{\beta_1\cdot F_d(\alpha)}\right)\cdot\left(\frac{L_d(\alpha)t^{-1} - (-1)^d t^{\beta_1-1}}{F_d(\alpha)}\right) \\
&= t^{\frac{L_d(\alpha)}{F_d(\alpha)}} \exp\left(\frac{(-1)^{d-1}t^{\beta_1}}{\beta_1\cdot F_d(\alpha)}\right)\cdot\left(y'(t) + y(t)\cdot\frac{L_d(\alpha)t^{-1} - (-1)^d t^{\beta_1-1}}{F_d(\alpha)}\right).
\end{aligned}$$

(Note that $L_d(\alpha)/F_d(\alpha)$ is always positive.) After substituting the value of $y'(t)$ from (4.5) and simplifying we obtain

$$z'(t) = t^{\frac{L_d(\alpha)}{F_d(\alpha)}} \exp\left(\frac{(-1)^{d-1}t^{\beta_1}}{\beta_1 \cdot F_d(\alpha)}\right) \cdot \frac{y(t) - (F_{r+1}(\alpha) + (-1)^{r+1}F_{d-r-1}(\alpha)t^{\beta_1})t^{\beta_0-\alpha}}{F_d(\alpha)t^{\beta_1+1}}.$$

By (4.6), the last equation is equivalent to

$$\begin{aligned} z'(t) &= \frac{z(t)}{F_d(\alpha)t^{\beta_1+1}} \\ &\quad -t^{\frac{L_d(\alpha)}{F_d(\alpha)}} \exp\left(\frac{(-1)^{d-1}t^{\beta_1}}{\beta_1 \cdot F_d(\alpha)}\right) \cdot \frac{(F_{r+1}(\alpha) + (-1)^{r+1}F_{d-r-1}(\alpha)t^{\beta_1})t^{\beta_0-\alpha}}{F_d(\alpha)t^{\beta_1+1}}. \end{aligned}$$

Comparing the coefficients of $t^{\beta_1\cdot n+\beta_0-\alpha+L_d(\alpha)/F_d(\alpha)-1}$ on both sides yields

$$\begin{aligned} s_n \cdot \left(\beta_1 \cdot n + \beta_0 - \alpha + \frac{L_d(\alpha)}{F_d(\alpha)}\right) &= \frac{s_{n+1}}{F_d(\alpha)} - \frac{F_{r+1}(\alpha)}{F_d(\alpha)} \cdot \frac{\left(\frac{(-1)^{d-1}}{\beta_1\cdot F_d(\alpha)}\right)^{n+1}}{(n+1)!} \\ &\quad + \frac{(-1)^r F_{d-r-1}(\alpha)}{F_d(\alpha)} \cdot \frac{\left(\frac{(-1)^{d-1}}{\beta_1\cdot F_d(\alpha)}\right)^{n}}{n!}. \end{aligned}$$

Note that the factor $(\beta_1\cdot n+\beta_0-\alpha+L_d(\alpha)/F_d(\alpha))$ in the last equation equals $\beta_1\cdot(n+\sigma)$, where σ is the magic sum, and that the magic quotient ρ appears twice on the right hand side. Since, by Corollary 2.4, $(\sigma+n)_{n+1}$ is not zero, we may divide both sides by $F_d(\alpha)^n\beta_1^{n+1}(\sigma+n)_{n+1}$, and obtain the following recurrence for $\widetilde{s}_n := s_n/(F_d(\alpha)^n\beta_1^n(\sigma+n-1)_n)$:

$$\widetilde{s}_{n+1} = \widetilde{s}_n + F_{r+1}(\alpha) \cdot \frac{\rho^{n+1}}{(n+1)!(\sigma+n)_{n+1}} - \frac{(-1)^r F_{d-r-1}(\alpha)}{F_d(\alpha)\beta_1} \cdot \frac{\rho^n}{n!(\sigma+n)_{n+1}}.$$

Considering the fact that $\widetilde{s}_0 = s_0 = F_{r+1}(\alpha)$ and that $(-1)^r F_{d-r-1}(\alpha)/(F_d(\alpha)\beta_1) = (-1)^{d-1-r}F_{d-r-1}(\alpha)F_d(\alpha)\beta_1\cdot\rho$, the last recurrence implies

$$\widetilde{s}_n = F_{r+1}(\alpha) \cdot \sum_{k=0}^{n} \frac{\rho^k}{k!(\sigma+k-1)_k} + (-1)^{d-r}F_{d-r-1}(\alpha)F_d(\alpha)\beta_1 \cdot \sum_{k=0}^{n-1} \frac{\rho^{k+1}}{k!(\sigma+k)_{k+1}}.$$

