TEMPERED STABLE DISTRIBUTIONS AND HIGH FREQUENCY FINANCIAL MODELING

by

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ABSTRACT

SINA SABA. Tempered stable distributions and high frequency financial modeling. (Under the direction of DR. MICHAEL GRABCHAK)

In this dissertation, we explore two applications of discrete tempered stable (DTS) distributions, a flexible class of distributions well-suited for modeling heavy-tailed and overdispersed data. DTS distributions are derived by tempering the tail of discrete stable distributions. They can also be represented as Poisson mixtures, for which the mixing distribution is a positive tempered stable (PTS) distribution. The first application addresses challenges in simulating positive tempered stable (PTS) distributions. Except for a few cases, there is no known simulation method for these distributions. We propose a novel simulation method using DTS distributions to approximate PTS simulations and establish a convergence rate for our estimation. This will make working with PTS distributions easier in many simulation-based methods. The second application focuses on modeling high-frequency financial data, also known as tick data. One characteristic of tick data is the discreteness of price changes in terms of tick size, which is the minimum price change allowed by an exchange. When considering price changes over a long time horizon, tick size usually does not play an important role and continuous models can be used for modeling. However, when working with high-frequency data over a small time horizon, price changes take only few values in term of tick size. This fact suggests the use of discrete distributions for modeling such price changes. We will compare several standard discrete distributions, specifically Poisson and negative binomial with the class of DTS distributions in modeling price changes in tick data. Additionally, we employ Monte Carlo methods to approximate the future distribution of portfolio values, utilizing these insights for risk assessment.

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CHAPTER 1: INTRODUCTION

A common practice in describing real world phenomena and modeling a population is to use probability distributions. Choosing the right distribution is often the most challenging part since, in most cases, just a sample, which is a small part of the entire population, is available. The main characteristics of a population should be reasonably explainable by the chosen distribution, e.g. , where the center of the population is, how much variation it has, or how extreme observations behave.

In some cases, it will be more favorable if the chosen distribution has some well behaved properties. For example, consider a study in which we are interested in modeling the loss from an investment in a one year time horizon. If X is a random variable that follows the *specified* chosen distribution, we can also model potential loss from the same investment in the monthly or weekly bases X can be written as the sum of 12 or 52 independent and identically distributed random variables. If one extends this idea for any natural number, then the property is called infinite divisibility. To name other preferable properties, we can refer to self-decomposability and stability, for which, roughly speaking, a random variable with these properties has the same distribution as a scaled copy of itself along with another distribution, and, has the same distribution as a scaled sum of copies of itself, respectively.

The other important fact when choosing a distribution is the amount of variation. Observing more variation in the sample suggests the use of distributions with heavier tails, in which the decay in the tails is slower, so it will be more probable to have observations farther from the center. This is the case in many real-world data sets, e.g. in financial data. In such cases, a class of distributions with heavier tails can be used.

Stable distributions are often good candidates for modeling in these cases, since they are infinitely divisible and heavy tailed. They are applicable to many fields, including finance, physics, biology, medicine, computer science [17]. However, their use is limited by two facts: 1) they have an infinite variance, i.e. their tails are too heavy, which is unrealistic in many applications as there are various real-world frictions preventing such heavy tails; and 2) they lack a closed form distribution function in almost all cases.

To address the first obstacle, there are some distributions obtained by modifying the tail behavior of a stable distribution. These modifications can be done by, for example, truncation, or tilting, in which the tails of the probability distribution function (pdf) vanish to zero starting from some point, or multiplying the pdf by some exponential function, respectively, and scaling the new pdf to integrate to one. These modifications speed up the decay of the pdf in the tails.

Another way to obtain a finite variance is by modifying in the Lévy measure in the characteristic function of a stable distribution. This leads to a subclass of infinitely divisible distributions, which are called tempered stable distributions (TS distributions). Indeed, if ν is the Lévy measure of a stable distribution on the real line, it has the following form [20]:

$$\nu(dx) = c_-|x|^{-1-\alpha}1_{(-\infty,0)}(x)dx + c_+x^{-1-\alpha}1_{(0,\infty)}(x)dx,$$

where $c_-, c_+ \geq 0$. Now, if we multiply this Lévy measure by a measurable, bounded, non-negative function g, while preserving it as a Lévy measure, which will be discussed soon, we get the Lévy measure of a TS distribution, which has lighter tails while leaving them essentially unchanged in the central portion [8]. Tweedie distributions are perhaps the first models developed in this way [23], using $g(x) = e^{-ax}$, for some $a > 0$ to modify the Lévy measure. By doing this, the tail of the heavy tailed stable distribution decays faster but not too fast, i.e., it leads to exponential tails.

In the other hand, to compare TS distributions to Gaussian distributions, which are popular in modeling in many fields, TS distributions have an advantage. It is well-known that the Gaussian's tails are too light to realistically describe the variability in many real-world data sets, but TS distributions have enough mass in their tails to explain this variability.

Choosing the best distribution with some well-behaved properties is not just applicable for samples with observations from a fixed point in time. In many studies, the sample consists of recorded observations of some quantity as it varies over time. These are known as time series and can be seen as a sample from a stochastic process, which is a sequence of random variables. One example of such a stochastic process is a Lévy process. Some of the main properties of a Lévy process are independent and stationary increments. If $\{X_t, t > 0\}$ is a stochastic process then, independent increment means for any $n > 0$ and any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables X_{t_0}, $X_{t_1} - X_{t_0}$, $\ldots$, $X_{t_n} - X_{t_{n-1}}$ are independent, and, stationary increment property means the distribution of $X_{s+t} - X_s$ does not depend on s. Starting almost surly at zero, i.e. $X_0 = 0$ a.s., and stochastically continuity and having sample path to be right continuous with the left limit with probability one, are other properties to define a Lévy process. These processes are closely related to the infinitely divisible distributions. One can refer to [20] for more details about Lévy processes and infinitely divisible distributions.

An interesting property of TS distributions is when they are used to define a Lévy process, i.e. X_1 is a TS distribution in $\{X_t, t > 0\}$. Under some general conditions, the behavior of Lévy processes with marginal TS distributions is similar to the Brownian motion in a long time frame and more like the stable Lévy process in a short time frame, thus it combines both Gaussian and stable trends [18].

However, TS distributions still suffer from the lack of a closed form for the distribution function. This may discourage practitioners to use these distributions. Also, except for a small number, there is not any well-developed way to simulate from TS distributions.

All stable distributions, except for the degenerated ones, are absolutely continuous and a subclass of infinitely divisible distributions and hence they have the nice properties of this class of distributions. In the seminal paper of Steutel and et al. [21], they introduced a discrete analogue of stable distributions which consists of a two parameters class of distributions for $\alpha \in (0,1)$ and $\eta \geq 0$. We denote these by $\mathrm{DS}_\alpha(\eta)$. A representation of these distributions is by means of positive stable (PS) distributions and Poisson processes, which are an example of Lévy process in which N_1 follows a Poisson distribution with mean 1, see page 371 Theorem 6.7 in [22]. Indeed, let $\alpha \in (0,1)$ and $\eta \geq 0$. If $\{N_t, t > 0\}$ is a Poisson process with rate 1, and $T \sim PS_\alpha(\eta)$ is independent of this process then $N_T \sim \mathrm{DS}_\alpha(\eta)$. In [8], an extension of this idea was used to define discrete tempered stable (DTS) distributions. These include many important integer-valued distributions, perhaps most prominently the Poisson-Tweedie distributions, that were introduced by Hougaard et al. in [10].

In this dissertation, we are specifically focused on the important class of DTS distributions. We consider two interesting applications of these discrete distributions. First, we use them to address the problem in simulations from TS distributions. We propose a simulation method based on DTS distributions to approximately simulate from TS distributions. This will make working with TS distributions easier in many simulation based methods. We also calculate a convergence rate for our estimation.

TS distribution that only take positive values are called positive tempered stable distributions (PTS distributions). PTS distributions on which we will be focusing on in our simulation study, are the building blocks from which almost all continuous TS distributions can be built.

In the second application, we use DTS distributions in modeling high-frequency financial data, also known as tick-by-tick data. One characteristic of tick data is the discreteness of price changes in terms of tick size, which is the minimum price change allowed by the exchange. This fact suggests the use of discrete distributions for modeling these price changes. We will compare some more standard discrete distributions, specifically Poisson and negative binomial with the class of DTS distributions in modeling price changes in tick data. These models can be used to simulate a sample path for the portfolio value. Using Monte Carlo methods, we approximate the distribution of the portfolio value for a future time horizon and use this approximation for risk assessment purposes.

CHAPTER 2: PRELIMINARIES

The aim of this chapter is to provide a summary of the required concepts for the following chapters. In this chapter, some concepts from probability theory, alongside with more specific areas, i.e., tempered stable distributions and discrete tempered stable distributions are discussed.

Throughout this writing, we use $\mathcal{B}(\mathbb{R})$ to denote the Borel σ-algebra on $\mathbb{R}$. This is the smallest σ-algebra that contains all of the open sets in $\mathbb{R}$. We use $(\Omega, \mathcal{F}, \mathbb{P})$ to denote a probability space, in which Ω is a set, $\mathcal{F}$ is a σ-algebra on Ω, and $\mathbb{P}$ is a probability measure. We use Pois, Pois_+, $U(a,b)$, $Ga(\alpha,\beta)$ to denote Poisson, zero-truncated Poisson, uniform and gamma distributions, respectively. For a set A, 1_A is the indicator function on A. We write $\mathrm{B}(x,s) = \int_0^1 t^{x-1}(1-t)^{s-1}dt$ and $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ to denote beta function and gamma function, respectively. We also use $\stackrel{\mathrm{D}}{=}$, $\stackrel{\mathrm{D}}{\to}$ to denote equality and convergence in distribution, respectively. The notation i.i.d. means independent and identically distributed. We also have $(x \wedge y) = \min\{x, y\}$.

2.1 Tempered Stable Distributions

One of the main results of this dissertation is to estimate a convergence rate for a sequence of properly modified DTS distributions to a PTS distribution. The convergence was proved in [8]. In this section, we recall some concepts from probability theory that are needed to define tempered stable distributions and we also explore some properties of these distributions.

Definition 2.1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A mapping X from Ω into $\mathbb{R}$ is a $\mathbb{R}$-valued random variable if $\{\omega : X(\omega) \in B\}$ is in $\mathcal{F}$ for each $B \in \mathcal{B}(\mathbb{R})$. A* *probability measure P_X defined on $\mathbb{R}$ using X,*

$$P_X(B) = \mathbb{P}(\{\omega : X(\omega) \in B\}), \qquad B \in \mathcal{B}(\mathbb{R}),$$

is called the distribution of X.

To study tempered stable distributions, we explore their characteristic functions, as they are often characterized using these. First, we recall the definition of the characteristic function of a distribution of a random variable.

Definition 2.1.2. *The characteristic function of the distribution P_X of a random variable X on $\mathbb{R}$ is denoted by $\hat{P}_X$ and it is defined as*

$$\hat{P}_X(z) = E[e^{izX}] = \int_{\mathbb{R}} e^{izx} P_X(dx), \quad z \in \mathbb{R}.$$

When working with non-negative real valued, or, non-negative integer valued random variables, it is more preferable to use probability Laplace-Stieltjes transforms (pLSt) and probability generating functions (pgf) [22], respectively.

Definition 2.1.3. *Let X be a non-negative real-valued random variable with distribution P_X. The probability Laplace-Stieltjes transform (pLSt) of X is a function $\pi_X : \mathbb{R}_+ \to (0, \infty)$:*

$$\pi_X(z) := E[e^{-zX}] = \int_{\mathbb{R}_+} e^{-zx} P_X(dx), \qquad z \in [0, \infty).$$

Definition 2.1.4. *Let X be a non-negative integer valued random variable with probability mass function (p_k), which means that X takes value k with the probability p_k. The probability generating function (pgf) of X is a function $\psi_X : [0,1] \to [0,1]$:*

$$\psi_X(z) := E[z^X] = \sum_{k=0}^{\infty} p_k z^k, \qquad z \in [0, 1].$$

We now give two important properties of characteristic functions, which are useful in the study of stable distributions.

Proposition 2.1.5. *Let X, Y be two random variables on $\mathbb{R}$,*

i) For any $a > 0$ and $b \in \mathbb{R}$

$$\hat{P}_{aX+b}(z) = e^{izb}\hat{P}_X(az), \quad z \in \mathbb{R}.$$

ii) If X and Y are independent, then

$$\hat{P}_{X+Y}(z) = \hat{P}_X(z)\hat{P}_Y(z), \quad z \in \mathbb{R}.$$

Proof. i) We have for $z \in \mathbb{R}$

$$\hat{P}_{aX+b}(z) = E[e^{iz(aX+b)}] = E[e^{izaX}e^{izb}] = e^{izb}E[e^{izaX}] = e^{izb}\hat{P}_X(az).$$

ii) We have for $z \in \mathbb{R}$

$$\hat{P}_{X+Y}(z) = E[e^{iz(X+Y)}] = E[e^{izX+izY}] = E[e^{izX}]E[e^{izY}] = \hat{P}_X(z)\hat{P}_Y(z),$$

in which the third equality follows by the independence of X and Y. □

One important concept regarding to sum of two independent random variables is the convolution of the distributions. We first recall the convolution of two distributions on $\mathbb{R}$, then we give a theorem about sum of independent random variables, in which, its proof can be found in [2].

Definition 2.1.6. *The convolution μ of two distributions μ_1 and μ_2 on $\mathbb{R}$, denoted* *by $\mu = \mu_1 * \mu_2$, is the distribution defined by*

$$\mu(B) = \iint_{\mathbb{R}\times\mathbb{R}} 1_B(x+y)\mu_1(dx)\mu_2(dy), \qquad B \in \mathcal{B}(\mathbb{R}).$$

Theorem 2.1.7. *If X and Y are independent, then $X+Y$ has distribution $P_X * P_Y$.*

Now, we define infinitely divisible distributions, which are a rich and interesting class of distributions. Stable and tempered stable distributions, on which we are focused in this dissertation, are infinitely divisible. For more details about these distributions, one can explore e.g. [20] or [22].

Definition 2.1.8. *A probability measure μ on $\mathbb{R}$ is infinitely divisible if, for any positive integer n, there is a probability measure μ_n on $\mathbb{R}$ such that $\mu = \mu_n^n$, where μ^n is the n-fold convolution of the probability measure μ with itself.*

The proof of the following theorem can be found in [20].

Theorem 2.1.9. *A $\mathbb{C}$ valued function ϕ on $\mathbb{R}$ is the characteristic function of an infinitely divisible distribution μ if and only if ϕ has the form*

$$\phi(z) = \exp\left(izb - \frac{1}{2}z^2\sigma^2 + \int_{\mathbb{R}}(e^{izx} - 1 - izx1_{\{|x|\le 1\}}(x))\nu(dx)\right),$$

where $b \in \mathbb{R}$, $\sigma^2 \ge 0$, and ν is a Borel measure on $\mathbb{R}$ satisfying

$$\nu(\{0\}) = 0, \qquad \int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty.$$

The triple (b, σ^2, ν) is uniquely determined by ϕ.

Definition 2.1.10. *We call (b, σ^2, ν) in Theorem 2.1.9 the generating triplet of μ. ν is called the Lévy measure of μ. When $\sigma^2 = 0$, μ is called purely non-Gaussian.*

Remark 2.1.11. *In Theorem 2.1.9, if the additional condition $\int_{|x|<1} |x|\nu(dx) < \infty$*

is satisfied, then we have

$$\phi(z) = \exp\left(izb' - \frac{1}{2}z^2\sigma^2 + \int_{\mathbb{R}}(e^{izx} - 1)\nu(dx)\right),$$

where $b' = b - \int_{\mathbb{R}} x1_{\{|x|\leq 1\}}(x)\nu(dx)$ *is called the drift of* μ. *In this case we write* $(b', \sigma^2, \nu)_0$ *to denote the generating triplet of* μ.

Proof of the following theorem can be found in [20]

Theorem 2.1.12. *If* μ *is infinitely divisible, then* $\hat{\mu}$ *has no zeros.*

Infinitely divisible distributions are closely related to Lévy process. One can refer to [20] for more information about Lévy processes and infinitely divisibility.

Definition 2.1.13. *A stochastic process* $\{X_t : t \geq 0\}$ *on* $\mathbb{R}$ *is called a Lévy process if the following conditions are satisfied:*

1. $X_0 = 0$ *a.s.*

2. *For any choice of* $n \geq 1$ *and* $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, *random variables* X_{t_0}, $X_{t_1} - X_{t_0}$, $\ldots$, $X_{t_n} - X_{t_{n-1}}$ *are independent. (independent increments property)*

3. *The distribution of* $X_{s+t} - X_s$ *does not depend on* s. *(stationary increments property)*

4. *It is stochastically continuous, i.e. for every* $t \geq 0$ *and* $\varepsilon > 0$,

$$\lim_{s\to t} \mathbb{P}[|X_s - X_t| > \varepsilon] = 0$$

5. *There is* $\Omega_0 \in \mathcal{F}$ *with* $\mathbb{P}[\Omega_0] = 1$ *such that, for every* $\omega \in \Omega_0$, $X_t(\omega)$ *is right continuous in* $t \geq 0$ *and has left limits in* $t > 0$.

In what follows we first recall the definition of a stable distribution and discuss how discrete stable distributions were defined. Then tempered stable distributions are defined by modifying the Lévy measure of a stable distribution.

Definition 2.1.14. *A probability measure μ on $\mathbb{R}$ is called stable if for any n and any $X, X_1, \dots X_n \overset{i.i.d.}{\sim} \mu$ there is $b_n \in \mathbb{R}$ such that*

$$X \overset{D}{=} n^{-1/\alpha} \sum_{k=1}^{n} X_k - b_n,$$

for some $\alpha \in (0, 2]$. In this case the measure is called α-stable distribution.

Proposition 2.1.15. *If μ is an α-stable distribution with $\alpha \in (0, 2)$, then it is infinitely divisible with generating triplet (b, σ^2, ν) in which the Lévy measure is*

$$\nu(dx) = \eta_- |x|^{-1-\alpha} 1_{(-\infty,0)}(x)dx + \eta_+ x^{-1-\alpha} 1_{(0,\infty)}(x)dx,$$

where $c_-, c_+ \geq 0$, $\sigma^2 = 0$, and $b \in \mathbb{R}$.

We denote the α-stable distribution by $\mathrm{S}_\alpha(\eta_-, \eta_+; b)$, and if $b = 0$ we denote it by $\mathrm{S}_\alpha(\eta_-, \eta_+) = \mathrm{S}_\alpha(\eta_-, \eta_+; 0)$, and if it is defined on $\mathbb{R}_+$, for which we will have $\alpha \in (0, 1)$ and $b \geq 0$, we denote it by $\mathrm{PS}_\alpha(\eta_+; b) = \mathrm{S}_\alpha(0, \eta_+; b)$

We note that stable distributions can not be defined by the above definition for discrete cases, since the discreteness will be lost by the multiplication. An analogy of stability in the case of discrete distributions was introduced in [21] by introducing thinning operation '$\circ$', for $\gamma \in (0, 1]$ and a random variable X on non-negative integers to be a random variable $\gamma \circ X$ with distribution:

$$\gamma \circ X \overset{\mathrm{D}}{=} \sum_{i=1}^{X} \varepsilon_i,$$

where $\varepsilon_1, \varepsilon_2, \dots$ are i.i.d. random variables independent of X and follow a Bernoulli distribution with $\mathbb{P}(\varepsilon_i = 1) = 1 - \mathbb{P}(\varepsilon = 0) = \gamma$ and assuming $\sum_{i=1}^{0} \varepsilon_i = 0$. Now, a discrete stable distribution on non-negative integers can be defined as follow:

Definition 2.1.16. *Let $\alpha \in (0, 1]$. A probability measure μ on $\mathbb{Z}_+$ is called discrete*

α-stable, if for any $n \in \mathbb{N}$, we have

$$X \stackrel{D}{=} n^{-1/\alpha} \circ (X_1 + X_2 + \cdots + X_n),$$

where $X, X_1, X_2, \ldots, X_n \stackrel{i.i.d.}{\sim} \mu$.

The pgf of the discrete stable distribution is

$$\psi_X(z) = \exp\left(-\eta \frac{\Gamma(1-\alpha)}{\alpha}(1-s)^\alpha\right), \qquad |s| \leq 1,$$

where $\eta \geq 0$ is a parameter [8]. We denote this two parameter class of distributions by $\mathrm{DS}_\alpha(\eta)$.

An interesting representation of a $\mathrm{DS}_\alpha(\eta)$ distribution is recalled from page 371 [22], where it is defined as a mixture of a Poisson distribution. Here, a Poisson process is a Lévy process $\{N_t, t \geq 0\}$ for which $N_1 \sim \mathrm{Pois}(1)$.

Proposition 2.1.17. *Fix $\alpha \in (0,1)$ and $\eta \geq 0$. If $\{N_t : t \geq 0\}$ is a Poisson process with rate 1 and $T \sim \mathrm{PS}_\alpha(\eta)$ is independent of this process, then $N_T \sim DS_\alpha(\eta)$.*

Now, we define TS distributions by making some changes in the Lévy measure of a stable distribution.

Definition 2.1.18. *An infinitely divisible distribution μ with generating triplet (b, σ^2, ν) is called a tempered stable (TS) distribution if $\sigma^2 = 0$, and*

$$\nu(dx) = \eta_- g(x)|x|^{-1-\alpha} 1_{\{x<0\}} dx + \eta_+ g(x) x^{-1-\alpha} 1_{\{x>0\}} dx,$$

where $\alpha \in (0,2)$, $\eta_-, \eta_+ \geq 0$ and $g : \mathbb{R} \mapsto \mathbb{R}_+$ is bounded, non-negative, Borel functions satisfying

$$\lim_{x \to 0} g(x) = 1, \qquad \int_{-\infty}^{\infty} (1 \wedge |x|^2) g(x)|x|^{-1-\alpha} dx < \infty.$$

We call g the tempering function and we write $\mu = \mathrm{TS}_\alpha(g, \eta, b)$. When $b = 0$, we write $\mathrm{TS}_\alpha(g, \eta) = \mathrm{TS}_\alpha(g, \eta, 0)$

The important special case of positive tempered stable distributions (PTS) is as follow.

Definition 2.1.19. *An infinitely divisible distribution μ with generating triplet (b, σ^2, ν) is called positive tempered stable (*PTS*) distribution if it is tempered stable with $b \geq 0$ and*

$$\nu(dx) = \eta g(x) x^{-1-\alpha} 1_{\{x>0\}} dx, \tag{2.1.1}$$

where $\alpha \in (0, 1)$, $\eta \geq 0$ and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a bounded, non-negative, Borel function satisfying

$$\lim_{x \downarrow 0} g(x) = 1, \quad \int_0^\infty (1 \wedge x) g(x) x^{-1-\alpha} dx < \infty.$$

We call g the tempering function and we write $\mu = \mathrm{PTS}_\alpha(g, \eta, b)$. When $b = 0$, we write $\mathrm{PTS}_\alpha(g, \eta) = \mathrm{PTS}_\alpha(g, \eta, 0)$*.*

Using the fact that PTS distributions are infinitely divisible and also (2.1.1), the characteristic function of a $\mathrm{PTS}_\alpha(g, \eta)$ is in the following form.

$$\gamma(z) = \exp\left(\int_0^\infty (\exp(ixz) - 1) \eta g(x) x^{-\alpha-1} dx \right). \tag{2.1.2}$$

In [8] discrete tempered stable distributions are defined in terms of PTS distributions as an analogy to the discrete stable distributions in 2.1.17.

Definition 2.1.20. *Fix $\alpha \in (0, 1)$ and $\eta \geq 0$. Let $T \sim \mathrm{PTS}_\alpha(g, \eta)$ and let $\{N_t : t \geq 0\}$ be a Poisson process with rate 1 and independent of T. The distribution of N_T is called a* DTS *distribution. We denote this distribution by* $\mathrm{DTS}_\alpha(g, \eta)$*.*

If $X \sim \mathrm{DTS}_\alpha(g, \eta)$ then the probability generating function of X is

$$E[z^X] = \exp\left\{ -\eta \int_0^\infty (1 - e^{-(1-z)x}) g(x) x^{-1-\alpha} dx \right\}, \quad z \in (0, 1]. \tag{2.1.3}$$

Indeed, if $T \sim \mathrm{PTS}_\alpha(g,\eta)$ and let the conditional distribution of X given T be Poisson with mean T, then the unconditional distribution of X is $\mathrm{DTS}_\alpha(g,\eta)$. First note that, since X given T follows a Pois(T) distribution, the pgf of X given T is

$$E[z^X|T] = \sum_{k=0}^{\infty} z^k e^{-T}\frac{T^k}{k!} = e^{-(1-z)T},$$

and, since $T \sim \mathrm{PTS}_\alpha(g,\eta)$, and the Lévy measure of T is $\nu(dx) = \eta g(x)x^{-1-\alpha}1_{x>0}dx$, we have

$$E[z^X] = E[E[z^X|T]] = E[e^{-(1-z)T}] = \exp\left\{-\eta\int_0^\infty (1-e^{-(1-z)t})g(t)t^{-1-\alpha}dt\right\},$$

in which, we use this fact that, the probability Laplace Stieltjes transform (pLSt) of a $\mathrm{PTS}_\alpha(g,\eta)$ with the Lévy measure of $\nu(dx) = \eta g(x)x^{-1-\alpha}1_{x>0}dx$ is

$$\pi_X(z) = E[e^{-zx}] = \exp\left\{-\eta\int_0^\infty (1-e^{-zx})g(x)x^{-1-\alpha}dx\right\}.$$

CHAPTER 3: SIMULATION AND ESTIMATION FOR DTS DISTRIBUTIONS

The aim of this chapter is to cover simulation methods for DTS distributions and give two examples. These methods are discussed in more details in [9]. These two examples are the two DTS distributions that will be used in the later sections for approximate simulation from PTS distributions. Simulation from a zero-truncated Poisson distribution is important for the simulation methods from DTS distributions, so different simulation methods will be considered for this distribution. In the last section, two different methods are discussed to estimate parameters of the DTS distributions from a sample.

3.1 Simulation from DTS Distributions

In this section, we review the simulation method in [9], which is based on rejection sampling method to simulate from DTS distributions. We begin by explaining some quantities.

Let $X \sim \mathrm{DTS}_\alpha(g, \eta)$. The sequence (ℓ_k) is defined as

$$\ell_k = \int_0^\infty e^{-x} g(x) x^{k-\alpha-1} dx, \quad k = 1, 2, \ldots,$$

and ℓ_+ is a quantity defined by

$$\ell_+ = \sum_{i=1}^\infty \frac{\ell_i}{i!} = \int_0^\infty (1 - e^{-x}) g(x) x^{-\alpha-1} dx.$$

From proposition 1 in [9] we have

Proposition 3.1.1. *1. The pgf of* $\mathrm{DTS}_\alpha(g,\eta)$ *is*

$$e^{-\eta\ell_+}\exp\left\{\eta\sum_{k=1}^{\infty}\frac{\ell_k}{k!}s^k\right\},\quad s\in(0,1].$$

2. $\mathrm{DTS}(g,\eta)$ *is infinitely divisible with the Lévy measure L, where L is concentrated on $\{1,2,\dots\}$ with*

$$L(\{k\})=\frac{\eta}{k!}\ell_k,\quad k=1,2,\dots,$$

and $L((0,\infty))=\eta\ell_+<\infty$.

3. Let L_1 be the probability measure on $\{1,2,\dots\}$ with

$$L_1(\{k\})=\frac{L(\{k\})}{L((0,\infty))}=\frac{\ell_k}{k!\ell_+},\quad k=1,2,\dots.$$

If $N\sim\mathrm{Pois}(\eta\ell_+)$ and $Y_1,Y_2,\dots\overset{i.i.d.}{\sim}L_1$ are independent of N, then

$$\sum_{i=1}^{N}Y_i\sim\mathrm{DTS}_\alpha(g,\eta).$$

Now, we restate the first part of Proposition 2 from [9]:

Proposition 3.1.2. *Let $X\sim\mathrm{DTS}_\alpha(g,\eta)$ and let $p_k=P(X=k)$ for $k=0,1,2,\dots$ then $p_0=e^{-\eta\ell_+}$ and*

$$p_k=\frac{\eta}{k}\sum_{i=0}^{k-1}\frac{\ell_{k-i}}{(k-i-1)!}p_i,\quad k=1,2,\dots. \tag{3.1.1}$$

We use the following algorithm to simulate from a DTS distribution:

Algorithm DTS: Simulation from $\mathrm{DTS}_\alpha(g,\eta)$

1. Simulate from $N\sim\mathrm{Pois}(\eta\ell_+)$

2. Simulate $Y_1,Y_2,\dots,Y_N\overset{\text{i.i.d.}}{\sim}L_1$ and return $\sum_{i=1}^{N}Y_i$

L_1 is a mixture of zero-truncated Poisson distribution [9]. Let $\mathrm{Pois}_+(\lambda)$ denote the zero truncated Poisson distribution with parameter λ and pmf

$$p_\lambda(k) = \frac{\lambda^k e^{-\lambda}}{k!(1-e^{-\lambda})},$$

then we have

$$\begin{aligned} L_1(\{k\}) &= \frac{1}{\ell_+}\int_0^\infty \frac{x^k e^{-x}}{k!(1-e^{-x})} g(x) x^{-\alpha-1}(1-e^{-x})dx \\ &= \int_0^\infty p_x(k) f(x) dx, \end{aligned}$$

where

$$f(x) = \frac{1}{\ell_+} g(x) x^{-\alpha-1}(1-e^{-x}), \quad x > 0.$$

Thus, to simulate from L_1, we can use the following Algorithm.

Algorithm L: Simulation from L_1.

1. Simulate $\lambda \sim f$
2. Simulate $Y \sim \mathrm{Pois}_+(\lambda)$ and return Y

To simulate from $\mathrm{Pois}_+(\lambda)$, we use 'rztpois' method in the R package 'actuar', see Dutand et al. [4], when $\lambda \geq 0.001$. We develop a new method and use it when $\lambda < 0.001$. This method will be discussed in a later section. Simulation from f, can be done using rejection sampling based on a trial distribution with pdf

$$g_\alpha(x) = \alpha(1-\alpha)x^{-1-\alpha}(x \wedge 1), \quad x > 0,$$

where $0 < \alpha < 1$ is a parameter. This is a type of Log-Laplace distribution which we denote by $\mathrm{LL}(\alpha)$ and if $U_1, U_2 \overset{i.i.d.}{\sim} U(0,1)$ and $Y = U_1^{1/(1-\alpha)} U_2^{-1/\alpha}$ then $Y \sim \mathrm{LL}(\alpha)$. Finally, we have

$$\phi_1(x) = C_q^{-1} \frac{g(x)(1-e^{-x})}{x \wedge 1},$$

where $C_q = \text{ess sup}_{x\geq 0} g(x) < \infty$. Now, we can use the following rejection sampling algorithm.

Algorithm F1: Simulation from f.

1. Independently simulate $U \sim \mathrm{U}(0,1)$ and $Y \sim \mathrm{LL}(\alpha)$
2. If $U \leq \phi(Y)$ return Y, otherwise go back to step 1

The probability of acceptance is $\ell_+\alpha(1-\alpha)/C_q$. In a case that probability of acceptance is low we can use a different trial distribution, which has a pdf of the form

$$h_{\beta,p,\zeta}(x) = \frac{p\zeta^{\beta/p}}{\Gamma(\beta/p)} x^{\beta-1} e^{-\zeta x^p}, \quad x > 0,$$

where β, p, ζ are parameters. We denote this parameter by $\mathrm{GGa}(\beta, p, \zeta)$. If $Y \sim \mathrm{Ga}(\beta/p, \zeta)$, then $Y^{1/p} \sim \mathrm{GGa}(\beta, p, \zeta)$. We use this distribution for rejection sampling whenever there exists $p, C > 0$ such that, for Lebesgue a.e. x,

$$g(x) \leq Ce^{-\zeta x^p}. \tag{3.1.2}$$

For this case we have

$$\phi_2(x) = \frac{g(x)(1-e^{-x})}{Cxe^{-\zeta x^p}}.$$

Thus, we use the following rejection sampling algorithm.

Algorithm F2: Simulation from f when (3.1.2) holds.

1. Independently simulate $U \sim \mathrm{U}(0,1)$ and $Y \sim \mathrm{GGa}(1-\alpha, p, \zeta)$
2. If $U \leq \phi(Y)$ return Y, otherwise go back to step 1

On a given iteration, the probability of acceptance is $\frac{\ell_+ p\zeta^{\frac{1-\alpha}{p}}}{C\Gamma(\frac{1-\alpha}{p})}$

Remark 3.1.3. *To enhance the speed of this method we vectorize the process. We simulate all the observations as a vector rather than simulate them one by one. However, in this method we always lose some of the observations due to rejection part of* *the method. To implement this method in a way that it can simulate a vector of a specific size, we need to estimate the number of rejections and account for this loss. To overcome this problem, we do as follow. If* p *is the probability of acceptance in either of the Algorithm F1 or F2, and if we need to simulate* k *observation from the* DTS *distribution, then we can estimate the number of simulations from* U$(0, 1)$ *and* LL *or* GGa *in Algorithm F1 or F2 to start the simulation and end up with* k *simulated* DTS *distribution after losing those rejected observations.*

1. *In Algorithm* DTS *simulate* k *observations* $N_1, N_2, \ldots, N_k$ *from* $\text{Pois}(\eta\ell_+)$
2. $S = N_1 + N_2 + \cdots + N_k$*, thus* S *is the number of observations we need from* $\text{Pois}_+(\lambda)$
3. *let* $n = S/p$*, thus* n *is an approximate to the number of observations we need to simulate from* $U(0, 1)$ *and* LL *or* GGa *in Algorithm F1 or F2 to end up with* k *simulated* DTS *distribution after rejecting some of these observations. (To make sure we have enough observations, we can multiply* n *by some number larger than* 1*, e.g.* 1.02*.)*

3.2 Simulation From Poisson-Tweedie and Beta-Prime Distributions

In this section, we use the simulation method in previous section to simulate from Poisson-Tweedie and Beta-Prime Tempering distributions.

Let X be a random variable that follows a Poisson-Tweedie distribution. This means that $X \sim \text{DTS}_\alpha(g, \eta)$ with $g(x) = e^{-ax}$, $a > 0$, $\eta \geq 0$ and $\alpha \in (0, 1)$.

To simulate from X, we also need to have c_g which is the essential supremum of the tempering function g, $C > 0$ which is the constant in the Algorithm F2, and the quantity ℓ_+. In this case ℓ_k, $k = 1, 2, 3, \ldots$, is of the form

$$\begin{aligned}
\ell_k &= (1 + a)^{-(k-\alpha)}\Gamma(k - \alpha), \\
\ell_+ &= \frac{\Gamma(1 - \alpha)}{\alpha}((a + 1)^\alpha - a^\alpha).
\end{aligned}$$

Now, we discuss the simulation method from previous section in more details and illustrate it with some examples. To simulate from the Poisson-Tweedie distribution, we consider the cases where $a = 1.5$, $\eta = 1$, $c_g = 1$, $C = 1$ and $\alpha \in \{0.25, 0.5, 0.75\}$. To start the simulation, we first calculate the probability of acceptance for Algorithms F1 and F2 for different values of α. Table 3.1 shows these probabilities. As we can see these probabilities are higher for Algorithm F2.

Table 3.1: Probability of acceptance in Algorithm F1 and F2 for $\alpha \in \{0.25, 0.5, 0.75\}$

Probability of Acceptance	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
Algorithm F1	0.1385501	0.3158459	0.5735478
Algorithm F2	0.8173162	0.8729833	0.9337058

To use Algorithm F2, first we need to simulate from $\mathrm{U}(0,1)$ and from $\mathrm{GGa}(1-\alpha, p, \zeta)$. For this case of the Poisson-Tweedie distribution, we have $C = p = 1$, $\zeta = a$, thus, we need to simulate from $\mathrm{Ga}(1-\alpha, a)$.

In this simulation study, our aim is to simulate 10^5 observations from Poisson-Tweedie distribution. by Remark 3.1.3, we can estimate the number of observations n, that we need to have from $\mathrm{U}(0,1)$ and $\mathrm{Ga}(1-\alpha, a)$ in Algorithm F2 to end up with $k = 10^5$ simulated observations from Poisson-Tweedie distribution after losing some of the observations due to rejections. The first row in Table 3.2 shows how many observations from each $\mathrm{U}(0,1)$ and $\mathrm{Ga}(1-\alpha, a)$ in Algorithm F2 we need to start with for different values of α to end up with 10^5 observations. The second row in this table shows the number of accepted observations in Algorithm F2, and the last row shows the proportion of accepted observations, for which they are very close to the probability of acceptance from Algorithm F2 in the Table 3.1.

Figure 3.1 shows the histogram of the simulated observation from the probability density function f and compare it to its actual value. In this case we have

$$f(x) = \frac{1}{\ell_+} e^{-x} x^{-\alpha-1}(1-e^{-x}), \quad x > 0$$

Table 3.2: First row shows the number of observations we need to start with in Algorithm F2. The second row shows the number of accepted observations in Algorithm F2, and the last row shows the proportion of acceptance using Algorithm F2

Acceptance rate	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
number of obs to start with in Algorithm F2	92865	147723	335418
number of accepted obs in Algorithm F2	75935	128813	312966
proportion of acceptance	0.8177	0.8720	0.9331

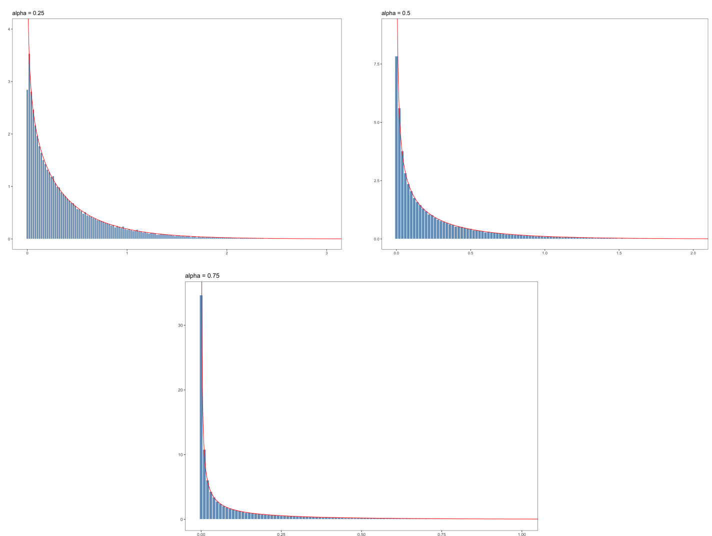

Figure 3.1: Histograms for simulated observations from f

Once we have the accepted observations from f in Algorithm F2, we use them as the mean of the zero-truncated Poisson distribution in Algorithm L to simulate $Y \sim \text{Pois}_{+}(\lambda)$, and then in the last step, in Algorithm DTS, we use each simulated observation N_k, $k = 1, 2, \ldots, 10^5$ from $\text{Pois}(\eta\ell_{+})$ to see how many of Ys should be added together to simulate an observation from Poisson-Tweedie distribution. Figure 3.2 compares the probability mass function and the histogram of simulated observations from Poisson-Tweedie distribution.

The second example in this section is to simulate from a Beta-Prime Tempering distribution. If X be a random variable with Beta-Prime Tempering distribution, we have $X \sim \text{DTS}_{\alpha}(g, \eta)$ and the pgf of X is in the form of (2.1.3) with the tempering

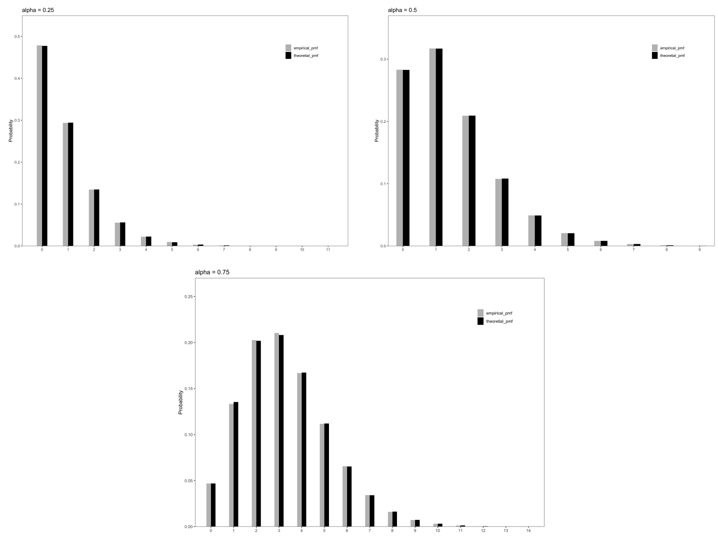

Figure 3.2: Probability mass function and histogram of simulated Poisson-Tweedie distribution

function

$$g(x) = \frac{1}{\mathrm{B}(\rho,\beta)} \int_0^\infty e^{-xs} s^{\rho-1} (1+s)^{-\beta-\rho} ds, \quad \beta > \alpha, \rho > 0,$$

where, $\eta \geq 0$, $\alpha \in (0,1)$, $\beta > \alpha$ and $\rho > 0$, are the parameters.

To simulate from X, we also need to have c_g which is the essential supremum of the tempering function g, and the quantity ℓ_+ [9]. We note that ℓ_k, $k = 1, 2, 3, \ldots$, is given by

$$\begin{aligned}
\ell_k &= \Gamma(k-\alpha)\mathrm{B}(\rho, \beta + k - \alpha)/\mathrm{B}(\rho,\beta), \\
\ell_+ &= \frac{\Gamma(1-\alpha)}{\alpha \mathrm{B}(\rho,\beta)} (\mathrm{B}(\rho, \beta-\alpha) - B(\alpha+\rho, \beta-\alpha).
\end{aligned}$$

Now, we consider some examples by simulating from a Beta-Prime Tempering where $\beta = 1$, $\eta = 1$, $c_g = 1$, $\rho = 0.5$ and $\alpha \in \{0.25, 0.5, 0.75\}$.

For Beta-Prime Tempering distribution, the condition in (3.1.2) does not hold. Thus, we use Algorithm F1 to simulate from f. Table 3.3 shows the acceptance probability for this algorithm.