Multiplying both sides by $(\sigma+n)_{n+1}$ yields

$$\begin{aligned} \frac{s_n}{F_d(\alpha)^n\beta_1^n} &= F_{r+1}(\alpha) \cdot \sum_{k=0}^{n} \frac{\rho^k(\sigma+n)_{n-k}}{k!} \\ &\quad +(-1)^{d-r}F_{d-r-1}(\alpha)F_d(\alpha)\beta_1 \cdot \sum_{k=0}^{n-1} \frac{\rho^{k+1}(\sigma+n)_{n-k-1}}{k!}. \end{aligned}$$

By the formulas given in Eq. (3.3), we may rewrite the previous equation in terms of the polynomials $R_n(x,y)$ and $S_n(x,y)$ as follows:

$$\frac{s_n}{F_d(\alpha)^n\beta_1^n} = F_{r+1}(\alpha)R_n(\rho,\sigma-1) + (-1)^{d-r}F_{d-r-1}(\alpha)F_d(\alpha)\beta_1 \cdot S_n(\rho,\sigma-1). \quad (4.7)$$

By (4.6) we have

$$y(t) = z(t) \cdot t^{-L_d(\alpha)/F_d(\alpha)} \exp\left(-\frac{(-1)^{d-1} t^{\beta_1}}{\beta_1 \cdot F_d(\alpha)}\right).$$

Comparing the coefficients of $t^{\beta_0+\beta_1 \cdot n-\alpha}$ yields

$$p_{nd+r-1} = \sum_{m=0}^{n} s_m \cdot \frac{\left(\frac{(-1)^{d-1}}{F_d(\alpha)\beta_1}\right)^{n-m}}{(n-m)!},$$

hence we have

$$\frac{p_{nd+r-1}}{F_d(\alpha)^n \beta_1^n} = \sum_{m=0}^{n} \frac{s_m}{F_d(\alpha)^m \beta_1^m} \cdot \frac{\left(\frac{(-1)^{d-1}}{F_d(\alpha)^2\beta_1^2}\right)^{n-m}}{(n-m)!} = \sum_{m=0}^{n} \frac{s_m}{F_d(\alpha)^m \beta_1^m} \cdot \frac{\rho^{n-m}}{(n-m)!}.$$

Substituting Eq. (4.7) into this last equation and using Eqs. (3.4) and (3.5) yields the formula for p_{nd+r-1} in Theorem 2.2.

For positive r, the formula for q_{nd+r-1} stated in Theorem 2.2 is an easy consequence of the formula for p_{nd+r-1}. Indeed, by Lemma 1.2, the denominator q_{nd+r-1} is the same as the numerator p_{nd+r-2} associated to $\xi(\alpha, \beta_0, \beta_1, d, r-1)$, thus we only need to replace r by $r-1$ in the formula stated for p_{nd+r-2}. It only remains to show the following lemma.

Lemma 4.1. *The equation stated for q_{nd+r-1} in Theorem 2.2 remains valid when we substitute $r = 0$.*

Proof. Since $F_0(\alpha) = 0$, substituting $r = 0$ in Theorem 2.2 gives

$$\frac{q_{nd-1}}{F_d(\alpha)^n \beta_1^n} = (-1)^{d+1} F_d(\alpha)^2 \cdot \beta_1 \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(n-k-1)!}{k!} \binom{n+\sigma-1-k}{n-2k-1} \rho^{k+1}.$$

Here we may replace $(-1)^{d+1} F_d(\alpha)^2 \cdot \beta_1$ by $\rho^{-1}\beta_1^{-1}$. After rearranging we obtain

$$q_{nd-1} = F_d(\alpha)^n \beta_1^{n-1} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{(n-k-1)!}{k!} \binom{n+\sigma-1-k}{n-2k-1} \rho^{k}. \tag{4.8}$$

We need to show the validity of this equation. Observe that

$$\begin{aligned} \xi(\alpha, \beta_0, \beta_1, d, 0) &= [\beta_0, \underbrace{\alpha, \ldots, \alpha}_{d-1}, \beta_0 + \beta_1, \ldots] = \beta_0 - \alpha + [\underbrace{\alpha, \ldots, \alpha}_{d}, \beta_0 + \beta_1, \ldots] \\ &= \beta_0 - \alpha + \xi(\alpha, \beta_0 + \beta_1, \beta_1, d, d) \end{aligned}$$