Table 3.3: Acceptance probability for Algorithm F1

Probability of Acceptance	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
Algorithm F1	0.3220412	0.5058543	0.6960968

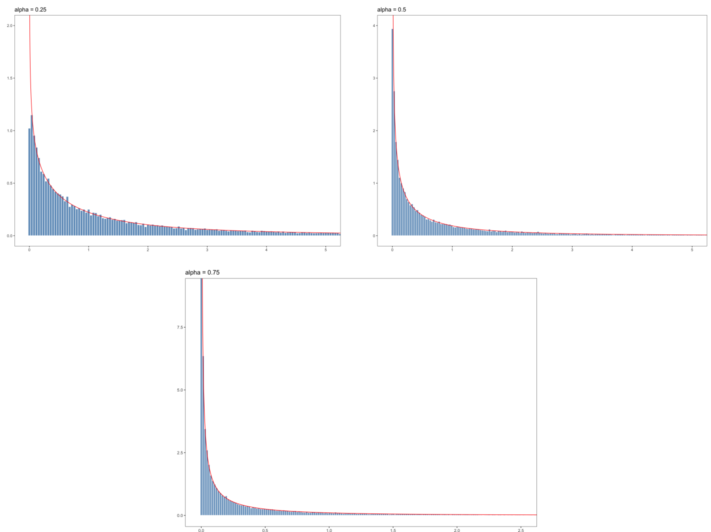

Figure 3.3: Histograms for simulated observations from f, for simulation from Beta-Prime Tempering distributions

To use Algorithm F1, we first need to simulate from U$(0, 1)$ and also from LL(α). By Remark 3.1.3, we can estimate n, the number of observations from U$(0, 1)$ and LL(α) in Algorithm F1 in order to simulate $k = 10^5$ observations from Beta-Prime Tempering distribution. The first row of Table 3.2 shows n for different values of α. Last two rows of Table 3.4 show the number of accepted observations and the proportion of acceptance in Algorithm F1, respectively.

Table 3.4: First row shows the number of observations we need to start with in Algorithm F1 from U$(0, 1)$ and also from LL(α) . The second row shows the number of accepted observations in Algorithm F1, and the last row shows the proportion of acceptance using Algorithm F1, to simulate form Beta-Prime Tempering distribution

Acceptance rate	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
number of obs to start with in Algorithm F1	1065205	800353	1067613
number of accepted obs in Algorithm F1	342700	404717	742763
proportion of acceptance	0.3217	0.5057	0.6957

Figure 3.3 shows the histogram of the simulated observation from the probability

density function f and compare it to its actual value. In this case we have

$$f(x) = \frac{1}{\ell_{+}\mathrm{B}(\rho,\beta)} \int_0^\infty e^{-xs} s^{\rho-1}(1+s)^{-\beta-\rho} ds \; x^{-\alpha-1}(1-e^{-x}), \quad x > 0.$$

The last few steps are very similar to the previous example, where we use accepted simulated observations from f and used them as the mean of a zero-truncated Poisson distribution to simulate Y in Algorithm L, and then, summing these observations, Ys in Algorithm DTS, to simulate from Beta-Prime Tempering distribution. Figure 3.4 compares the probability mass function and the histogram of simulated observations from Beta-Prime Tempering distribution.

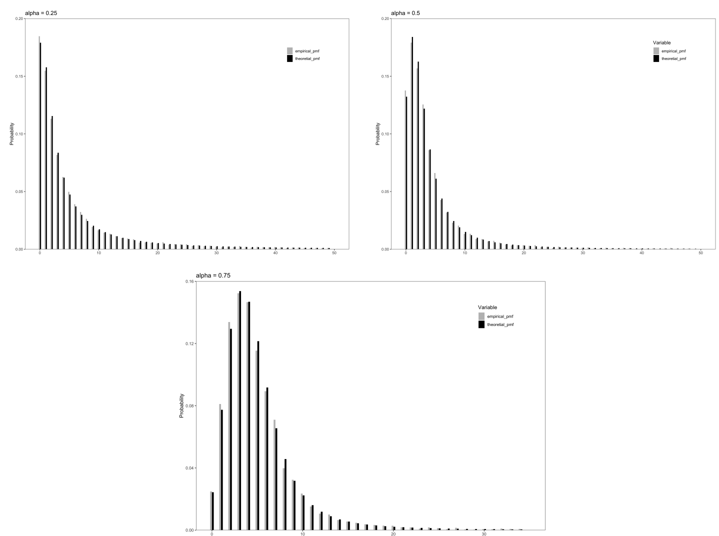

Figure 3.4: Probability mass function and histogram of simulated Beta-Prime Tempering distribution

3.3 Simulating From Zero-Truncated Poisson Distribution

Pois_{+} distributions are used in the simulation method for DTS distributions we discussed before. They can also be used for simulations from a zero-truncated DTS. In this section, we consider two algorithms for simulating from a zero-truncated Poisson distribution with the parameter λ i.e. $\mathrm{Pois}_{+}(\lambda)$. Different implementations are available for simulating from a zero-truncated Poisson distribution. One is implemented in the 'rztpois' method of the R package 'actuar'. The issue with this implementation is that it is unstable when λ is close to zero. We are interested in implementing a more stable method to be used in this case.

In the first algorithm, we use the quantile function of a Pois_+ distribution. Let $F(x)$ be the distribution function of a random variable X, i.e. $F(x) = \mathbb{P}(X \leq x)$. The quantile function can be defined as:

$$F^{\leftarrow}(u) = \inf\{x : F(x) \geq u\}.$$

In this method, one first simulates from $U(0,1)$ and then uses the quantile function of the Pois_+ distribution in a comparison to get a value from Pois_+. A simple conditioning can be used to find the probability mass function of the $X \sim \mathrm{Pois}_+(\lambda)$. Let $Y \sim \mathrm{Pois}(\lambda)$. The pmf of Y is

$$\mathbb{P}(Y = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, 3, \ldots$$

and to find the pmf of X we can write for $k = 1, 2, 3, \ldots$,

$$\begin{aligned} P_X(k) = \mathbb{P}(X = k) = \mathbb{P}(Y = k | Y \neq 0) &= \frac{\mathbb{P}(Y = k \cap Y \neq 0)}{\mathbb{P}(Y \neq 0)} \\ = \frac{\mathbb{P}(Y = k)}{(1 - \mathbb{P}(Y = 0))} &= e^{-\lambda}\frac{\lambda^k}{k!(1 - e^{-\lambda})}. \end{aligned}$$

Now, the cdf and the quantile functions can be calculated to be used in the following algorithm.

Algorithm 1: Simulation from Pois_+ using quantile function.

1. Simulate $U \sim \mathrm{U}(0,1)$
2. Return $Y = F^{\leftarrow}(U)$

To implement Algorithm 1, for uniform observations close to 1, one may need to go far in the tail of the distribution to find a value x for which $F(x) \geq u$, so, this can be computationally expensive.

The next algorithm uses a rejection sampling method for simulating from Pois_+ distribution. This method is more efficient when the parameter of the Pois_+ distribution is small. The method is based on this fact that if $\bar{X} \sim \text{Pois}(\lambda)$ then $Y = \bar{X}+1$ can serve as the dominating distribution for $\text{Pois}_+(\lambda)$. Indeed, let $P_X(k) = e^{-\lambda}\lambda^k/k!(1-e^{-\lambda})$, $k = 1, 2, 3, \ldots$ and $P_Y(k) = e^{-\lambda}\lambda^{k-1}/(k-1)!$, $k = 1, 2, 3, \ldots$ be the probability mass functions of $X \sim \text{Pois}_+(\lambda)$ and Y, respectively. We have

$$P_X(k) = \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}\frac{\lambda}{k(1-e^{-\lambda})} \leq \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}\frac{\lambda}{(1-e^{-\lambda})} = P_Y(k)c,$$

in which $c = \lambda/(1-e^{-\lambda})$. This method can be implemented by the following algorithm.

Algorithm 2: Simulation from Pois_+ using rejection sampling method.

1. Independently simulate $U \sim \text{U}(0,1)$ and $X \sim \text{Pois}(\lambda)$
2. Set $Y = X + 1$
3. If $U \leq P_X(Y)/(cP_Y(Y))$ return Y, otherwise go back to step 1

The third step actually just needs to check if $U \leq 1/k$, if $Y = k$, since in this case, we have

$$\begin{aligned}\frac{P_X(k)}{cP_Y(k)} &= \frac{\frac{e^{-\lambda}\lambda^k}{k!(1-e^{-\lambda})}}{\frac{\lambda}{1-e^{-\lambda}}\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}} \\ &= \frac{(1-e^{-\lambda})e^{-\lambda}(k-1)!\lambda^k}{(1-e^{-\lambda})e^{-\lambda}k!\lambda^{k-1}\lambda} = \frac{1}{k}.\end{aligned}$$

To verify that a sample from Algorithm 2 follows a Pois_+ distribution, let Z be the sample returned by the algorithm. Then

$$\begin{aligned}
\mathbb{P}(Z = k) &= \mathbb{P}(Y = k | U \leq P_X(Y)/(cP_Y(Y))) \\
&= \frac{\mathbb{P}(Y = k, U \leq P_X(Y)/(cP_Y(Y)))}{\mathbb{P}(U \leq P_X(Y)/cP_Y(Y))} \\
&= c\mathbb{P}(Y = k, U \leq P_X(Y)/(cP_Y(Y))) \\
&= c\frac{P_X(k)}{cP_Y(k)} P_Y(k) = P_X(k) = \mathbb{P}(X = k),
\end{aligned}$$

in which the third equality follows from the fact that U is uniform, thus

$$\mathbb{P}\left(U \leq \frac{P_X(Y)}{cP_Y(Y)}\right) = \frac{P_X(Y)}{cP_Y(Y)} = \sum_{k=1}^{\infty} \frac{P_X(k)}{cP_Y(k)} P_Y(k) = 1/c.$$

Equation (3.3) also shows the probability of acceptance in a given iteration is

$$\frac{P_X(Y)}{cP_Y(Y)} = 1/c = \frac{1 - e^{-\lambda}}{\lambda},$$

which shows this method is more efficient when λ is close to zero. The probability of acceptance can also be used to estimate the number of observations to start the first step. i.e. if n observations are needed one can start with $n \times c$ in the first step and then add more observations if needed to reach n accepted sample from Pois_+.

3.4 Parameter Estimation

In this section, we consider different methods to estimate the parameters of a DTS distribution given a sample. Regardless of the choice of the tempering function, a DTS distribution always has at least two parameters. α which is the stability parameter and η with is a constant in the corresponding Lévy measure. Additional parameters may be present based on the tempering function that we choose. Here, we consider two methods for estimation. In the first method, we use likelihood function which is $L(\theta | X_1, \ldots X_n) = \prod_{i=1}^{n} p(X_i | \theta)$, given a sample $X_1, \ldots, X_n$. The idea is to numerically maximize likelihood function to estimate the parameters (MLE). The second method is based on minimizing some distance between empirical and theoretical probability generating functions.

- Parameter Estimation Based on MLE: The idea in this method is to numerically maximizing the log likelihood function of a DTS given a sample $X_1, \ldots, X_n$. Let $X \sim$ DTS be a random variable. The pmf of X can be calculated by the recursive equation (3.1.1), thus by proposition 3.1.2, we have $p_k = p(k|\theta) = \mathbb{P}(X = k|\theta)$, $k = 0, 1, 2, \ldots$ and θ in the parameter vector. The log likelihood function can be written in the following form:

$$\begin{aligned} \ln L(\theta|X_1, \ldots X_n) &= \ln\left(\prod_{i=1}^{n} p(X_i|\theta)\right) \\ &= \ln\left(\prod_{j=1}^{k} n_j p(j|\theta)\right) \\ &= \sum_{j=0}^{k} \ln(n_j p(j|\theta)), \end{aligned}$$

where $k = \max(X_1, \ldots, X_n)$ and $n_j = \sum_{i=1}^{n} 1_j(X_i)$. In case when the sample that is used for fitting is assumed to be from a zero-truncated distribution, one can use the pmf of such a zero-truncated distribution, or change the log likelihood function in the following form to be used in the zero-truncated case:

$$\begin{aligned} \ln L(\theta|X_1, \ldots, X_n) &= \ln\left(\prod_{i=1}^{n} \frac{p(X_i|\theta)}{1 - p(0|\theta)}\right) \\ &= \sum_{i=1}^{n} \ln\left(\frac{p(X_i|\theta)}{1 - p(0|\theta)}\right) \\ &= \sum_{i=1}^{n} \ln(p(X_i|\theta)) - n\ln(1 - p(0|\theta)) \\ &= \sum_{j=0}^{k} n_j \ln(p(j|\theta)) - n\ln(1 - p(0|\theta)), \end{aligned}$$

Now, 'optim' method in statistical software R can be used to maximizing desired log likelihood function to estimate θ.

- Parameter estimation based on pgf: parameter estimation for DTS distributions given a sample can be done using probability generating functions (pgf)s. The idea is to find the parameters at which the distance between the theoretical and the empirical pgfs are minimized. This minimization can be done numerically, using e.g. the 'optim' method in the statistical software R. We recall from equation (2.1.3) that the pgf of a random variable $X \sim \mathrm{DTS}_\alpha(g, \eta)$ is

$$P_\theta = E[z^X] = \exp\left\{-\eta \int_0^\infty (1 - e^{-(1-z)x}) g(x) x^{-1-\alpha} dx\right\}, \quad z \in (0, 1]. \quad (3.4.1)$$

Here θ is the parameter vector, including α, η and any other parameters specified by g, and the empirical pgf given a sample $X_1, X_2, \ldots, X_n$ is given by

$$\hat{P}(z) = \frac{1}{n} \sum_{i=1}^{n} z^{x_i}, \quad z \in (0, 1]. \quad (3.4.2)$$

Now, one can consider a sequence of numbers z_j, $j = 1, 2, , \ldots, m$ in $(0, 1]$, and estimate parameter vector θ which minimizes $\sum_{j=1}^m |P_\theta(z_j) - \hat{P}(z_j)|$ or $\sum_{j=1}^m (P_\theta(z_j) - \hat{P}(z_j))^2$.

To compare these estimation methods, we consider Poisson-Tweedie distribution. Let $X \sim \mathrm{DTS}_\alpha(e^{-ax}, \eta)$, for $\alpha \in \{0.25, 0.5, 0.75\}$, $a \in \{0.5, 1, 2\}$ and $\eta \in \{0.5, 1, 2\}$.

In this case the theoretical pgf can be write in terms of gamma function as:

$$\begin{aligned} P_\theta(z) = E[z^X] &= \exp\left\{-\eta \int_0^\infty (1 - e^{-(1-z)x})e^{-ax}x^{-1-\alpha}dx\right\} \\ &= \exp\left\{-\eta\left(\int_0^\infty e^{-ax}x^{-1-\alpha}dx - \int_0^\infty e^{-(1-z)x-ax}x^{-1-\alpha}dx\right)\right\} \\ &= \exp\left\{-\eta\Gamma(-\alpha)\left(a^\alpha - (1-z+a)^\alpha\right)\right\}, \qquad z \in (0,1]. \end{aligned}$$

Table 3.5 compares the estimated parameters from a sample of size 10^5, generated from any combination of above values for parameters using above method. Error here calculated by adding the absolute values of the difference between parameter values and its estimate. Based on the results represent here, MLE method has the minimum values of errors in most of the cases.

Table 3.5: Parameter estimation for three parameters α, η and a in Poisson-Tweedie distribution using different methods.

Actual Values			MLE				pgf(L_1 norm)				pgf(L_2 norm)			
α	η	a	$\hat{\alpha}$	$\hat{\eta}$	$\hat{a}$	Error	$\hat{\alpha}$	$\hat{\eta}$	$\hat{a}$	Error	$\hat{\alpha}$	$\hat{\eta}$	$\hat{a}$	Error
0.25	0.5	0.5	0.259	0.491	0.491	**0.027**	0.343	0.419	0.408	0.266	0.349	0.413	0.399	0.288
0.25	0.5	1	0.264	0.478	0.951	**0.085**	0.358	0.38	0.788	0.439	0.355	0.384	0.796	0.425
0.25	0.5	2	0.202	0.587	2.216	**0.351**	0.001	1.057	2.922	1.728	0.001	1.038	2.864	1.652
0.25	1	0.5	0.238	1.019	0.514	0.045	0.292	0.921	0.459	0.163	0.256	0.986	0.497	**0.023**
0.25	1	1	0.259	0.982	0.99	**0.037**	0.533	0.48	0.515	1.287	0.347	0.792	0.836	0.469
0.25	1	2	0.236	1.034	2.011	**0.059**	0.489	0.463	1.184	1.591	0.486	0.474	1.237	1.525
0.25	2	0.5	0.267	1.952	0.49	**0.076**	0.001	3.06	0.752	1.56	0.001	3.055	0.75	1.554
0.25	2	1	0.243	2.025	1.005	**0.037**	0.545	0.928	0.496	1.872	0.352	1.565	0.825	0.712
0.25	2	2	0.313	1.657	1.805	**0.601**	0.703	0.417	0.591	3.446	0.675	0.474	0.692	3.259
0.5	0.5	0.5	0.483	0.521	0.525	0.063	0.499	0.499	0.5	**0.003**	0.498	0.499	0.5	**0.003**
0.5	0.5	1	0.517	0.476	0.968	0.073	0.518	0.484	1.016	0.05	0.512	0.483	0.981	**0.049**
0.5	0.5	2	0.492	0.499	1.922	**0.088**	0.499	0.459	1.663	0.379	0.523	0.442	1.776	0.305
0.5	1	0.5	0.507	0.989	0.5	**0.018**	0.505	0.988	0.494	0.023	0.497	1.016	0.514	0.033
0.5	1	1	0.475	1.094	1.091	**0.209**	0.648	0.604	0.658	0.887	0.647	0.604	0.647	0.896
0.5	1	2	0.387	1.51	2.515	**1.138**	0.752	0.318	0.702	2.233	0.763	0.309	0.782	2.172
0.5	2	0.5	0.49	2.034	0.506	**0.05**	0.001	5.995	1.212	5.206	0.001	6.006	1.214	5.219
0.5	2	1	0.5	2.036	1.041	**0.077**	0.277	4.034	1.666	2.924	0.367	3.054	1.38	1.566
0.5	2	2	0.179	5.795	3.281	5.397	0.742	0.665	0.735	2.842	0.719	0.767	0.923	**2.529**
0.75	0.5	0.5	0.772	0.453	0.469	**0.1**	0.707	0.618	0.639	0.301	0.663	0.758	0.827	0.673
0.75	0.5	1	0.754	0.487	0.941	**0.076**	0.784	0.398	0.672	0.464	0.759	0.47	0.9	0.14
0.75	0.5	2	0.894	0.167	0.842	1.635	0.821	0.281	0.669	1.62	0.843	0.261	1.066	**1.265**
0.75	1	0.5	0.756	0.972	0.493	**0.041**	0.353	4.409	1.76	5.066	0.217	6.474	2.118	7.625
0.75	1	1	0.715	1.194	1.143	**0.372**	0.507	3.097	2.399	3.739	0.503	3.117	2.381	3.744
0.75	1	2	0.808	0.686	1.537	0.835	0.719	1.206	2.36	**0.598**	0.601	2.243	3.482	2.874
0.75	2	0.5	0.749	2.018	0.513	**0.032**	0.478	5.581	1.209	4.562	0.366	7.798	1.488	7.17
0.75	2	1	0.623	3.579	1.499	**2.205**	0.572	4.622	1.926	3.725	0.448	7.472	2.515	7.288
0.75	2	2	0.707	2.52	2.233	**0.796**	0.668	2.885	2.045	1.012	0.501	6.788	3.926	6.963

CHAPTER 4: CONVERGENCE OF DISCRETE TEMPERED STABLE DISTRIBUTIONS

4.1 Main Results

In this chapter, we find a convergence rate for a sequence of properly scaled DTS distributions to a PTS distribution. The convergence was proved in [8] and is given in the following proposition.

Proposition 4.1.1. *Let $Y \sim \mathrm{PTS}_\alpha(g, \eta)$ with $E[Y] < \infty$ and $X_a \sim \mathrm{DTS}_\alpha(g_{1/a}, a^{-\alpha}\eta)$ where $g_{1/a}(x) = g(ax)$, $a > 0$. Then,*

$$aX_a \xrightarrow{D} Y \quad as \qquad a \downarrow 0.$$

Let $\mathfrak{F}$ be the space of all distribution functions on the real line. Some useful distances on $\mathfrak{F}$ are the L^p metrics for $p \in [1, \infty]$ and, the Lévy distance [3]. L^p metric where $p = \infty$ is known as the Kolmogorov distance and when $p = 1$, it is also called the Kantorovich or Wasserstein distance. These metrics for two distribution functions F and G are defined as follow.

Definition 4.1.2. *Let F and G be two distribution functions on $\mathbb{R}$. The Lévy distance between F and G is defined as*

$$d_L(F, G) = \inf\{h \geq 0 : G(x - h) - h \leq F(x) \leq G(x + h) + h, \forall x \in \mathbb{R}\}.$$

Definition 4.1.3. *Let F and G be two distribution functions on $\mathbb{R}$. The L^p distance* *between F and G is defined as*

$$\|F-G\|_p = \left(\int_{-\infty}^{\infty} |F(x)-G(x)|^p dx\right)^{1/p}, \qquad 1 \le p < \infty$$

and if $p=\infty$,

$$\|F-G\|_\infty = \sup_x |F(x)-G(x)|.$$

If F and G are two distribution functions, then it is well-known that

$$d_L(F,G) \le \|F-G\|_\infty \le 1,$$

and also, if G is absolutely continuous we have

$$\|F-G\|_\infty \le \left(1+\sup_x |G'(x)|\right) d_L(F,G).$$

One can refer to [3] for more information about the relationships between these metrics. The main result of this section is the following theorem, in which we find a convergence rate in L^p, $p \in [1,\infty]$ for a sequence of properly scaled DTS distributions to a PTS distribution. To show the convergence rate we use $\mathcal{O}(a^r)$ notation which means the expression divided by a^r remains bounded as $a \to 0$.

Theorem 4.1.4. *Let $Y \sim \mathrm{PTS}_\alpha(g,\eta)$ with $E[Y] < \infty$ and $X_a \sim \mathrm{DTS}_\alpha(g_{1/a}, a^{-\alpha}\eta)$ where $g_{1/a}(x) = g(ax)$, $a > 0$ and, let $P_Y(y)$ and $P_{aX_a}(x)$ be the distribution functions of Y and aX_a respectively. Then, for any $a > 0$,*

$$\|P_Y - P_{aX_a}\|_\infty \le \mathcal{O}\left(a^{\frac{1}{2-\alpha}}\right), \tag{4.1.1}$$

and if $p \in [2,\infty)$ and any $a > 0$

$$\|P_Y - P_{aX_a}\|_p \le \mathcal{O}\left(a^{\frac{1}{p(2-\alpha)}}\right). \tag{4.1.2}$$

If we also have $E[X^2] < \infty$, *then for any* $p \in [1, \infty)$ *and any* $a > 0$,

$$\|P_Y - P_{aX_a}\|_1^{1/p} \le \mathcal{O}\left(a^{\frac{1}{p(2-\alpha)}}\right). \tag{4.1.3}$$

Theorem 4.1.4 gives a rate of convergence and measures the distance between two distribution functions P_Y and P_{aX} on the real line with respect to L^p norms for $p \in [1, \infty]$.

Lemma 4.1.5. *If* $\gamma(z)$ *is the characteristic function of a* $\mathrm{PTS}_\alpha(g, \eta)$ *distribution, then there is* $A > 0$ *such that for any* $|z| \ge 1$, $z \in \mathbb{R}$

$$|\gamma(z)| \le \exp(-A|z|^\alpha).$$

Proof. Let $\gamma(z)$ be a characteristic function of a tempered stable distribution. i.e.

$$\gamma(z) = \exp\left(\int_0^\infty (\exp(ixz) - 1)\eta g(x)x^{-\alpha-1}dx\right).$$

Using Euler's formula, $\exp(ixz) = \cos(xz) + i\sin(xz)$, we can write the absolute value of $\gamma(z)$ as

$$\begin{aligned}
|\gamma(z)| &= \exp\left(\Re\left(\int_0^\infty (\cos(xz) + i\sin(xz) - 1)\eta g(x)x^{-\alpha-1}dx\right)\right) \\
&= \exp\left(\Re\left(\int_0^\infty (\cos(xz) - 1)\eta g(x)x^{-\alpha-1}dx + i\int_0^\infty \sin(xz)\eta g(x)x^{-\alpha-1}dx\right)\right) \\
&= \exp\left(\int_0^\infty (\cos(x|z|) - 1)\eta g(x)x^{-\alpha-1}dx\right),
\end{aligned}$$

by substituting $u = x|z|$, we have

$$\begin{aligned}
|\gamma(z)| &= \exp\left(|z|^\alpha \int_0^\infty (\cos(u) - 1)\eta g(u/|z|)u^{-\alpha-1}du\right) \\
&= \exp\left(-|z|^\alpha\eta \int_0^\infty (1 - \cos(u))g(u/|z|)u^{-1-\alpha}du\right).
\end{aligned}$$

Since $\lim_{u\downarrow 0} g(u) = 1$, there exists $\varepsilon > 0$ such that $g(u) \geq 1 - \delta$ if $0 < u < \varepsilon$, thus if $|z| \geq 1$ we have

$$\begin{aligned}|\gamma(z)| &\leq \exp\left(-\eta|z|^\alpha \int_0^\varepsilon (1-\cos(u))g(u/|z|)u^{-1-\alpha}du\right)\\ &\leq \exp\left(-\eta|z|^\alpha(1-\delta)\int_0^\varepsilon (1-\cos(u))u^{-1-\alpha}du\right).\end{aligned}$$

Now if $c = \int_0^\varepsilon (1-\cos(u))u^{-1-\alpha}du$ and $A = \eta(1-\delta)c$, we have

$$|\gamma(z)| \leq \exp(-A|z|^\alpha).$$

□

To prove Theorem 4.1.4, we recall some useful results in the Lemma 4.1.6. See [3] for a survey about these results. Before proceeding, let γ be a characteristic function of a PTS distribution. Lemma 4.1.5 proves an upper bound for the absolute value of γ, and in particular it proves that it is integrable. It what follows we write $r_0 = \int_0^\infty |\gamma(z)|dz$.

Lemma 4.1.6. *Let F and G be two probability distribution functions and F has derivative f for which has essential superimum m and G is with vanishing expectation and characteristic function $\hat{G}$. Also, let F has a continuously differentiable characteristic function $\hat{F}$ with $\hat{F}(0) = 1$ and $\hat{F}'(0) = 0$, and let $r_0 = \int_0^\infty |\hat{F}(z)|dz < \infty$. Then for any $T > 0$, we have*

$$\|F - G\|_\infty \leq \frac{1}{\pi}\int_{-T}^{T} \left|\frac{\hat{F}(z) - \hat{G}(z)}{z}\right| dz + \frac{12r_0}{\pi^2 T}, \tag{4.1.4}$$

and for any $T > 0$ and any $p \in [1, \infty)$

$$\|F-G\|_p^p \leq \|F-G\|_1 \leq \left(\int_{-T}^{T} \left|\frac{\hat{F}(z)-\hat{G}(z)}{z}\right|^2 dz\right)^{1/2}$$
$$+\left(\int_{-T}^{T} \left|\frac{\frac{d}{dz}\hat{F}(z)-\frac{d}{dz}\hat{G}(z)}{z}\right|^2 dz\right)^{1/2}$$
$$+\left(\int_{-T}^{T} \left|\frac{\hat{F}(z)-\hat{G}(z)}{z^2}\right|^2 dz\right)^{1/2} + \frac{4\pi}{T}, \tag{4.1.5}$$

and, for $p \in [2, \infty)$, if $q = \frac{p}{p-1}$, then for any $T > 0$

$$\|F-G\|_p \leq \frac{1}{2}\left(\int_{-T}^{T} \left|\frac{\hat{F}(z)-\hat{G}(z)}{z}\right|^q dz\right)^{1/q} + \frac{4(p-1)}{T^{1/p}}. \tag{4.1.6}$$

Proof. First, we show (4.1.4). From Esseen's smooting lemma, page 538 in [6], we have,

$$\|F-G\|_\infty \leq \frac{1}{\pi}\int_{-T}^{T} \left|\frac{\hat{F}(z)-\hat{G}(z)}{z}\right| dz + \frac{24m}{\pi T}, \tag{4.1.7}$$

in which m is the essential supremum of $f = F'$. Now to prove (4.1.4), we just need to use the fact that m is upper bounded by $r_0/2\pi$. Indeed, using the fact that $r_0 < \infty$ and the inversion formula for characteristic functions, see section 26 in [2], we have

$$f(x) = \frac{1}{2\pi}\int_0^\infty e^{-izx}\hat{F}(z)dz,$$

and thus

$$|f(x)| \leq \frac{1}{2\pi}\int_0^\infty |e^{-izx}||\hat{F}(z)|dz = r_0/2\pi,$$

and hence m can be upper bounded by $r_0/2\pi$. To show (4.1.5), by corollary 8.3 in

[3], and also theorem 1.5.4 in [11], we have

$$\|F-G\|_1 \le \left(\int_{-T}^{T}\left|\frac{\hat{F}(z)-\hat{G}(z)}{z}\right|^2 dz\right)^{1/2} + \left(\int_{-T}^{T}\left|\frac{d}{dz}\frac{\hat{F}(z)-\hat{G}(z)}{z}\right|^2 dz\right)^{1/2} + \frac{4\pi}{T}. \tag{4.1.8}$$

Now, using quotient rule gives,

$$\left(\int_{-T}^{T}\left|\frac{d}{dz}\frac{\hat{F}(z)-\hat{G}(z)}{z}\right|^2 dz\right)^{1/2} = \left(\int_{-T}^{T}\left|\frac{\hat{F}'(z)-\hat{G}'(z)}{z} - \frac{\hat{F}(z)-\hat{G}(z)}{z^2}\right|^2 dz\right)^{1/2},$$

and now by Minkowski's inequality, the right hand side of the above inequality is less than or equal to

$$\left(\int_{-T}^{T}\left|\frac{\hat{F}'(z)-\hat{G}'(z)}{z}\right|^2 dz\right)^{1/2} + \left(\int_{-T}^{T}\left|\frac{\hat{F}(z)-\hat{G}(z)}{z^2}\right|^2 dz\right)^{1/2},$$

From here, (4.1.5) follows by combining these results. The last equation, i.e., (4.1.6) is given in Corollary 7.1 of [3]. $\square$

In the following two lemmas, we review some useful upper bounds related to complex numbers, that we use later in our proofs.

Lemma 4.1.7. *Fix $z \in \mathbb{C}$. We have*

1. $|e^z - 1| \le |z|e^{|z|}$;

2. *if* $|z| \le 1$, *then* $|e^z - 1| \le 2|z|$;

3. *if* $\Re z \le 0$ *then* $|e^z - 1| \le 2\min\{|z|, 1\}$.

Proof. We have

$$\begin{aligned}|e^z - 1| = \left|\sum_{n=1}^{\infty} \frac{z^n}{n!}\right| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{n!} \leq |z| \sum_{n=1}^{\infty} \frac{|z|^{n-1}}{n!} \\ = |z| \sum_{n=0}^{\infty} \frac{|z|^n}{(n+1)!} \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = |z|e^{|z|}.\end{aligned}$$

If $|z| \leq 1$, then

$$|e^z - 1| = \left|\sum_{n=1}^{\infty} \frac{z^n}{n!}\right| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{n!} \leq |z| \sum_{n=1}^{\infty} \frac{1}{n!} = |z|(e-1) \leq 2|z| \leq 2.$$

This gives the second part and the third part for $|z| \leq 1$. Now assume that $|z| > 1$ and $\Re(z) \leq 0$. We have $z = -a + ib$, where $a = -\Re(z)$ and $b = \Im(z)$. We have

$$|e^z - 1| = \left|e^{-a} - e^{-ib}\right| \leq |e^{-a}| + |e^{-ib}| \leq 2 \leq 2|z|.$$

From here the third part follows.

$\square$

A proof of the following lemma can be found in section 26 of [2].

Lemma 4.1.8.

$$|e^{ix} - (1 + ix)| \leq \min\left\{\frac{1}{2}x^2, 2|x|\right\}.$$

Lemma 4.1.9. *Let* $X \sim \mathrm{PTS}_\alpha(g, \eta)$*. For any* $n \geq 1$*,*

$$\int_0^{\infty} g(x)x^{n-1-\alpha}dx < \infty \quad \textit{if and only if} \quad E[X^n] < \infty.$$

Proof. Let $f(x) = x^n \vee 1$, $x > 0$, by Theorem 25.3 in [20] we have

$$E[f(x)] < \infty \qquad \text{iff} \qquad \int_1^{\infty} f(x)\nu(dx) < \infty.$$

where $\nu(dx) = \eta g(x)x^{-1-\alpha}dx$ is the Lévy measure of the X. Thus

$$E[f(x)] < \infty \qquad \text{iff} \qquad \int_1^\infty \eta g(x)x^{n-1-\alpha}dx < \infty. \tag{4.1.9}$$

Now, let $\int_0^\infty g(x)x^{n-1-\alpha}dx < \infty$ In this case we have

$$\int_1^\infty g(x)x^{n-1-\alpha}dx < \infty.$$

so, we have

$$\begin{aligned} E[X^n] &= \int_0^\infty x^n P_X(dx) \\ &= \int_0^1 x^n P_X(dx) + \int_1^\infty x^n P_X(dx) \\ &\leq \int_0^1 P_X(dx) + \int_1^\infty x^n P_X(dx) \\ &= E[f(X)] < \infty. \end{aligned}$$

Now, let $E[X^n] < \infty$. We have

$$\int_1^\infty x^n P_X(dx) \leq E[X^n] < \infty,$$

thus

$$E[f(X)] = \int_0^1 P_X(dx) + \int_1^\infty x^n P_X(dx) < \infty.$$

Now, from (4.1.9) we have

$$\int_1^\infty g(x)x^{n-1-\alpha}dx < \infty,$$

and since $\int_0^1 g(x)x^{n-1-\alpha}dx < \infty$, we have

$$\int_0^\infty g(x)x^{n-1-\alpha}dx < \infty.$$

$\square$

Lemma 4.1.10. *Let* $Y \sim \mathrm{PTS}_\alpha(g,\eta)$ *with* $E[Y] < \infty$ *and* $X \sim \mathrm{DTS}_\alpha(g_{1/a}, a^{-\alpha}\eta)$ *where* $g_{1/a} = g(ax)$*. Then,*

1. *For any* $a > 0$

$$|\hat{P}_Y(z) - \hat{P}_{aX}(z)| \leq |\hat{P}_Y(z)|\eta a C_1 z^2 \exp(\eta a C_1 z^2), \tag{4.1.10}$$

2. *If* $E[Y^2] < \infty$*, then for any* $a > 0$

$$\left|\frac{d}{dz}\hat{P}_Y(z) - \frac{d}{dz}\hat{P}_{aX}(z)\right| \leq |\hat{P}_Y(z)|a\eta\left(2C_1 z + C_2 z^2 + C_1^2 z^2 \exp(\eta a C_1 z^2)\right) \tag{4.1.11}$$

where $C_1 = \int_0^\infty g(x)x^{-\alpha}dx$ *and* $C_2 = \int_0^\infty g(x)x^{1-\alpha}dx$*.*

Proof. To prove part 1, we first recall that

$$\begin{aligned}
\hat{P}_Y(z) &= \exp\left\{-\int_0^\infty (1 - e^{izx})\eta g(x)x^{-1-\alpha}dx\right\}, \\
\hat{P}_{aX}(z) &= \exp\left\{-\int_0^\infty \left(1 - e^{-(1-e^{iza})x}\right) a^{-\alpha}\eta g(ax)x^{-1-\alpha}dx\right\}.
\end{aligned}$$

By substitution of $u = ax$, $\hat{P}_{aX}(z)$ can be written in the following form:

$$\hat{P}_{aX}(z) = \exp\left\{-\int_0^\infty \left(1 - e^{-\frac{(1-e^{iza})}{a}x}\right)\eta g(x)x^{-1-\alpha}dx\right\}.$$

By Theorem 2.1.12, we have

$$|\hat{P}_Y(z) - \hat{P}_{aX}(z)| = |\hat{P}_Y(z)| \left|1 - \frac{\hat{P}_{aX}(z)}{\hat{P}_Y(z)}\right|.$$

Now, the properties of the exponential can be used to write

$$\left|1 - \frac{\hat{P}_{aX}(z)}{\hat{P}_Y(z)}\right| = \left|1 - \exp\left(\int_0^\infty \eta g(x) x^{-1-\alpha}(e^{-\frac{(1-e^{iza})}{a}x} - e^{izx})dx\right)\right|. \tag{4.1.12}$$

By Lemma 4.1.7

$$\begin{aligned}\left|1 - \frac{\hat{P}_{aX}(z)}{\hat{P}_Y(z)}\right| &\leq \left|\int_0^\infty \eta g(x) x^{-1-\alpha}(e^{-\frac{(1-e^{iza})}{a}x} - e^{izx})dx\right| \\ &\qquad \exp\left|\int_0^\infty \eta g(x) x^{-1-\alpha}(e^{-\frac{(1-e^{iza})}{a}x} - e^{izx})dx\right| \\ &\leq \int_0^\infty \eta g(x) x^{-1-\alpha} \left|e^{-\frac{(1-e^{iza})}{a}x} - e^{izx}\right| dx \\ &\qquad \exp\left\{\int_0^\infty \eta g(x) x^{-1-\alpha} \left|e^{-\frac{(1-e^{iza})}{a}x} - e^{izx}\right| dx\right\},\end{aligned}$$

now, since $e^{izx} \neq 0$, $|e^{izx}| = 1$ we have

$$\begin{aligned}&\int_0^\infty \eta g(x) x^{-1-\alpha} \left|e^{-\frac{(1-e^{iza})}{a}x} - e^{izx}\right| dx \\ &\qquad \exp\left\{\int_0^\infty \eta g(x) x^{-1-\alpha} \left|e^{-\frac{(1-e^{iza})}{a}x} - e^{izx}\right| dx\right\}, \\ &= \int_0^\infty \eta g(x) x^{-1-\alpha} \left|e^{izx}\right| \left|1 - e^{(\frac{(e^{iza}-1)}{a} - iz)x}\right| dx \\ &\qquad \exp\left\{\int_0^\infty \eta g(x) x^{-1-\alpha} \left|e^{izx}\right| \left|1 - e^{(\frac{(e^{iza}-1)}{a} - iz)x}\right| dx\right\},\end{aligned} \tag{4.1.13}$$

since $\Re((\frac{(e^{iza}-1)}{a} - iz)x) = (\cos za - 1)x/a \leq 0$, by Lemma 4.1.7, (4.1.13) can be upper bound by

$$
\begin{aligned}
&\int_0^\infty \eta g(x)x^{-1-\alpha}2\min\left\{\left|(\frac{e^{iza}-1}{a}-iz)x\right|,1\right\}dx \\
&\qquad \exp\left\{\int_0^\infty \eta g(x)x^{-1-\alpha}2\min\left\{\left|(\frac{e^{iza}-1}{a}-iz)x\right|,1\right\}dx\right\} \\
&= \int_0^\infty \eta g(x)x^{-1-\alpha}2\min\left\{\frac{x}{a}\left|e^{iza}-1-iza\right|,1\right\}dx \\
&\qquad \exp\left\{\int_0^\infty \eta g(x)x^{-1-\alpha}2\min\left\{\frac{x}{a}\left|e^{iza}-1-iza\right|,1\right\}dx\right\}. \qquad (4.1.14)
\end{aligned}
$$

Now, by Lemma 4.1.8, (4.1.14) can be upper bounded by

$$
\begin{aligned}
&\int_0^\infty \eta g(x)x^{-1-\alpha}2\min\left\{\frac{xaz^2}{2},2x|z|,1\right\}dx \\
&\qquad \exp\left\{\int_0^\infty \eta g(x)x^{-1-\alpha}2\min\left\{\frac{xaz^2}{2},2x|z|,1\right\}dx\right\} \\
&\le \int_0^\infty \eta g(x)x^{-1-\alpha}2(\frac{xaz^2}{2})dx \\
&\qquad \exp\left\{\int_0^\infty \eta g(x)x^{-1-\alpha}2(\frac{xaz^2}{2})dx\right\} \\
&\le \eta az^2\int_0^\infty g(x)x^{-\alpha}dx \\
&\qquad \exp\left\{\eta az^2\int_0^\infty g(x)x^{-\alpha}dx\right\} \\
&\le \eta az^2C_1\exp\left\{\eta az^2C_1\right\},
\end{aligned}
$$

where $C_1 = \int_0^\infty x^{-\alpha}g(x)dx < \infty$ by Lemma 4.1.9. Thus (4.1.10) follows. Now, we prove (4.1.11).

$$\begin{aligned}\left|\frac{d}{dz}\hat{P}_Y(z)-\frac{d}{dz}\hat{P}_{aX}(z)\right| &= \left|\frac{d}{dz}\exp\left\{-\int_0^\infty (1-e^{izx})\eta g(x)x^{-1-\alpha}dx\right\}\right.\\ &\qquad \left.-\frac{d}{dz}\exp\left\{-\int_0^\infty (1-e^{-\frac{(1-e^{iza})}{a}x})\eta g(x)x^{-1-\alpha}dx\right\}\right|\\ &= \left|\left(\int_0^\infty ixe^{izx}\eta g(x)x^{-1-\alpha}dx\right)\hat{P}_Y(z)\right.\\ &\qquad \left.-\left(\int_0^\infty ixe^{iza-(1-e^{iza})\frac{x}{a}}\eta g(x)x^{-1-\alpha}dx\right)\hat{P}_{aX}(z)\right|\\ &= |K(z)\hat{P}_Y(z)-K_a(z)\hat{P}_{aX}(z)|,\end{aligned}$$

where

$$\begin{aligned}K(z) &= \int_0^\infty ixe^{izx}\eta g(x)x^{-1-\alpha}dx\\ K_a(z) &= \int_0^\infty ixe^{iza-(1-e^{iza})\frac{x}{a}}\eta g(x)x^{-1-\alpha}dx.\end{aligned}$$

If we use following inequality

$$\begin{aligned}|K_a(z)\hat{P}_{aX}-K(z)\hat{P}_Y| &= |K_a(z)\hat{P}_{aX}-K(z)\hat{P}_Y+K_a(z)\hat{P}_Y-K_a(z)\hat{P}_Y|\\ &\le |K_a(z)-K(z)||\hat{P}_Y|+|K_a(z)||\hat{P}_Y-\hat{P}_{aX}|,\end{aligned}$$

we have

$$\left|\frac{d}{dz}\hat{P}_Y(z)-\frac{d}{dz}\hat{P}_{aX}(z)\right| \le |K_a(z)-K(z)||\hat{P}_Y|+|K_a(z)||\hat{P}_Y-\hat{P}_{aX}|.$$

Now, we find an upper bound for $|K_a - K|$.

$$
\begin{aligned}
|K_a(z) - K(z)| &= \left| \int_0^\infty ixe^{izx} \eta g(x) x^{-1-\alpha} dx - \int_0^\infty ixe^{iza-(1-e^{iza})\frac{x}{a}} \eta g(x) x^{-1-\alpha} dx \right| \\
&\leq \int_0^\infty \eta g(x) x^{-\alpha} \left| e^{izx} - e^{iza-(1-e^{iza})\frac{x}{a}} \right| dx \\
&= \int_0^\infty \eta g(x) x^{-\alpha} \left| e^{izx} \right| \left| 1 - e^{iza-izx-(1-e^{iza})\frac{x}{a}} \right| dx,
\end{aligned}
$$

Now, since $|e^{izx}| = 1$ and $\Re(iza - izx - (1-e^{iza})\frac{x}{a}) = \frac{x}{a}(\cos az - 1) \leq 0$, by part (3) of Lemma 4.1.7, we have

$$
\begin{aligned}
|K_a(z) - K(z)| &\leq 2\eta \int_0^\infty g(x) x^{-\alpha} \left| iza - izx - (1-e^{iza}) \frac{x}{a} \right| dx \\
&\leq 2\eta za \int_0^\infty g(x) x^{-\alpha} dx + 2\eta \int_0^\infty g(x) x^{-\alpha} \frac{x}{a} \left| e^{iza} - (1 + iza) \right| dx \\
&\leq 2\eta za \int_0^\infty g(x) x^{-\alpha} dx + 2\eta \int_0^\infty g(x) x^{-\alpha} \frac{x}{2a} z^2 a^2 dx \\
&\leq 2\eta za \int_0^\infty g(x) x^{-\alpha} dx + \eta a z^2 \int_0^\infty g(x) x^{1-\alpha} dx \\
&\leq 2\eta za C_1 + \eta a z^2 C_2,
\end{aligned}
$$

where $C_1 = \int_0^\infty g(x) x^{-\alpha} dx$ and $C_2 = \int_0^\infty g(x) x^{1-\alpha} dx$. We also have

$$
\begin{aligned}
|K_a(z)| &= \left| \int_0^\infty ixe^{iza-(1-e^{iza})\frac{x}{a}} \eta g(x) x^{-1-\alpha} dx \right| \\
&\leq \int_0^\infty |xe^{iza-(1-e^{iza})\frac{x}{a}} \eta g(x) x^{-1-\alpha}| dx \\
&= \eta \int_0^\infty |e^{iza-(1-e^{iza})\frac{x}{a}}| g(x) x^{-\alpha} dx \leq \eta C_1,
\end{aligned}
$$

in which the last inequality follows from the fact that $\Re(e^{iza} - 1) = \cos(za) - 1 < 0$, indeed

$$
|e^{iza-(1-e^{iza})\frac{x}{a}}| = |e^{\Re((e^{iza}-1)\frac{x}{a})}||e^{i(za+\Im((e^{iza})\frac{x}{a})}| \leq 1.
$$

Now, by results from part (1) we have

$$\begin{aligned}\left|\frac{d}{dz}\hat{P}_Y(z)-\frac{d}{dz}\hat{P}_{aX}(z)\right| &\le \left(2\eta z a C_1+\eta a z^2 C_2\right)|\hat{P}_Y(z)|+\eta C_1|\hat{P}_Y(z)|\eta a C_1 z^2\exp(\eta a C_1 z^2)\\ &\le |\hat{P}_Y(z)|a\eta\left(2C_1 z+C_2 z^2+C_1^2 z^2\eta\exp(\eta a C_1 z^2)\right).\end{aligned}$$

□

Now, we give a proof for Theorem 4.1.4.