Thus the denominator q_{nd-1} associated to $\xi(\alpha, \beta_0, \beta_1, d, 0)$ is the same as the same as the denominator $q(n-1)d-1$ associated to $\xi(\alpha, \beta_0+\beta_1, \beta_1, d, d)$. We may apply the already shown part of Theorem 2.2. Since the magic quotient ρ depends only on α, β_1 and d, it is the same for $\xi(\alpha, \beta_0, \beta_1, d, 0)$ and for $\xi(\alpha, \beta_0+\beta_1, \beta_1, d, d)$. For the magic sums we get

$$\sigma(\alpha, \beta_0+\beta_1, \beta_1, d, d) = \frac{\beta_0+\beta_1-\alpha}{\beta_1} + \frac{L_d(\alpha)}{\beta_1 F_d(\alpha)} = \sigma(\alpha, \beta_0, \beta_1, d, 0) + 1.$$

Therefore we may obtain an equation for the q_{nd-1} associated to $\xi(\alpha,\beta_0,\beta_1,d,0)$ by replacing n with $n-1$, r with d and σ with $\sigma+1$ in the formula for q_{nd-1} in Theorem 2.2. Since $F_0(\alpha)=0$, the second sum vanishes and we get

$$\frac{q_{nd+r-1}(\alpha,\beta_0,\beta_1,d,0)}{F_d(\alpha)^{n-1}\beta_1^{n-1}}=F_d(\alpha)\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(n-1-k)!}{k!}\binom{n+\sigma-1-k}{n-1-2k}\rho^k,$$

which is obviously equivalent to (4.8). □

5. Special cases leading to elementary expressions

In this section we describe all instances of Theorem 2.9 for which the magic number σ is the half of an odd integer, forcing all (modified) Bessel functions in the statement to be known elementary functions. There is not much to say about the case $d=1$: as seen in Remark 2.10, we have $\sigma=\beta_0/\beta_1$ thus σ depends only on β_0 and β_1 in a very simple fashion. In the case when $d\geq 2$ we give a similarly simple description.

Theorem 5.1. *If $d\geq 2$ then σ is the half of an odd integer if and only if one of the following conditions holds:*

1. *$d=3$, $\alpha=1$ and $(\beta_0+1)/\beta_1$ is the half of an odd integer;*
2. *$d=2$, $\alpha=1$ and $(\beta_0+2)/\beta_1$ is the half of an odd integer;*
3. *$d=2$, $\alpha=2$ and $(\beta_0+1)/\beta_1$ is the half of an odd integer;*
4. *$d=2$, $\alpha=4$ and $(2\beta_0+1)/\beta_1$ is an integer.*

Proof. First we show that even assuming that σ is the half of an (even or odd) integer implies that d is at most 3. The fact that σ belongs to $(1/2)\cdot\mathbb{Z}$ implies the same for $\beta_1\sigma+\alpha-\beta_0=L_d(\alpha)/F_d(\alpha)$. We may subtract any integer multiple of $F_d(\alpha)$ from $L_d(\alpha)$ and still have element of $(1/2)\cdot\mathbb{Z}$. Let us select $r_d(\alpha):=L_d(\alpha)-\alpha F_d(\alpha)$ and consider the fraction $r_d(\alpha)/F_d(\alpha)\in(1/2)\cdot\mathbb{Z}$. We claim that

$$0<\frac{r_d(\alpha)}{F_d(\alpha)}<1\quad\text{holds for }\alpha\geq 2\text{ and }d\geq 3. \tag{5.1}$$

Indeed, for $d=3$, we have $r_3(\alpha)=L_3(\alpha)-\alpha F_3(\alpha)=(\alpha^3+3\alpha)-\alpha(\alpha^2+1)=2\alpha$ and $F_3(\alpha)=\alpha^2+1$. Both $r_3(\alpha)$ and $F_3(\alpha)$ are positive and $r_3(\alpha)<F_3(\alpha)$ follows from

$$F_3(\alpha)-r_3(\alpha)=(\alpha-1)^2>0\quad\text{for }\alpha\geq 2.$$

For $d=4$ we have $r_4(\alpha)=L_4(\alpha)-\alpha F_4(\alpha)=\alpha^4+4\alpha^2+2-\alpha(\alpha^3+2\alpha)=2\alpha^2+2$ and $F_4(\alpha)=\alpha^3+2\alpha$. Both $r_4(\alpha)$ and $F_4(\alpha)$ are positive and $r_4(\alpha)<F_4(\alpha)$ follows from

$$F_4(\alpha)-r_4(\alpha)=\alpha^3-2\alpha^2+2\alpha-2=\alpha^2(\alpha^1-1)+2(\alpha-1)>0\quad\text{for }\alpha\geq 2.$$

For larger values of d, we may show that

$$0<r_d(\alpha)<F_d(\alpha)\quad\text{holds when }d\geq 3\text{ and }\alpha\geq 2$$

by induction on d, using the fact that the statement is valid for $d\in\{3,4\}$ and that $r_d(\alpha)$ satisfies the same recurrence as $F_d(\alpha)$, allowing to express the inequality for the next value of d as a positive combination of the inequalities for the current and the previous values of d. This concludes the proof of (5.1).