Proof of Theorem 4.1.4.

First we note that by Lemma 4.1.10 for any $q\ge 1$ we have

$$\begin{aligned}\left(\int_{-T}^{T}\left|\frac{\hat{P}_Y(z)-\hat{P}_{aX}(z)}{z}\right|^q dz\right)^{1/q} &\le \left(\int_{-T}^{T}\frac{\left(|\hat{P}_Y(z)|\eta a C_1|z|^2\exp(\eta a C_1|z|^2)\right)^q}{|z|^q}dz\right)^{1/q}\\ &\le \eta a C_1\left(\int_{-T}^{T}|\hat{P}_Y(z)|^q|z|^q\exp(q\eta a C_1|z|^2)dz\right)^{1/q}.\end{aligned}$$

Now, since $|\hat{P}_Y(z)|\le 1$ and Minkowski's inequality and Lemma 4.1.5, we have

$$\begin{aligned}&\left(\int_{-T}^{T}|\hat{P}_Y(z)|^q|z|^q\exp(q\eta a C_1 z^2)dz\right)^{1/q}\\ &\le 2\left(\int_0^1 z^q\exp(q\eta a C_1 z^2)dz\right)^{1/q}+2\left(\int_1^T z^q\exp(-Aqz^\alpha)\exp(q\eta a C_1 z^{2-\alpha}z^\alpha)dz\right)^{1/q}\\ &\le 2\left(\int_0^1 z^q\exp(q\eta a C_1 z^2)dz\right)^{1/q}+2\left(\int_1^T z^q\exp(-Aqz^\alpha)\exp(q\eta a C_1 z^{2-\alpha}z^\alpha)dz\right)^{1/q}\\ &\le 2\exp(\eta a C_1)+2\left(\int_1^T z^q\exp(-Aqz^\alpha)\exp(q\eta a C_1 T^{2-\alpha}z^\alpha)dz\right)^{1/q}\\ &\le 2\exp(\eta a C_1)+2\left(\int_1^\infty z^q\exp(-Aqz^\alpha)\exp(Aq\eta a C_1 T^{2-\alpha}z^\alpha/A)dz\right)^{1/q}. \quad (4.1.15)\end{aligned}$$

Thus we have

$$\left(\int_{-T}^{T}\left|\frac{\hat{P}_Y(z)-\hat{P}_{aX}(z)}{z}\right|^q dz\right)^{1/q} \leq$$
$$2\eta aC_1\left(\exp(\eta aC_1)+\left(\int_1^\infty z^q\exp(-Aqz^\alpha)\exp(Aq\eta aC_1T^{2-\alpha}z^\alpha/A)dz\right)^{1/q}\right). \tag{4.1.16}$$

Now, to prove (4.1.1), be Lemma 4.1.6 and take $q=1$, we have

$$\begin{aligned}
\|P_Y-P_{aX_a}\|_\infty &\leq \frac{1}{\pi}\int_{-T}^{T}\left|\frac{\hat{P}_Y(z)-\hat{P}_{aX}(z)}{z}\right|dz+\frac{12r_0}{\pi^2T}\\
&\leq \frac{2\eta aC_1}{\pi}\left(\exp(\eta aC_1)+\int_1^\infty z\exp(-Az^\alpha)\exp(A\eta aC_1T^{2-\alpha}z^\alpha/A)dz\right)+\frac{12r_0}{\pi^2T}\\
&\leq a\frac{2\eta C_1}{\pi}\exp(\eta aC_1)\\
&\quad + a\frac{2\eta C_1}{\pi}\int_1^\infty z\exp(-Az^\alpha)\exp\left(A\eta aC_1\left(\frac{A}{2\eta aC_1}\right)z^\alpha/A\right)dz\\
&\quad + a^{1/(2-\alpha)}\frac{12r_0}{\pi^2(\frac{A}{2\eta C_1})^{1/(2-\alpha)}}\\
&\leq a\frac{2\eta C_1}{\pi}\exp(\eta aC_1)\\
&\quad + a\frac{2\eta C_1}{\pi}\int_1^\infty z\exp(-Az^\alpha/2)dz+a^{1/(2-\alpha)}\frac{12r_0}{\pi^2(\frac{A}{2\eta C_1})^{1/(2-\alpha)}}=\mathcal{O}\left(a^{1/(2-\alpha)}\right),
\end{aligned}$$

where in the third inequality $T=(\frac{A}{2a\eta C_1})^{1/(2-\alpha)}$.

To prove (4.1.2), from Lemma 4.1.6 and (4.1.16), we have

$$
\begin{aligned}
\|P_Y - P_{aX_a}\|_p &\le \frac{1}{2}\left(\int_{-T}^{T}\left|\frac{\hat{P}_Y(z)-\hat{P}_{aX}(z)}{z}\right|^q dz\right)^{1/q} + \frac{4(p-1)}{T^{1/p}} \\
\le &\quad a\eta C_1\exp(\eta a C_1) + a\eta C_1\left(\int_1^\infty z^q\exp(-Az^\alpha)\exp(Aq\eta a C_1 T^{2-\alpha}z^\alpha/A)dz\right)^{1/q} \\
&\quad + \frac{4(p-1)}{T^{1/p}} \\
\le &\quad a\eta C_1\exp(\eta a C_1) + a\eta C_1\left(\int_1^\infty z^q\exp(-Az^\alpha)\exp(Aq\eta a C_1\frac{A}{aq2\eta C_1}z^\alpha/A)dz\right)^{1/q} \\
&\quad + \frac{4(p-1)}{\frac{A}{2aq\eta C_1}^{1/p(2-\alpha)}} \\
\le &\quad a\eta C_1\exp(\eta a C_1) + a\eta C_1\left(\int_1^\infty z^q\exp(-Az^\alpha/2)dz\right)^{1/q} \\
&\quad + a^{1/(p(2-\alpha))}\frac{4(p-1)}{\left(\frac{A}{2q\eta C_1}\right)^{1/p(2-\alpha)}} = \mathcal{O}\left(a^{1/p(2-\alpha)}\right),
\end{aligned}
$$

where in the third inequality $T = (\frac{A}{aq2\eta C_1})^{1/(2-\alpha)}$.

To prove (4.1.3), from Lemma 4.1.6 we have

$$
\begin{aligned}
\|P_Y - P_{aX}\|_1 \le &\left(\int_{-T}^{T}\left|\frac{\hat{P}_Y(z)-\hat{P}_{aX}(z)}{z}\right|^2 dz\right)^{1/2} \\
&+ \left(\int_{-T}^{T}\left|\frac{\frac{d}{dz}\hat{P}_Y(z)-\frac{d}{dz}\hat{P}_{aX}(z)}{z}\right|^2 dz\right)^{1/2} \\
&+ \left(\int_{-T}^{T}\left|\frac{\hat{P}_Y(z)-\hat{P}_{aX}(z)}{z^2}\right|^2 dz\right)^{1/2} + \frac{4\pi}{T}.
\end{aligned}
$$

Now, we find upper bounds for second and third terms in the right hand side of the above inequality. First, by Lemma 4.1.10, we have

$$
\begin{aligned}
&\left(\int_{-T}^{T}\left|\frac{\frac{d}{dz}\hat{P}_Y(z)-\frac{d}{dz}\hat{P}_{aX}(z)}{z}\right|^2 dz\right)^{1/2}\\
&\leq \left(\int_{-T}^{T}\left|\frac{|\hat{P}_Y(z)|a\eta\left(2C_1z+C_2z^2+C_1^2z^2\exp(\eta aC_1z^2)\right)}{z}\right|^2 dz\right)^{1/2}\\
&\leq \left(\int_{-T}^{T}\left||\hat{P}_Y(z)|a\eta\left(2C_1+C_2z+C_1^2z\exp(\eta aC_1z^2)\right)\right|^2 dz\right)^{1/2}\\
&\leq \left(\int_{-T}^{T}\left||\hat{P}_Y(z)|a\eta 2C_1\right|^2 dz\right)^{1/2}+\left(\int_{-T}^{T}\left||\hat{P}_Y(z)|a\eta C_2 z\right|^2 dz\right)^{1/2}\\
&\quad+\left(\int_{-T}^{T}\left||\hat{P}_Y(z)|a\eta C_1^2z\exp(\eta aC_1z^2)\right|^2 dz\right)^{1/2}\\
&\leq a\eta 2C_1\left(\int_{-T}^{T}\exp(-2A|z|^\alpha)dz\right)^{1/2}+a\eta C_2\left(\int_{-T}^{T}\exp(-2A|z|^\alpha)|z|^2dz\right)^{1/2}\\
&\quad+a\eta C_1^2\left(\int_{-T}^{T}\left||\hat{P}_Y(z)|z\exp(\eta aC_1z^2)\right|^2 dz\right)^{1/2}\\
&\leq a\eta 2C_1\left(\int_{-\infty}^{\infty}\exp(-2A|z|^\alpha)dz\right)^{1/2}+a\eta C_2\left(\int_{-\infty}^{\infty}\exp(-2A|z|^\alpha)|z|^2dz\right)^{1/2}\\
&\quad+a\eta C_1^2\left(\int_{-T}^{T}|\hat{P}_Y(z)|^2|z|^2\exp(2\eta aC_1z^2)dz\right)^{1/2}\\
&\leq a\eta 2C_1\left(\int_{-\infty}^{\infty}\exp(-2A|z|^\alpha)dz\right)^{1/2}+a\eta C_2\left(\int_{-\infty}^{\infty}\exp(-2A|z|^\alpha)|z|^2dz\right)^{1/2}\\
&\quad+2a\eta C_1^2\left(\exp(\eta aC_1)+\left(\int_1^{\infty}z^2\exp(-2Az^\alpha)\exp(2A\eta aC_1T^{2-\alpha}z^\alpha/A)dz\right)^{1/2}\right),
\end{aligned}
$$

where fourth inequality is by Lemma 4.1.5 and last one is by (4.1.15). Now, by similar calculations we have

$$\left(\int_{-T}^{T}\left|\frac{\hat{P}_Y(z)-\hat{P}_{aX}(z)}{z^2}\right|^2 dz\right)^{1/2} \leq \left(\int_{-T}^{T}\left|\frac{||\hat{P}_Y(z)|\eta a C_1 z^2 \exp(\eta a C_1 z^2)|}{z^2}\right|^2 dz\right)^{1/2}$$
$$\leq \eta a C_1 \left(\int_{-T}^{T} |\hat{P}_Y(z)|^2 \exp(2\eta a C_1 z^2) dz\right)^{1/2}$$
$$\leq 2\eta a C_1 \left(\exp(\eta a C_1) + \left(\int_1^T \exp(-2Az^\alpha)\exp(2\eta a C_1 z^2)\right)^{1/2}\right)$$
$$\leq 2\eta a C_1 \left(\exp(\eta a C_1) + \left(\int_1^\infty \exp(-2Az^\alpha)\exp(A2\eta a C_1 T^{2-\alpha} z^\alpha / A)\right)^{1/2}\right).$$

Now, by taking $q = 2$ in (4.1.16), and combining these upper bounds we have

$$\begin{aligned}
\|P_Y - P_{aX}\|_1 &\leq 2\eta a C_1 \exp(\eta a C_1) \\
&+ 2\eta a C_1 \left(\int_1^\infty z^2 \exp(-A2z^\alpha)\exp(A2\eta a C_1 T^{2-\alpha} z^\alpha/A) dz\right)^{1/2} \\
&+ 2a\eta C_1 \left(\int_{-\infty}^{\infty} \exp(-2A|z|^\alpha) dz\right)^{1/2} + a\eta C_2 \left(\int_{-\infty}^{\infty} \exp(-2A|z|^\alpha)|z|^2 dz\right)^{1/2} \\
&+ 2a\eta C_1^2 \left(\exp(\eta a C_1) + \left(\int_1^\infty z^2 \exp(-2Az^\alpha)\exp(A2\eta a C_1 T^{2-\alpha} z^\alpha/A) dz\right)^{1/2}\right). \\
&+ 2\eta a C_1 \left(\exp(\eta a C_1) + \left(\int_1^\infty \exp(-2Az^\alpha)\exp(A2\eta a C_1 T^{2-\alpha} z^\alpha/A)\right)^{1/2}\right) \\
&+ \frac{4\pi}{T}.
\end{aligned}$$

Now, the result follows by taking $T = \left(\frac{A}{2\eta a C_1}\right)^{1/(2-\alpha)}$. □

4.2 Simulation Method for PTS Based on DTS

In this section, we propose a method for approximate simulation from PTS distributions based on the idea suggested in [8]. In this method, we simulate from a PTS distribution by *modifying* a simulation from a DTS distribution.

We focus on two examples, specifically simulating from Tweedie and Power-Tempering distributions, as PTS distributions, by modifying simulations from Poisson-Tweedie and Beta-Prime Tempering distributions as DTS cases, respectively, to show how this approach works.

In Section 3.1, we reviewed the simulation method for DTS distribution in [9]. In Section 3.2, we recalled Poisson-Tweedie and Beta-Prime Tempering distributions and, we use the method described in Section 3.1 to simulate from these two distributions. In Section 4.2.1, we describe our simulation method to simulate from PTS distributions, i.e. how to modify a simulation from a DTS distribution to approximate a simulation from a PTS distribution, then, we focus on Tweedie and Power-Tempering distributions and, show the results of approximating simulations from these two distributions.

4.2.1 Simulation From Tweedie and Power-Tempering Distributions

In this section, we propose a method to approximate a simulation from a PTS distribution. The idea is based on Proposition 2 in [8], which we have already restated in Proposition 4.1.1. Indeed, let $Y \sim \mathrm{PTS}_\alpha(g, \eta)$. We approximate a simulation from Y by simulating from $X_a \sim \mathrm{DTS}_\alpha(g_{1/a}, a^{-\alpha}\eta)$, and then scale it by multiplying this simulation by $a > 0$ to get aX_a. Here $g_{1/a}(x) = g(ax)$ and $a > 0$ should be small enough to have a reasonable approximation.

We provide two examples to show how this method works. We consider several choices of a and compare the scaling of modified Poisson-Tweedie distributions with Tweedie distribution and also different scaling of modified Beta-Prime Tempering distributions with Power Tempering distribution. To show how well these approximations work, we compare the kernel density estimators of these simulations with the probability density function of Tweedie and Power Tempering distributions. These pdfs are implemented in the 'SymTS' package for R. We also consider q-q plots, and perform Kolmogorov-Smirinov and Cramer-Von Mises tests.

As the first example, we show how we can approximately simulate from a Tweedie distribution. They have $g(x) = e^{-ax}$, for some $a > 0$, as the tempering function in their Lévy measure. Indeed, If X is a $\mathrm{PTS}_\alpha(g,\eta)$ distribution in which $g(x)=e^{-ax}$ for some $a>0$, $\eta>0$ and $\alpha\in(0,1)$, then X has a Tweedie distribution. Here, we can observe the similarity in the tempering functions of Tweedie and Poisson-Tweedie distributions. Tweedie distributions are well known and there are some well studied algorithms for simulation. For many other PTS distributions there is no known simulation method. In these cases, our method can be used for approximate simulation.

We use symmetric CTS (classical tempered stable) distribution in 'SymTS' package in statistical software R for Tweedie distribution. The method to evaluate the pdf of this distribution is

$$\mathrm{dCTS}(x,\alpha,c,\ell,\mu),$$

with the Lévy measure of

$$\mathrm{M}(dx)=c\ \ell^\alpha e^{-\frac{|x|}{\ell}}|x|^{-1-\alpha}dx.$$

These are a symmetrical class of distributions and coincide with Tweedie distributions when defined on the non-negative real numbers.

In what follows, we approximate a simulation from a Tweedie distribution and then symmetrize it to compare it to the CTS distribution. For this study we choose $\alpha\in\{0.25,0.5,0.75\}$, $c=1$, $\ell=0.5$ and $\mu=0$. By comparing the Lévy measure of a CTS distribution to the Lévy measure of a TS distribution,

$$\mathrm{M}(dx)=\eta g(x)x^{-1-\alpha}1_{x>0}dx,$$

we choose the following values for parameters in a DTS distribution

$$\eta=c\ \ell^\alpha,\qquad g(x)=e^{-\frac{x}{\ell}},$$

which is a Poisson-Tweedie distribution for the discrete case. We can get the desired Tweedie distribution by modifying this simulation and then to be able to compare it to the CTS distribution we symmetrize it, i.e., we subtract half of the observations from the other half.

Let Y be the Tweedie distribution with $g(x) = e^{-\frac{x}{\ell}}$ and $\eta = c\ \ell^{\alpha}$, i.e. $Y \sim \mathrm{PTS}(g, \eta) = \mathrm{PTS}(e^{-\frac{x}{\ell}}, c\ell^{\alpha})$. We need to simulate from a Poisson-Tweedie distribution $X_a \sim \mathrm{DTS}_{\alpha}(g_{1/a}, a^{-\alpha}\eta)$, $g_{1/a}(x) = g(ax)$ and then scale it by multiplying it by a to approximate a simulation from Y. By our choice of parameters for Tweedie distribution, we first simulate from

$$X_a \sim \mathrm{DTS}_{\alpha}(g_{1/a}, a^{-\alpha}\eta) = \mathrm{DTS}_{\alpha}(e^{-\frac{ax}{\ell}}, a^{-\alpha}c\ \ell^{\alpha}) = \mathrm{DTS}_{\alpha}(e^{-2ax}, a^{-\alpha}(0.5)^{\alpha}),$$

where $\alpha \in \{0.25, 0.5, 0.75\}$ and $a \in \{0.5, 0.1, 0.01, 0.0001\}$. Then we multiply it by a to get aX_a. Finally, we symmetrize it and compare it to the CTS distribution to show how aX_a will converge to the Tweedie distribution as $a \to 0$ for different values for α.

Table 4.1 shows probability of acceptance in F1 and F2 algorithms for different values of α and a. Table 4.2 shows the total number of observations to start with, accepted observations and the proportion of accepted observations in which algorithm that was used.

Figure 4.1 compares the pdf of the CTS distribution with the kernel density estimation of the simulated scaled Poisson-Tweedie distributions for different values for α and a after symmetrization. In each of these plots, the kernel density estimations of these simulations get closer to the CTS distribution as $a \to 0$.

Figure 4.2 compares the q-q plots from CTS and symmetrized scaled Poisson-Tweedie distributions. Table 4.3 shows the average of p-values of Kolmogorov-Smirinov and Cramer-Von Mises tests from 100 simulations for any choice of a and α in which

Table 4.1: Probability of acceptance in F1 and F2 algorithms for different values of α and a.