As a consequence of the inequality (5.1), whenever $d\geq 3$ and $\alpha\geq 2$, the only way for $r_d(\alpha)/F_d(\alpha)$ to be an integer is to have $r_d(\alpha)/F_d(\alpha)=1/2$. In other words, in this case, we must have $2r_d(\alpha) = F_d(\alpha)$ which is equivalent to $2L_d(\alpha) = (2\alpha + 1)F_d(\alpha)$. To show that $\alpha \geq 2$ and $d \geq 3$ can not hold simultaneously, it suffices to show that $2L_d(\alpha)$ can never be equal to $(2\alpha + 1)F_d(\alpha)$. To do so, first we observe that

$$2L_d(\alpha) > (2\alpha + 1)F_d(\alpha) \quad \text{holds for } \alpha \in \{2, 3\} \text{ and } d \geq 3.$$

Indeed, for $\alpha = 2$ we have $28 = 2L_3(2) > (2 \cdot 2 + 1)F_3(2) = 25$ and $68 = 2L_4(2) > (2 \cdot 2 + 1)F_4(2) = 60$ and we may prove the same inequality for higher values of d by induction. Similarly, for $\alpha = 3$ we have $72 = 2L_3(3) > (2 \cdot 3 + 1)F_3(3) = 70$ and $238 = 2L_4(3) > (2 \cdot 3 + 1)F_4(3) = 231$ and we may proceed again by induction on d. Finally, to exclude $\alpha \geq 4$, we will show

$$2L_d(\alpha) < (2\alpha + 1)F_d(\alpha) \quad \text{holds for } \alpha \geq 4 \text{ and } d \geq 3.$$

For $d = 3$ we have

$$(2\alpha + 1)F_3(\alpha) - 2L_3(\alpha) = \alpha^2 - 4\alpha + 1 = \alpha(\alpha - 4) + 1 > 0,$$

and for $d = 4$ we have

$$(2\alpha + 1)F_4(\alpha) - 2L_4(\alpha) = \alpha^3 - 4\alpha^2 + 2\alpha - 4 = \alpha^2(\alpha - 4) + 2\alpha(\alpha - 2) > 0,$$

and again we may proceed by induction on d.

We obtained that, for $d \geq 3$, $\sigma \in (1/2) \cdot \mathbb{Z}$ is only possible if $\alpha = 1$. In that case, for $d \geq 4$ we have $L_d(1) - 2F_d(1) = F_{d-3}(1)$ (this may be shown by induction). Clearly $L_d(1)/F_d(1)$ is the half of an integer, if and only if the same holds for $F_{d-3}(1)/F_d(1)$. Now we may exclude the case $d \geq 4$ completely, after observing that

$$0 < \frac{F_{d-3}(1)}{F_d(1)} < \frac{1}{2} \quad \text{holds for } d \geq 4.$$

Indeed, the above inequality is equivalent to $0 < 2F_{d-3}(1) < F_d(1)$ which may be easily shown by induction.

We have shown that σ can only be the half of an integer if $d = 2$ or $d = 3$. In the case, when $d = 3$, we have also shown that only $\alpha = 1$ is possible, and we get

$$\sigma = \frac{\beta_0 - 1}{\beta_1} + \frac{L_3(1)}{\beta_1 F_3(1)} = \frac{\beta_0 - 1}{\beta_1} + \frac{4}{2\beta_1} = \frac{\beta_0 + 1}{\beta_1}.$$

Consider finally the case when $d = 2$. As before, $\sigma \in (1/2)\mathbb{Z}$ implies that $L_d(2)/F_d(2) = (\alpha^2 + 2)/\alpha \in (1/2)\mathbb{Z}$. This implies that α must be a divisor of 4, that is, an element of $\{1, 2, 4\}$. We have

$$\sigma = \frac{\beta_0 - \alpha}{\beta_1} + \frac{L_2(\alpha)}{\beta_1 F_2(\alpha)} = \frac{\alpha\beta_0 - \alpha^2}{\beta_1\alpha} + \frac{\alpha^2 + 2}{\beta_1\alpha} = \frac{\alpha\beta_0 + 2}{\beta_1\alpha}.$$