α	a	Algorithm F1	Algorithm F2
	0.5	0.1739	0.7568
0.25	10^{-1}	0.3473	0.4521
	10^{-2}	0.5780	0.1338
	10^{-4}	0.8098	0.0059
	0.5	0.3671	0.8284
0.50	10^{-1}	0.5745	0.5798
	10^{-2}	0.7697	0.2457
	10^{-4}	0.8738	0.0279
	0.5	0.6180	0.9091
0.75	10^{-1}	0.7681	0.7556
	10^{-2}	0.8718	0.4822
	10^{-4}	0.9050	0.1583

Table 4.2: Number of observations to start the simulation, number of accepted observations and the proportion of accepted observation for different values of α and a

α	a	observations to start with	accepted observations	proportion
	0.5	12299	9382	0.7628
0.25	10^{-1}	62777	28515	0.4542
	10^{-2}	144953	83425	0.5755
	10^{-4}	458218	371112	0.8099
	0.5	17707	14589	0.8239
0.50	10^{-1}	90669	52619	0.5803
	10^{-2}	288777	222379	0.7701
	10^{-4}	2888714	2524964	0.8741
	0.5	37013	33567	0.9069
0.75	10^{-1}	181852	139535	0.7673
	10^{-2}	1022948	891319	0.8713
	10^{-4}	32338742	29261959	0.9049

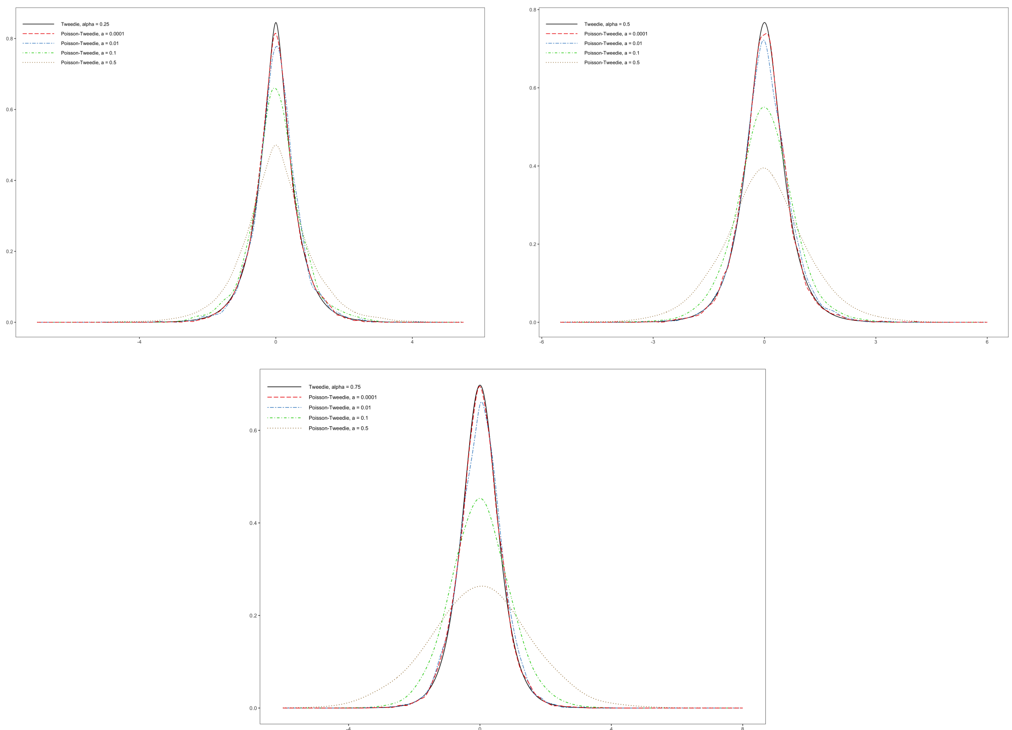

Figure 4.1: Solid black curves show the pdf of the Tweedie distribution for $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.75$. Long-dash red, two-dash blue, dotdash green and dotted orange curves show the density estimation of the scaled Poisson-Tweedie distribution with the scaled value $a = 0.0001, 0.01, 0.1, 0.5$ respectively.

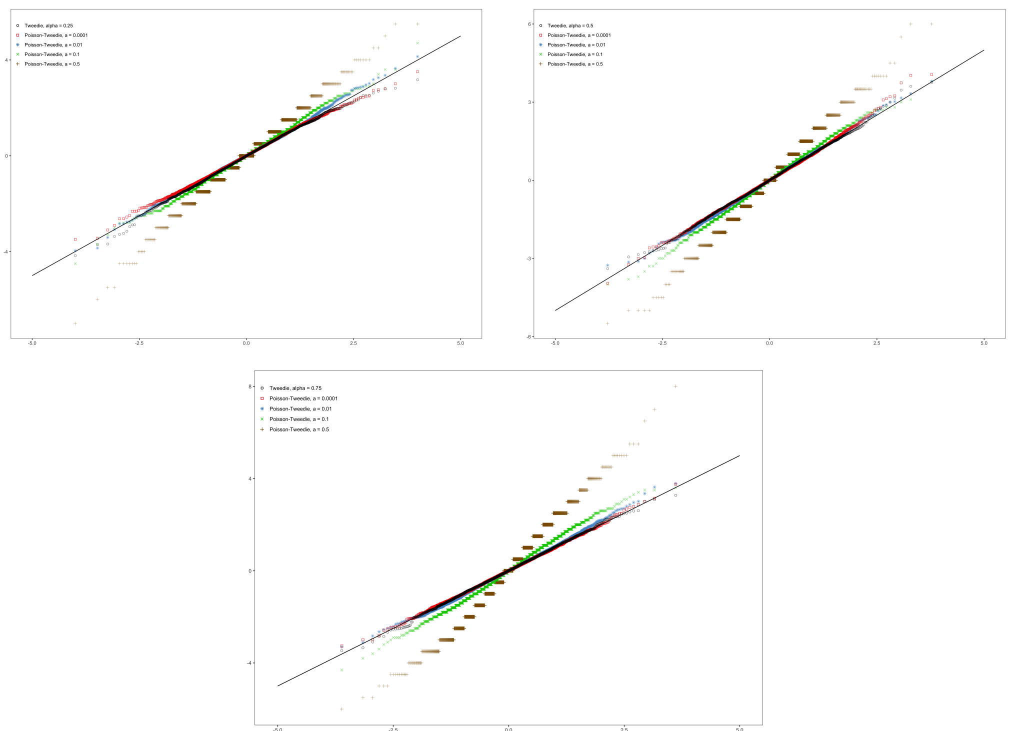

Figure 4.2: Black circles show q-q plot of CTS distribution for $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.75$. Red squares, blue asterisks, green crosses and orange pluses show q-q plots for symmetrized scaled Poisson-Tweedie distributions with the scaled value $a = 0.0001, 0.01, 0.1, 0.5$ respectively.

Table 4.3: P-values for Kolmogorov-Smirinov and Cramer-Von Mises test for different values of α and a in approximating a simulation from Tweedie distribution by Poisson-Tweedie distribution.

α	a	Kolmogorov-Smirinov	Cramer-Von Mises
	0.5	0	0
0.25	10^{-1}	$\leq 10^{-11}$	$\leq 10^{-6}$
	10^{-2}	0.2620	0.4019
	10^{-4}	0.4722	0.4643
	0.5	0	0
0.50	10^{-1}	$\leq 10^{-17}$	0
	10^{-2}	0.2221	0.3773
	10^{-4}	0.4644	0.4626
	0.5	0	0
0.75	10^{-1}	0	0
	10^{-2}	0.0564	0.0800
	10^{-4}	0.5113	0.5125

each has 5000 observations. Since under the null hypothesis the p-value has $U(0,1)$ distribution, we expect to see these average values close to 0.5, and this happens when a is small enough, in this case $a = 10^{-4}$. This shows that the scaled simulation from Poisson-Tweedie is well approximating a simulation from Tweedie distribution.

As the second example, we approximate a simulation from Power-Tempering distribution. The Lévy measure of this distribution is

$$\mathrm{M}(dx) = c_- g(|x|^p, -1)|x|^{-1-\alpha} 1_{x<0} dx + c_+ g(x^p, 1) x^{-1-\alpha} 1_{x>0} dx, \tag{4.2.1}$$

where

$$\begin{aligned} g(r,-1) &= p^{-1} \int_0^\infty e^{-ru} (1 + u^{-1/p})^{-\nu_- - 1} u^{-(1+\alpha)/p - 1} du, \\ g(r,1) &= p^{-1} \int_0^\infty e^{-ru} (1 + u^{-1/p})^{-\nu_+ - 1} u^{-(1+\alpha)/p - 1} du. \end{aligned}$$

Here we consider the case when $p = 1$. For more information about this class of distributions, one can refer to [7]. We use a modification of Beta-Prime Tempering distribution to approximate a simulation from Power-Tempering distribution, since the tempering functions, g, are similar in both cases. To evaluate our method, we compare the kernel density estimations of simulations from different modifications of Beta-Prime Tempering distributions with the probability density function of the Power-Tempering distribution, already available in 'SymTS' package in R, we also consider the q-q plots, and run Kolmogorov-Smirinov and Cramer-Von Mises tests.

We use symmetric PowTS distribution in 'SymTS' package in statistical software R for Power Tempering distribution. The method to evaluate the pdf of this distribution is

$$\mathrm{dPowTS}(x, \alpha, c, \ell, \mu).$$

The implementation of this distribution in this package uses a different parametrization, which uses the Rosinski measure. For more informations about Rosinski measure one can refer to [18] or [7]. Section 6.2.1 in [7] discusses this measure and how it relates to the Lévy measure. Here, we mostly interested in the Lévy measure, so we mention how the desired parameters can be calculated to be used based on the Lévy measure.

The Rosinski measure of the Power Tempering distribution in the package is

$$\mathrm{R}(dx) = c(\alpha + \ell + 1)(\alpha + \ell)(1 + |x|)^{-2-\alpha-\ell}(dx). \tag{4.2.2}$$

By comparing (4.2.2) to the Rosinski measure of a p-tempering α-stable distribution, section 6.2.1 in [7],

$$\mathrm{R}(dx) = c_-(1 + |x|)^{-\nu_- -1} 1_{x<0} dx + c_+(1 + x)^{-\nu_+ -1} 1_{x>0} dx, \tag{4.2.3}$$

where $c_-, c_+ \geq 0$ and $\nu_-, \nu_+ > \alpha \vee 0$, the Lévy measure of this distribution can be calculated from its Rosinski measure. Thus, we have the Lévy measure of this distribution as (4.2.1).

Let Y be a Power-Tempering distribution, i.e. $Y \sim \mathrm{PTS}(g, \eta)$. Since we are using PowTS method from 'SymTS' package for Power-Tempering distribution, we need to first find η and $g(x)$ in terms of parameters in PowTS method, which are α, c, ℓ and μ. To find $g(x)$, from (4.2.1) since we consider $p = 1$, we have

$$\begin{aligned} g(x,1) &= \int_0^\infty e^{-xu}(1+u^{-1})^{-\nu_+-1}u^{-(1+\alpha)-1}du \\ &= \int_0^\infty e^{-xu}(1+u)^{-\nu_+-1}u^{-1-\alpha+\nu_+}du, \end{aligned}$$

and by comparing (4.2.3) with (4.2.2) we have

$$\begin{aligned} -\nu_+ - 1 &= -2 - \alpha - 1, \\ \nu_+ &= \alpha + \ell + 1. \end{aligned}$$

Thus g, the tempering function in terms of parameters from 'SymTS' package is

$$g(x) = \int_0^\infty e^{-xu}(1+u)^{-\alpha-\ell-2}u^\ell du.$$

Now, we can find η in the same way by comparing the (4.2.3) with (4.2.2). We have

$$\eta = c(\alpha + \ell + 1)(\alpha + \ell).$$

Given g and η we can write

$$Y \sim \mathrm{PTS}_\alpha(g, \eta) = \mathrm{PTS}_\alpha\left(\int_0^\infty e^{-xu}(1+u)^{-\alpha-\ell-2}u^\ell du, c(\alpha+\ell+1)(\alpha+\ell)\right).$$

By [8], we know that if $Y \sim \mathrm{PTS}_\alpha(g, \eta)$ and $X_a \sim \mathrm{DTS}_\alpha(g(ax), a^{-\alpha}\eta)$ then $aX_a \xrightarrow{\mathrm{D}} \mathrm{PTS}_\alpha(g, \eta)$

Now, we find the parameters required to simulate from a scaled Beta-Prime Tempering distribution. We have X_a in terms of 'SymTS' package parameters

$$X_a \sim \mathrm{DTS}_\alpha\Big(\int_0^\infty e^{-axs} s^\ell (1+s)^{-\alpha-\ell-2} ds, c(\alpha+\ell+1)(\alpha+\ell)a^{-\alpha}\Big). \quad (4.2.4)$$

After we simulate from X_a, we get an approximation of a simulation from Y by multiplying X_a by a. In our case, since Power-Tempering distribution in 'SymTS' package is symmetric, we need to also symmetrize aX_a by subtracting half of the observations from the other half.

We now perform a small simulation study. We choose $\alpha \in \{0.25, 0.5, 0.75\}$, $c = 1$, $\ell = 1$ and $\mu = 0$ to use PowTS method in the 'SymTS' package to get the pdf of the Power-Tempering distribution. Then, we compare this distribution with aX_a, which is a scaled Beta-Prime Tempering distribution.

Figure 4.3 compares the pdf of Power-Tempering distribution with the kernel density estimation of the simulated scaled Beta-Prime Tempering distribution for different values for α. In each of these figures, we can see these scaled Beta-Prime Tempering distributions converge to the Power-Tempering distribution as $a \to 0$.

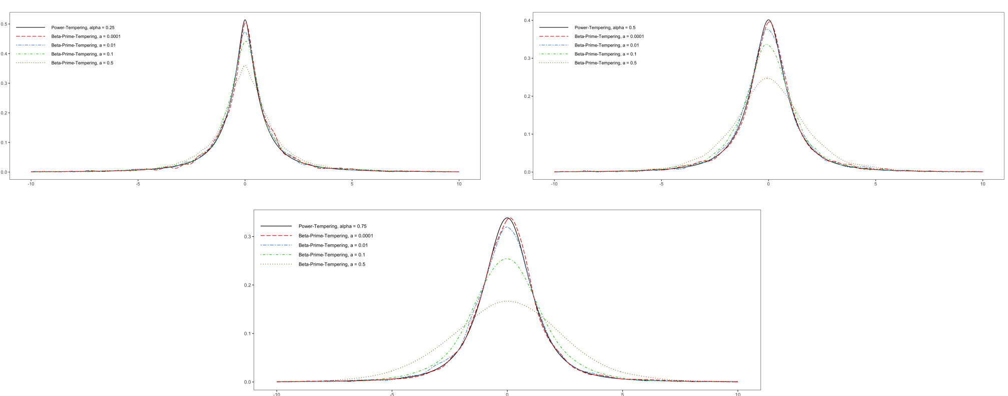

Figure 4.3: Solid black curves show the pdf of the Power-Tempering distribution with $\alpha = 0.25$, $\alpha = 0.5$ amd $\alpha = 0.75$. Long-dash red, two-dash blue, dotdash green and dotted orange curves show the density estimation of the scaled Beta-Prime Tempering distribution with the scaled value $a = 0.0001, 0.01, 0.1, 0.5$ respectively.

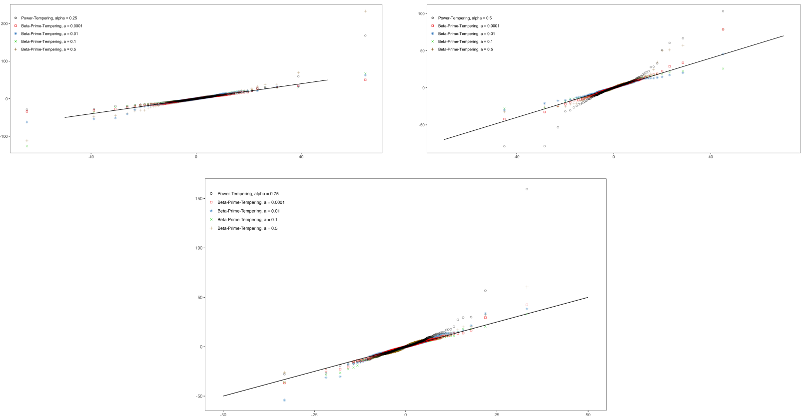

Figure 4.4: Black circles show q-q plot of Power-Tempering distribution for $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.75$. Red squares, blue asterisks, green crosses and orange pluses show q-q plots for scaled Beta-Prime Tempering distributions with the scaled value $a = 0.0001, 0.01, 0.1, 0.5$ respectively.

Figure 4.4 considers q-q plots for Power-Tempering distribution and compare it to the scaled Beta-Prime Tempering distribution. Table 4.4 shows the average of p-values of Kolmogorov-Smirinov and Cramer-Von Mises tests from 100 simulations for any choice of a and α in which each has 5000 observations. Here, again $a = 10^{-4}$ gives the best approximation based on these tests as the average of p-values is close to 0.5.

Table 4.4: P-values for Kolmogorov-Smirinov and Cramer-Von Mises test for different values of α and a in approximating a simulation from power-tempering distribution by Beta-Prime Tempering distributions

α	a	Kolmogorov-Smirinov	Cramer-Von Mises
0.25	0.5	0	0
	10^{-1}	$\leq 10^{-4}$	0.0005
	10^{-2}	0.3797	0.4506
	10^{-4}	0.5142	0.5057
0.50	0.5	0	0
	10^{-1}	$\leq 10^{-5}$	$\leq 10^{-3}$
	10^{-2}	0.3918	0.4664
	10^{-4}	0.4805	0.4887
0.75	0.5	0	0
	10^{-1}	$\leq 10^{-11}$	$\leq 10^{-8}$
	10^{-2}	0.3484	0.4026
	10^{-4}	0.4886	0.4853

CHAPTER 5: HIGH FREQUENCY FINANCIAL MODELING

An important application of DTS and TS distributions is to the modeling of financial data. The aim of this chapter is to show how these distributions can be used to model tick data in the context of high-frequency financial data.

By high frequency we mean that the records in the data set are mostly within fractions of a second and they are in terms of a unit that is called the tick size. Tick size is the minimum price change allowed by an exchange on a given market.

The emergence of new technologies provides easier access to high-frequency data. Some characteristics of these data are irregular temporal property and discreteness of price changes. Irregular temporal property here refers to the fact that the recorded observations are not equally spaced in time. These add more challenges in modeling while it can help to understand more about microstructure behavior of price changes and also help to enhance decision making. [16]

5.1 High Frequency Financial Data

In this section, we consider high-frequency financial data readily available on Dukascopy Swiss Banking Group[1].

We mainly consider the currency exchange rate between Euro and Dollar in Forex which is an foreign exchange market for the trading of currencies. One can also consider different instruments e.g. crypto currency exchange rate for Bitcoin and Dollar, USA 500 Index, etc. In each of these data sets there are five variables, time of the record that can be in local time or GMT, bid and ask prices and volumes.

Figure 5.1 shows ask prices for July 2024 in currency exchange rate between Euro

[1] The data can be downloaded from historical data feed provided at https://www.dukascopy.com/swiss/english/marketwatch/historical/.

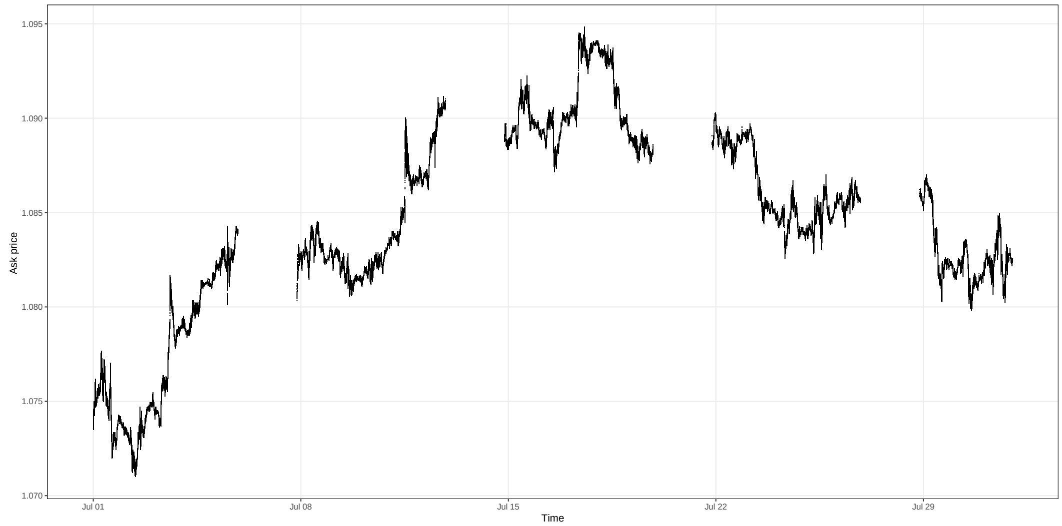

Figure 5.1: Ask price for July 2024

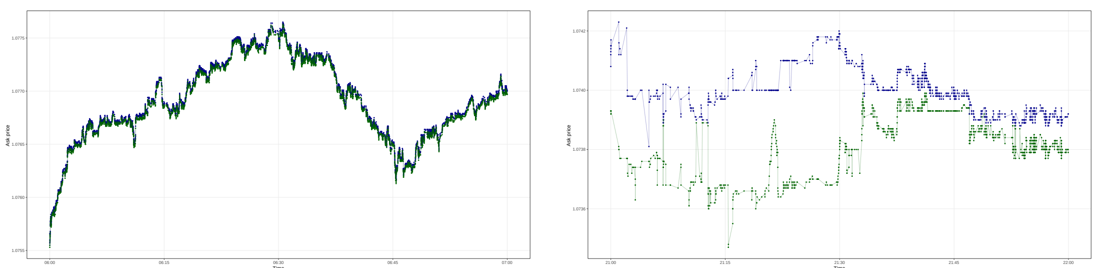

Figure 5.2: Ask and bid prices from currency exchange rate between Euro and Dollar. Left figure shows one hour of the data from 6:00 to 7:00 in GMT and the right figure show the same data from 21:00 to 22:00 GMT.

and Dollar. This marker is active 24 hours a day except on weekends. The intensity of market activity is not uniform during a given day. Figure 5.2 shows ask and bid prices for two different hours during the first day of July 2024. It is clear that during some hours in a given day the intensity of trades is higher than at other times. Ask-bid spread is the difference between ask and bid prices in a given time and one can observe that how intensity of the trades and the ask-bid spread can be related.