Therefore, for $\alpha = 1$ we get $\sigma = (\beta_0 + 2)/\beta_1$, for $\alpha = 2$ we get $\sigma = (\beta_0 + 1)/\beta_1$ and for $\alpha = 4$ we get $\sigma = (2\beta_0 + 1)/(2\beta_1)$. □

A nice example of the case when $d = 3$ and $\alpha = 1$ in Theorem 5.1 above is the case when $\alpha = 1$, $\beta_0 = 3m - 1$, $\beta_1 = 2m$, $d = 3$, $r = 2$ for some $m > 0$. In this case we get $\sigma = 3/2$ and $\rho = 1/(16m^2)$. Theorem 2.9 gives

$$\xi(1, 3m - 1, 2m, 3, 2) = \frac{2I_{1/2}(1/(2m))}{I_{1/2}(1/(2m)) + I_{3/2}(1/(2m))}$$

Using (1.8) and (1.10) we may rewrite the preceding equation as

$$[\overline{1, 1, 3m-1+2mn}]_{n=0}^{\infty} = \frac{2\sinh(1/(2m))}{\cosh(1/(2m)) - (2m-1)\sinh(1/(2m))}.$$

For $m = 1$ we obtain

$$[1, 1, 2, 1, 1, 4, \ldots] = \frac{e^{1/2} - e^{-1/2}}{e^{1/2}} = e - 1.$$

A similarly nice example for the case when $d = 2$ and $\alpha = 1$ in Theorem 5.1 above is

$$\xi(1, 3m-2, 2m, 2, 1) = [\overline{1, 3m-2+2mn}]_{n=0}^{\infty}$$

for some $m > 0$. In this example $\sigma = 3/2$ and $\rho = (-1)/(4m^2)$ hold. Theorem 2.9 gives

$$\xi = \frac{J_{1/2}(2\sqrt{-\rho})}{J_{1/2}(2\sqrt{-\rho}) - J_{3/2}(2\sqrt{-\rho})} = \frac{J_{1/2}(1/m)}{J_{1/2}(1/m) - J_{3/2}(1/m)}.$$

Using (1.9) and (1.10) we may rewrite ξ above as

$$\xi = \frac{\sin(1/m)}{\cos(1/m) - (m-1)\sin(1/m)}.$$

Substituting $m = 1$ yields $\tan(1) = [1, 1, 1, 3, 1, 5, 1, 7, \ldots]$.

We conclude this section with an example which is not likely to be found in the literature, due to its "sheer ugliness". Let us set $\alpha = 4$, $\beta_0 = 7m + 3$, $\beta_1 = 2m + 1$, $d = 2$ and $r = 1$, where m is any nonnegative integer. For this example we have $\sigma = 7/2$ and $\rho = (-1)/16(2m + 1)^2$, yielding $2\sqrt{-\rho} = 1/(4m + 2)$. Using the fact that

$$J_{5/2}(z) = \sqrt{\frac{2}{\pi z}}\left(\left(\frac{3}{z^2} - 1\right)\sin(z) - \frac{3}{z}\cos(z)\right)$$

and

$$J_{7/2}(z) = \sqrt{\frac{2}{\pi z}}\left(\left(\frac{15}{z^3} - \frac{6}{z}\right)\sin(z) - \left(\frac{15}{z^2} - 1\right)\cos(z)\right)$$

(see [10, List of formulæ: 46, 47]), Theorem 2.9 gives

$$\xi = \frac{4\left((12(2m+1)^2 - 1)\sin\left(\frac{1}{4m+2}\right) - 6(2m+1)\cos\left(\frac{1}{4m+2}\right)\right)}{(240m^2 + 228m + 53)\cos\left(\frac{1}{4m+2}\right) - (960m^3 + 1392m^2 + 648m + 97)\sin\left(\frac{1}{4m+2}\right)}.$$

The above formula was calculated from Theorem 2.9 with the help of Maple. The same program was used to double-check its correctness for selected values of m. For example, for $m = 0$ we obtain

$$\frac{4\left(11\sin\left(\frac{1}{2}\right) - 6\cos\left(\frac{1}{2}\right)\right)}{53\cos\left(\frac{1}{2}\right) - 97\sin\left(\frac{1}{2}\right)} = [4, 3, 4, 4, 4, 5, 4, 6, 4, 7, 4, \ldots].$$

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