Before exploring more about the activity of the market, we follow the instructions in [16] to calculate pure-mid prices, which is a price between ask and bid prices, calculated in a way to keep changes in prices to a minimum.

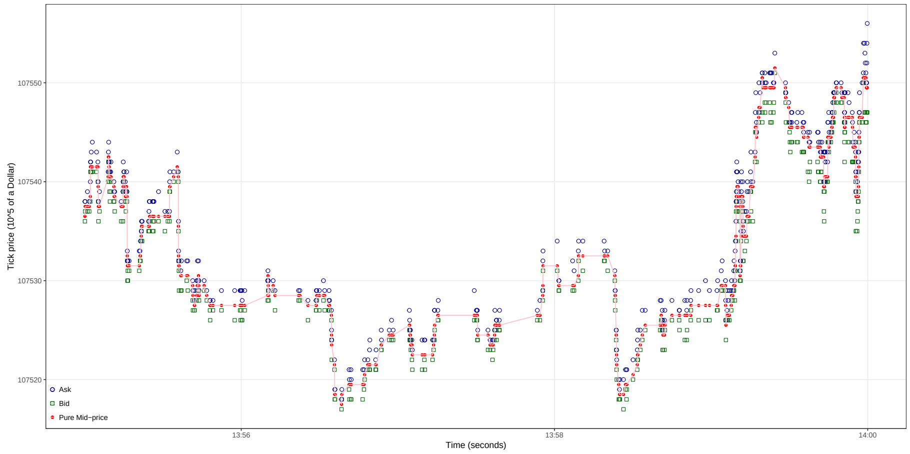

Figure 5.3: One minute of ask, bid and pure-mid prices.

Tick size in this data set is 0.00001 of a unit, thus we get integer ask and bid prices if we divide these prices by the tick size. Choosing the average of ask and bid prices, mid price, results in integer and half integers and thus adds complexities to the modeling of price changes [16]. Pure-mid price however, is defined in a way to allow just half integers and hence, the price changes will be always integers. Indeed, if we show pure-mid and the mid (average of ask and bid) prices by $p_{\text{pure},t}$, $p_{\text{mid},t}$ respectively, then

- Find the initial value $p_{\text{pure},t_1} = p_{\text{mid},t_1} = \frac{p_{\text{ask},t_1} + p_{\text{bid},t_1}}{2}$.
- If $p_{\text{bid},t_{i+1}} < p_{\text{pure},t_i} < p_{\text{ask},t_{i+1}}$, then $p_{\text{pure},t_{i+1}} = p_{\text{pure},t_i}$.
- If $p_{\text{pure},t_i} < p_{\text{bid},t_{i+1}}$, then $p_{\text{pure},t_{i+1}} = p_{\text{bid},t_{i+1}} + 0.5$.
- If $p_{\text{ask},t_{i+1}} < p_{\text{pure},t_i}$, then $p_{\text{pure},t_{i+1}} = p_{\text{ask},t_{i+1}} - 0.5$.

Figure 5.3 shows 1 minute of ask, bid and pure-mid prices. We will just consider pure-mid price for the rest of the analysis, since it simplifies modeling and also, shows a fair price of the exchange rate at the time.

As we noticed before, the market activity is not uniform during a given day. Just like the ask price, changes in pure-mid prices vary during the day. The two upper pictures in Figure 5.4 show the number of changes in pure-mid price during a month and a week in an hourly basis, respectively. The bottom figure shows the standard deviation of the pure-mid prices during a week in an hourly basis. A repeated pattern like a seasonal effect is obvious in the intensity of price changes. Here, we do not see a peak as large as other days in 4th of July, which is a holiday in the United State, and maybe, this is the reason. Other peaks might be explained based on the times when different markets open and close from east Asia to Europe and America.

5.2 Modeling High Frequency Financial Data

In this section, we consider different approaches in modeling high-frequency financial data. As was discussed earlier, one of the main characteristics of these data is that they are in nature irregular time series, i.e. the times between observations are not equally spaced. This may be the first obstacle in modeling high-frequency data, and one should decide if this characteristic should be a part of the modeling or not. One can always use interpolation to filter the data and extract a regular time series from it. There are pros and cons in deciding to work with either one. If we choose to work with irregular time series, then we must also model these times, which makes modeling more challenging. It gets even more complicated when modeling a complex portfolio is in interest, because of the time synchronization issues coming from the fact that observations in each data set are recorded at different times. However, these times duration contain useful information about the micro-structure properties of high-frequency data sets and modeling these times make the final model more informative. In the other hand, if regular time series are chosen for modeling, one should decide on a fixed time interval to be used and also decide on how interpolation should be done for the time points when there are no recorded observations. There are some well-developed mathematical models to be used in this case [5].

The other aspect of high-frequency financial data is the discreteness of price changes based on the tick size. When considering price changes over a long time horizon,

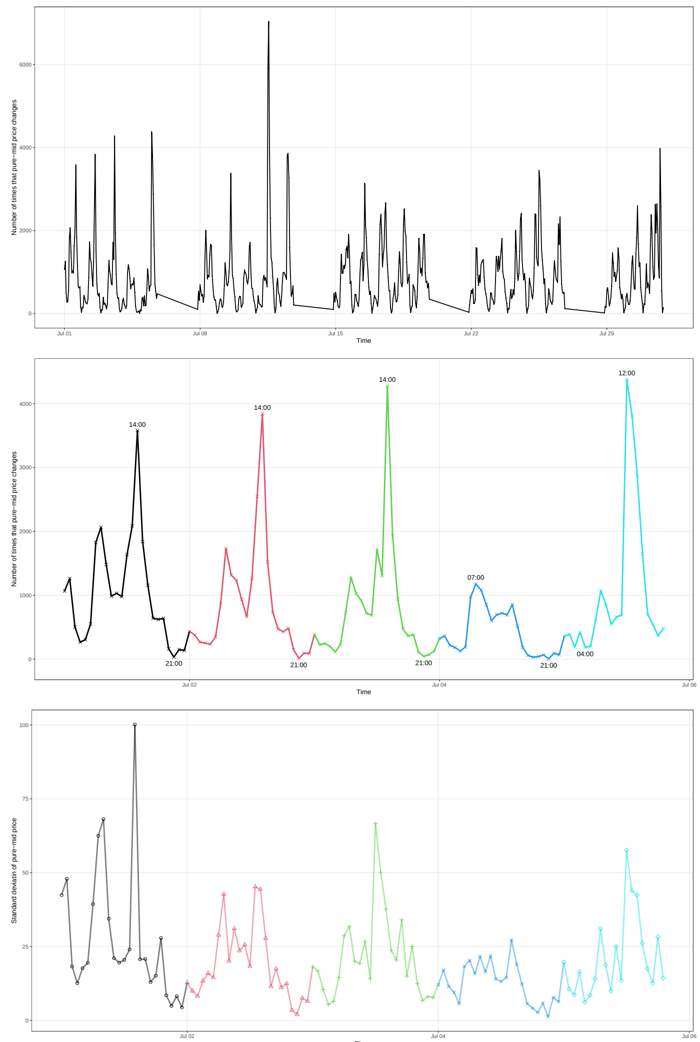

Figure 5.4: Upper and middle: number of price changes in an hourly basis during July 24 and the first week on July 24, respectively. Bottom: standard deviation of pure-mid prices in an hourly basis during a week.

tick size usually do not play an important role and by some degree of smoothness, continuous models can be used for modeling. However, when working with high-frequency data over a small time horizon, price changes take only few values in term of tick size. This fact suggest the use of discrete distributions for modeling such price changes [5]. Poisson, negative binomial and DTS distributions are the discrete distributions that we consider for the modeling these price changes. Since these distributions are positive, some mixture or subtraction of the distributions can be used to model positive and negative jumps.

Modeling jumps with the specific assumption of i.i.d. exponentially distributed observations for interarrival times, will let us to use Lévy processes for modeling, since in this case, a compound Poisson process with innovations from the given discrete distribution, which is a Lévy process, can be used. As we can see in the following sections, these assumptions are far from the reality of the tick data. However, these assumptions make modeling much simpler. In the last section, we consider modeling with these assumptions and we evaluate the models.

In the next two sections, we first compare different models for jumps and then we model the interarrival times. In each section, we show how the modeling works by considering the exchange rate between Euro and Dollar data set from the first day of July 2024. The data that we use to estimate the parameters of the distributions is nine hours from 03:00 to 12:00 GMT.

5.2.1 Modeling Jumps

In this section, we consider the price changes, i.e. jumps, for modeling. These are integer valued and discrete distributions like Poisson, negative binomial and DTS distributions can be used. Figure 5.5 shows price changes in the pure-mid price for 1 minute period.

High-frequency financial data can be seen as a sample in random times from a continuous process. In this regard, one can model this process by considering the

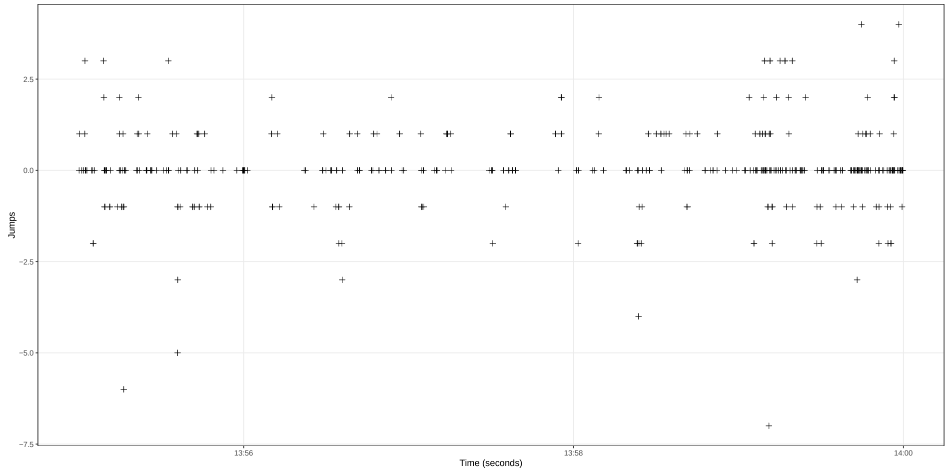

Figure 5.5: Jumps in 1 minute of high-frequency data.

recorded observations as innovations in which the innovations can be zero when there is no change in the price, some models that use compound Poisson process with such a innovations are explained in [16] and [14]. In another method, one can model the observed changes in the prices and just consider non-zero innovations for modeling.

In the first technique, Skellam, ΔNB (delta negative binomial) and delta Poisson tempered stable distributions, which are the difference of independent Poisson, negative binomial and Poisson tempered stable distributions which coincide with the class of Poisson-Tweedie distributions, respectively, can be used for modeling [16].

Here, we use the second technique and we model non-zero jumps or price changes as a mixture of zero-truncated discrete distributions as follow:

$$J \stackrel{\mathrm{D}}{=} \begin{cases} X_1 & \text{w. p. } p_1 \\ -X_2 & \text{w. p. } p_2 \end{cases}$$

where X_1 and X_2 are zero-truncated discrete distributions, and p_1 and p_2 are proportion of positive and negative jumps. We will consider zero truncated Poisson, negative binomial and some DTS distributions specifically, Poisson-Tweedie and Pareto tempering distributions, which is also a special case of beta-prime tempering where $\rho = 1$,

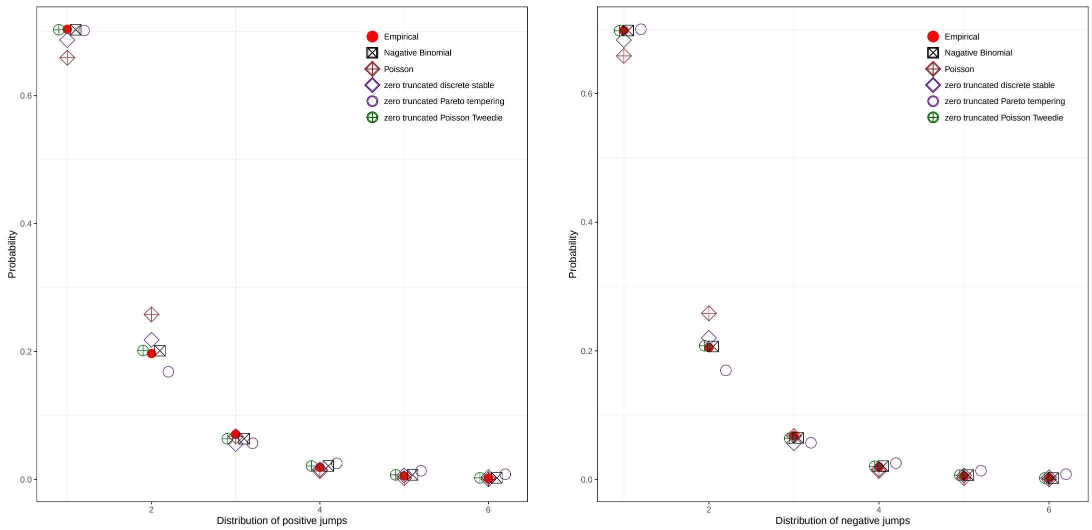

Figure 5.6: Distribution of positive and negative jumps

discussed in the previous sections. To see how these distributions are related one can observe that the class of $\mathrm{DS}_\alpha(\eta)$ distributions coincides with the class of $\mathrm{Pois}(\eta)$ distributions when $\alpha = 1$, and Poisson-Tweedie distributions, i.e. $\mathrm{DTS}_\alpha(e^{-ax}, \eta)$, coincides with the class of negative binomial distributions when $\alpha = 0$. One can refer to [1] for more information about the relations between these distributions. On the other hand, Poisson-Tweedie and Pareto tempering are DTS distributions, i.e. they are obtained by tempering a $\mathrm{DS}_\alpha(\eta)$ distribution by tempering functions, $g(x) = e^{-ax}$, $a > 0$ and $g(x) = \beta e^x x^\beta \Gamma(-\beta, x)$, $\beta > \alpha$, respectively. For more details one can refer to [9].

Given the training data, one can estimate the parameters of these distributions for positive and negative jumps and use them to simulate the jumps for future time horizons. Figure 5.6 compares some fitted discrete distributions on the training data with the frequency of empirical positive and negative jumps on the training data. These are zero-truncated versions of Poisson, negative binomial, DTS(Poisson-Tweedie) and DTS(Pareto tempering) and also discrete stable (with no tempering i.e. g(x) = 1) distributions. One can see that tempering generally improves the performance in the case of DTS distributions.

5.2.2 Modeling Interarrival Times

In this section, we consider modeling the interarrival times, which are continuous and non-negative. Some continuous distributions like exponential, gamma or positive tempered stable distributions can be used for modeling. One can also use bootstrap sampling to sample from historical observations.

Figure 5.7 shows the first 150 interarrival times in the train data from the exchange rate between Euro and Dollar. This figure shows the variability in the interarrival times. Although most of these interarrival times are less than a second, there are some that are more than one minute.

One of the common characteristics of these times is a persistent dependency with a long memory. Figure 5.7 shows the auto correlation function (Acf) for the interarrival times in exchange rate between Euro and Dollar interarrival times and also it shows a kernel density estimation and a frequency histogram.

To account for dependencies in interarrival times, we use a GARCH model. To model the innovations in the GARCH model one can use a continuous distribution or it can be done by bootstrap sampling from historical observations.

A GARCH(p,q) model is a time series model defined as follow:

$$S_n = \sigma_n \varepsilon_n,$$

where

$$\sigma_n^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i S_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2.$$

The interarrival times are denoted by S_n, and we use some continuous distributions to model the innovations ε_n. The inference from a GARCH model is based on a Maximum Likelihood Estimation (MLE) with the conditional Gaussian assumption on the innovation distribution. However, many empirical studies suggest heavy-tailed distributions for ε_n, and this may lead to inconsistency of the parameter estimations

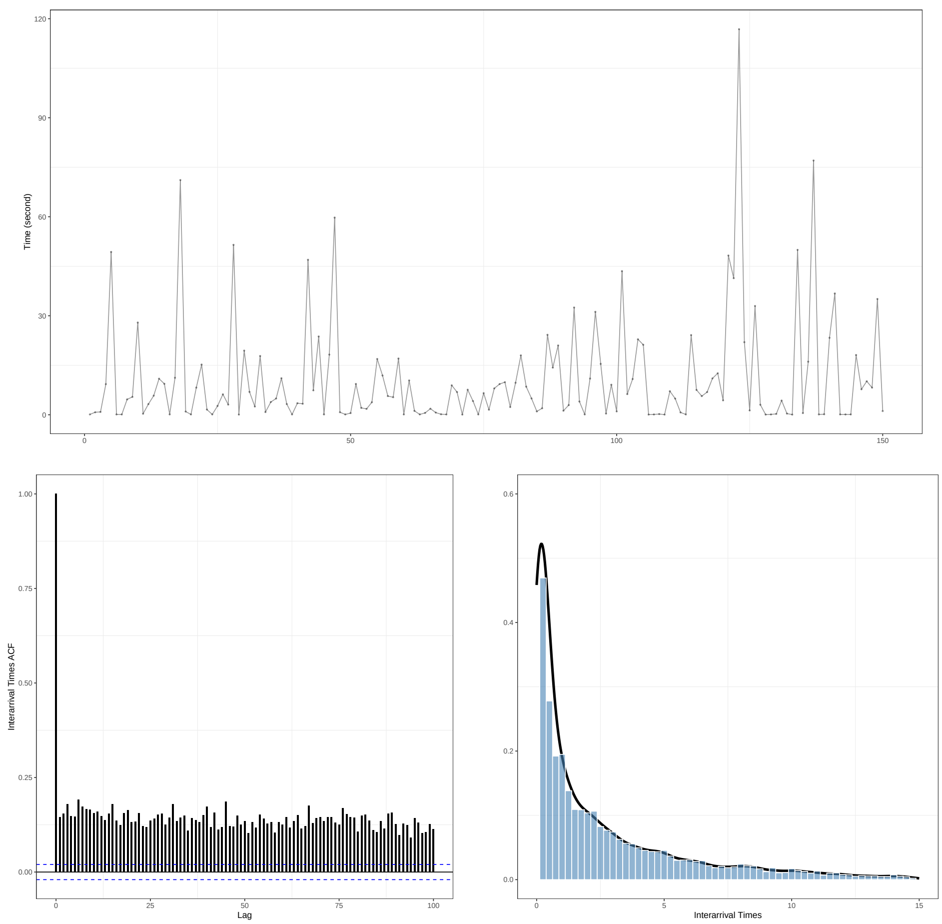

Figure 5.7: The top figure shows the first 150 interarrival times in the train data from exchange rate between Euro and Dollar. Bottom left one is the Acf and the right one is a kernel density estimation and the frequency histogram for these observations.

when the innovations models are misspecified. Gaussian Quasi-Maximum Likelihood Estimator (GQMLE) was introduced to account for this problem. It requires a finite fourth moment for innovations and also holds when innovation distribution is far from the Gaussian. One can find more details in [12] and the references there in.

When fitting this GARCH model, The standardize residuals, $\hat{\varepsilon}$ are the residuals divided by its estimated conditional standard deviation $\hat{\sigma}$, and it estimates ε_n. If the model fits well, there should be no serial correlation in the standardize residuals or in the squared standardize residuals [19].

For modeling, we fit a GARCH(1,1) model to the interarrival times with the assumption that the standardized residuals follow a continuous distribution, e.g. gamma, exponential or Tweedie distributions. Figure 5.8 shows a the qq-plot and Acf for the standard residuals from a GARCH(1,1) model fitted on the interarrival times. One can observe that the standardized residual distribution is far from Gaussian. Also fitting a GARCH(1,1) model explained the dependencies in the interarrival times.

In the context of counting process, e.g. the Poisson process, there is a well know relationship between these processes and the exponential distribution. Indeed, the interarrival times in this case are i.i.d exponential. Here, we have already observed that these interarrival times are highly correlated, for which a GARCH model was fitted to account for this dependency. Also, we observed that the distribution of innovations in the GARCH model is far from the Gaussian distribution. Here, we fit exponential, gamma and Tweedie distributions to the standardized residuals, and this finalizes modeling for interarrival times. Figure 5.8 also compares a kernel density estimation of interarrival times with the fitted exponential, gamma and the Tweedie distributions.

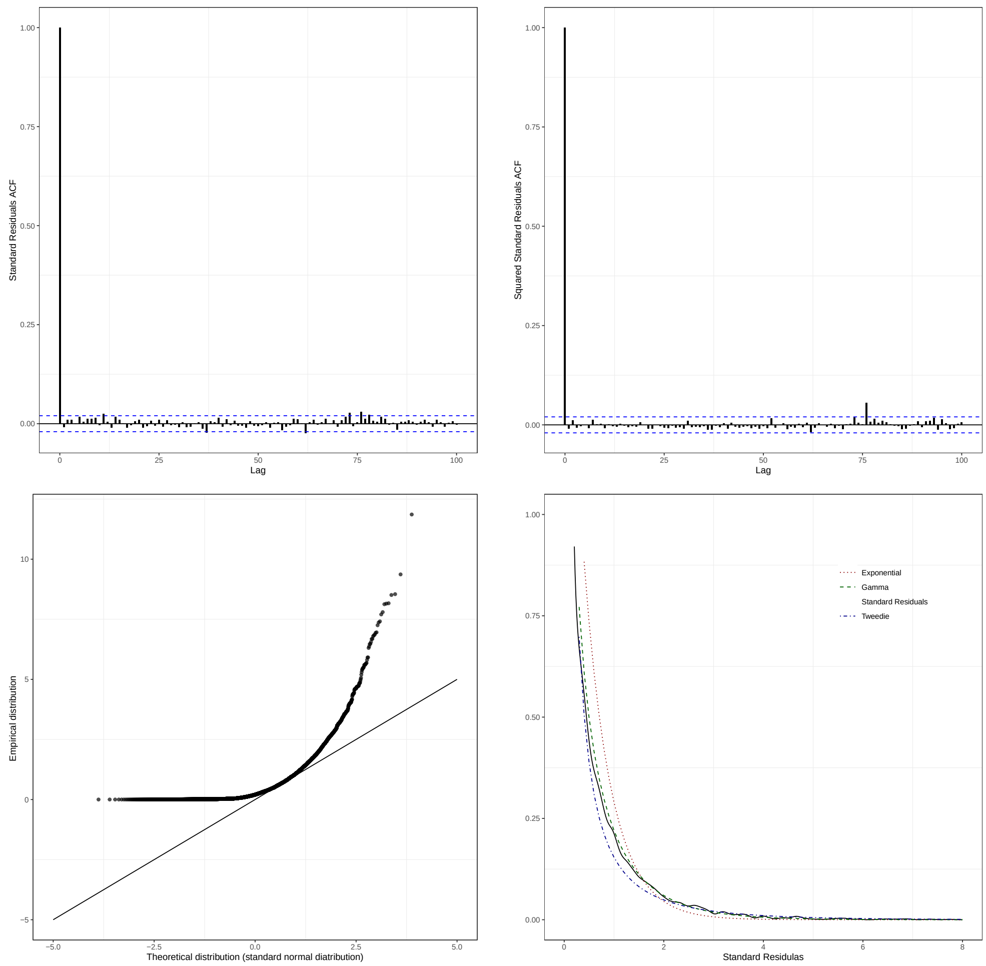

Figure 5.8: Top: Acf for standard residuals and square of standardized residuals from the GARCH(1,1) model. Bottom left: q-q plot that compares the standardized residuals to the standard normal distribution. Bottom right: solid black line is a kernel density estimation of standardized residuals. Dotted, dashed and dot-dashed lines are pdf of fitted exponential, gamma and the Tweedie distributions.

5.2.3 Simulating Sample Paths

The aim for this section is to show how a sample path can be simulated for tick data for a time horizon H. First, we need to simulate a sequence of interarrival times until the sum of these is larger than H. This will give us the number of jumps. For each jump, a Bernoulli distribution can be used to decide if it should be a positive or a negative jump. To simulate the GARCH model for interarrival times one need to simulate from the non-negative continuous distribution to be used as standardized residuals. The parameters of this distribution can be estimated from the standardized residuals from the fitted GARCH model on interarrival times in the train data. The GARCH model can be simulated in the following way. Let $S_0, S_1, \ldots, S_N$ be the interarrival times from the train data. We want to simulate $S_{N+1}, S_{N+2}, \ldots$ for the next H minutes. If $\varepsilon_1, \varepsilon_2, \ldots$ are simulated observations from the distribution of residuals, then

$$S_{N+1} = \sigma_{N+1}\varepsilon_1, \quad S_{N+2} = \sigma_{N+2}\varepsilon_2, \quad \cdots$$

where

$$\begin{aligned}
\sigma_{N+1}^2 &= \alpha_0 + \alpha_1 S_N^2 + \beta_1 \sigma_N^2 \\
\sigma_{N+2}^2 &= \alpha_0 + \alpha_1 S_{N+1}^2 + \beta_1 \sigma_{N+1}^2
\end{aligned}$$

and, α_0, α_1 and β_1 are estimated by fitting the GARCH model on the interarrival times in the train data. To simulate a sample path, let $\{S_{N+1}, S_{N+2}, \ldots\}$, $\{P_1, P_2, \ldots\}$ and $\{N_1, N_2, \ldots\}$, be simulated interarrival times, simulated positive jumps and simulated negative jumps, respectively. We simulate from jumps with the estimated parameters from the train data. To estimate the parameters for positive jumps and negative jumps, we use estimation method based on MLE separately on these jumps in the training data.

Now, we start with the last observed value in the train data and consider the time S_{N+1} to be the time for the first jump. To decide for each jump to be a positive or negative, we use a Bernoulli distribution with proportion of positives jumps in the train data to be the probability of a positive jump.

We continue this process until we exceed H minutes as the sum of interarrival times. Figure 5.9 compares different sample paths simulated from different distributions for jumps, while bootstrap sampling was used to simulate interarrival times. By bootstrap sampling here we mean sampling with replacement from the standard residuals from the GARCH model. One can observe that discrete stable and DTS (Pareto tempering) are more prone to produce large jumps since these are distributions with heavier tails. Tempering the tail of these distributions will results smaller jumps, like the ones in the DTS (Poisson-Tweedie).

Figure 5.10 shows different sample paths for which DTS (Poisson-Tweedie) distribution used to simulate jumps. Here, using gamma and exponential distributions generated more interarrival times and consequently more jumps. On the other hand, Tweedie distribution is more similar to the bootstrap sampling.

5.3 Risk Measurements and Backtesting

In this section, we use our model for risk assessment purposes. Most of the risk measures are depending on probabilistic properties of the loss distribution of a portfolio. Value at risk (VaR) and expected shortfall (ES) are some examples of risk measures that are depending on the quantiles of the loss distribution.

VaR has two parameters, the time horizon H and the confidence level $1-\alpha$. If $\mathcal{L}_H$ is the loss over the holding period H, then $\text{VaR}(\alpha, H)$ is the αth upper quantile of $\mathcal{L}_H$. Thus, for any loss distribution

$$\text{VaR}(\alpha, H) = \inf\{x : P(\mathcal{L}_H > x) \leq \alpha\}$$

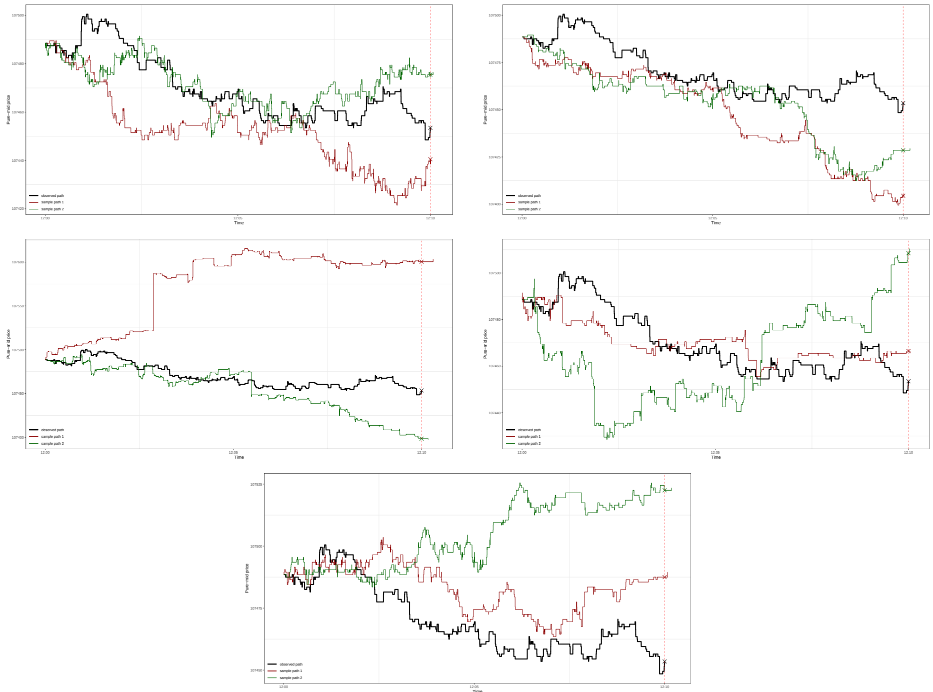

Figure 5.9: Solid black line is the observed path and two other colored and thinner ones are simulated paths using bootstrap sampling for standard residuals in the GARCH model for interarrival times and different distributions for jumps as follow; top left: negative binomial, top right: DTS (Poisson-Tweedie) middle left: discrete stable, middle right: DTS (Pareto tempering), bottom: Poisson.

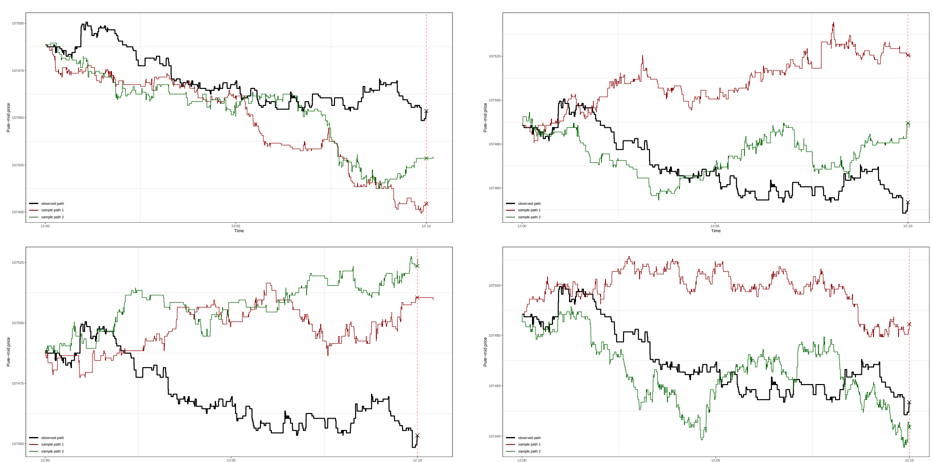

Figure 5.10: Solid black line is the observed path and two other colored and thinner ones are simulated paths using DTS (Poisson-Tweedie) distribution for jumps and different distributions for standard residuals in the GARCH model for interarrival times as follow; top left: bootstrap sampling, top right: gamma, bottom left: Tweedie, Bottom right: exponential.

ES is another important and widely used risk measure. It is the expected loss given that the loss exceeds VaR. It can be calculated by

$$\mathrm{ES}(\alpha, H) = \frac{\int_0^\alpha \mathrm{VaR}(u, H)du}{\alpha}$$

Proposition 8.13 in [15] gives alternative expressions for $\mathrm{ES}(\alpha, H)$ and it can be use in cases where the loss distribution is discrete.

$$\mathrm{ES}(\alpha, H) = \frac{1}{\alpha} E\left(\left(\mathcal{L}_H; \mathcal{L}_H > q_{1-\alpha}(\mathcal{L}_H)\right) + q_{1-\alpha}(\mathcal{L}_H)(\alpha - P(\mathcal{L}_H > q_{1-\alpha}(\mathcal{L}_H)\right)$$

in which $q_c(\mathcal{L}_H) = F_{\mathcal{L}}^{\leftarrow}(c) = \inf\{x : \mathcal{L}(x) \geq c\}$ For more details about VaR and ES one can refer to [19] or [15].

VaR can be estimated using different methods, e.g., using historical data to find the sample quantile of the losses, or using the parametric method on which a proposed distribution for the loss can be used to find the upper α quantile. On the other hand, VaR can be used to test the performance of a proposed model. This is called backtesting and the idea is to compare any model with the historical behavior of the portfolio price to test its performance. For example, if one is interested in testing the performance of the parametric method, the proposed distribution can be used to estimate the VaR for a given time horizon (e.g. one day) for a period of time (one year) and compare it to the historical losses and count the exceptions, which are the events on which the historical loss is greater than the estimated VaR value from the parametric method. For example, if we calculate the parametric $\mathrm{VaR}(0.05, \text{one day})$, and the historical losses for a period of one year, we will have 252 loss values, each for every day. The exceptions will be those days on which the loss is greater than the estimated VaR values. If the parametric model works well, then the proportion of exceptions, which we call exception rate, should be close to 0.05, so the number of exceptions should be about 13 days. After reporting the exception rate, one should also consider two facts, one is to test if the difference between exception rate and α is statistically significant, and the other is how random is the occurrence of these exceptions, in other word, to test for exception clustering.

A hypothesis test can be performed to check the significance of the difference between exception rate and α. Let T be the total number of time horizons for a period, e.g. $T = 252$ for one day horizon in a year, and N be the number of exceptions, then, N follows a binomial distribution with parameter T and α, $\mathrm{B}(T, \alpha)$ to be the total number of times the trial is repeated and the probability of success, respectively. Then, if T is large (one rule of thumb is $T\alpha > 5$ and $T(1-\alpha) > 5$) then

$$Z = \frac{N - n\alpha}{\sqrt{n\alpha(1-\alpha)}} \approx \mathrm{N}(0, 1)$$

Now, one can set up a test in which the null hypothesis is that the model works well and the alternative hypothesis is that the model has too many or too few exceptions. So, a small p-value here means the model does not work well, or the difference between the exception rate and α is statistically significant. This test is called the *unconditional coverage* test. A more powerful test can be performed based on the following test statistics which is based on likelihood ratio test, and under the null hypothesis when T is large, follow a χ^2_1 chi-squared with 1-degree of freedom.

$$\mathrm{LR}_{\mathrm{uc}} = -2\ln\left((1-\alpha)^{T-N}\alpha^{N}\right) + 2\ln\left((1-N/T)^{T-N}(N/T)^{N}\right)$$

The other test is to check the randomness of occurrence of the exceptions. In other word we want to check the independence of these exceptions' occurrence. The exceptions should be spread over time. If there is some clustering in the exceptions during time, the model is not working well in estimating VaR. This is the test of independence of exception occurrence. We consider two states of 0 and 1, on which 1 shows an exception. Let π_i be the probability of observing an exception in the next move, conditional on being in state i. If the model works well, we should have $\pi_0 = \pi_1$.

Let T_{ij} be the number of times that state j occurred after state i, and, $T = T_{00} + T_{01} + T_{10} + T_{11}$. We can estimate π_0 and π_1 by

$$\hat{\pi}_0 = \frac{T_{01}}{T_{01} + T_{00}}, \qquad \hat{\pi}_1 = \frac{T_{11}}{T_{10} + T_{11}}.$$

If the model is working well, we should have $\pi_0 = \pi_1$, for which can be estimated by

$$\hat{\pi} = \frac{T_{11} + T_{01}}{T}$$

Now, a hypothesis test can be set up. The test statistic is

$$\mathrm{LR}_{\mathrm{ind}} = -2\ln\left((1-\hat{\pi})^{T_{00}+T_{10}}\hat{\pi}^{T_{01}+T_{11}})\right) + 2\ln\left((1-\hat{\pi}_0)^{T_{00}}\hat{\pi}_0^{T_{01}}(1-\hat{\pi}_1)^{T_{10}}\hat{\pi}_1^{T_{11}}\right),$$

which follows asymptotically a χ_1^2 distribution as T gets large.

Finally, the *conditional coverage* test can be performed by to following test statistics, which follows a χ_2^2 distribution, using the fact that $\mathrm{LR}_{\mathrm{uc}}$ and $\mathrm{LR}_{\mathrm{ind}}$ are asymptotically independent.

$$\mathrm{LR}_{\mathrm{cc}} = \mathrm{LR}_{\mathrm{uc}} + \mathrm{LR}_{\mathrm{ind}}.$$

For more details about conditional coverage test one can refer to [13]

Here, we consider VaR not as an upper α quantile of the loss distribution $\mathcal{L}$ calculated from pure-mid price, but as the lower α quantile of the distribution of the pure-mid price value itself. We do this since, in the context of the tick data, price changes are calculated rather than returns or log returns, and we model these price changes by integer-valued probability models. Thus, the exceptions for us will be when the pure-mid price drops under the estimated lower α quantile if the distribution of the pure-mid price.

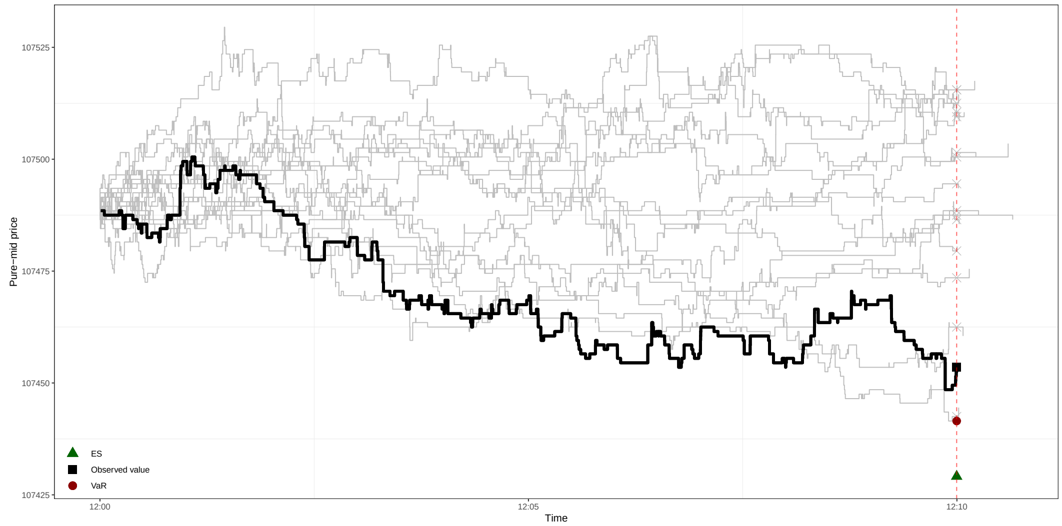

Figure 5.11: Solid black line is the observed path and gray paths are simulated ones from Tweedie and Poisson Tweedie distribution for interarrival times and jumps, respectively. Black square and red circle are observed value and VaR value, respectively.

We use Monte Carlo method to estimate the distribution of pure-mid price or more specifically, the lower α quantile of this distribution. For any choice of distribution for modeling the jumps (these are negative binomial, Poisson, discrete stable, DTS (Pareto tempering) and DTS (Poisson-Tweedie)) and standardized residuals in the GARCH model for interarrival times (these are exponential, gamma and Tweedie), we simulate 5000 sample paths for a given time horizon (these are 10, 20 and 60 minutes), then we estimate VaR for different confidence levels ($\alpha \in \{0.01, 0.025, 0.05, 0.1\}$) by sample quantiles.

Figure 5.11 shows 15 sample path out of 5000 ones, and also shows the estimated $\mathrm{VaR}(10, 0.05)$. Figure 5.12 shows a kernel density estimation of simulated pure-mid prices on which the observed value and VaR are marked. On the right hand side, one can see how pure-mid price, VaR and ES are changing during a day.

To test our proposed models, we consider 3rd and 4th weeks of July 2024, for which we will have 10 days. For any combination of distributions for jumps and standard residuals in the GARCH model for interarrival times, time horizon and confidence

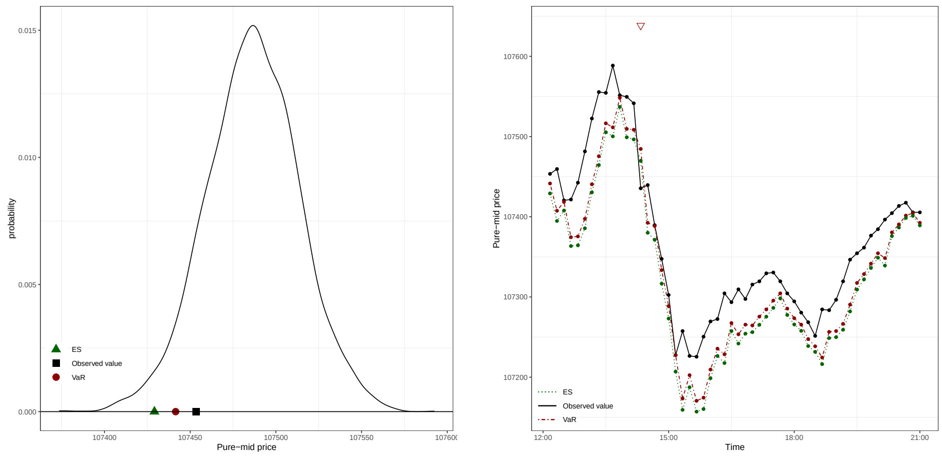

Figure 5.12: Left figure shows a kernel density estimation of simulated pure-mid prices on which the observed value, VaR and ES are marked. Right figure shows the observed values, VaR and ES for one day on July 1st 2024. One can see when exceptions occurred.

level, we perform conditional coverage test. Figure 5.13 shows $\mathrm{VaR}(10, \alpha)$ for three cases in which discrete stable, Poisson and Poisson-Tweedie was used for jumps, bootstrap sampling was used for standard residuals in the GARCH model for interarrival times, and $\alpha \in \{0.01, 0.025, 0.05, 0.1\}$. One can also see where the exceptions occurred. Tables 5.1, 5.2, 5.3 and 5.4 show the results from exceptions counting and conditional coverage tests for all combinations of models and time horizons for different values of α. In most cases, negative binomial has the most satisfactory results. However, exception rates from DTS (Poisson-Tweedie) have an acceptable results for which in most of the cases the the reported p-values are high enough, thus we fail to reject the null hypothesis in those cases. In general, one can see discrete stable distributions may not be reliable in modeling jumps in the tick data, however tempering these distributions will result in a much better distributions for modeling jumps in the tick data.

In some cases, mostly for discrete stable and DTS (Pareto tempering) the rates of exceptions are zero. In those cases, we will not be interested in checking the dependency of the occurrences since there are no exceptions. However, the p-values are high, but these models do not possess a good rate of exceptions to be interesting models.

Figures 5.14 and 5.15 show box plots of the maximum values of the log-likelihood functions fitted on standard residuals of the GARCH model and also on the positive and negative jumps using different distributions. In Figure 5.15 Tweedie distribution has a higher value for log-likelihood function during all 10 days of our analysis and also for different time horizons 10, 20 and 60 minutes. Refering to Figure 5.15 negative binomial has the best performance followed by DTS (poisson-Tweedie) and Poisson in fitting the jumps as we have the same performance base on the Tables 5.1 to 5.4.

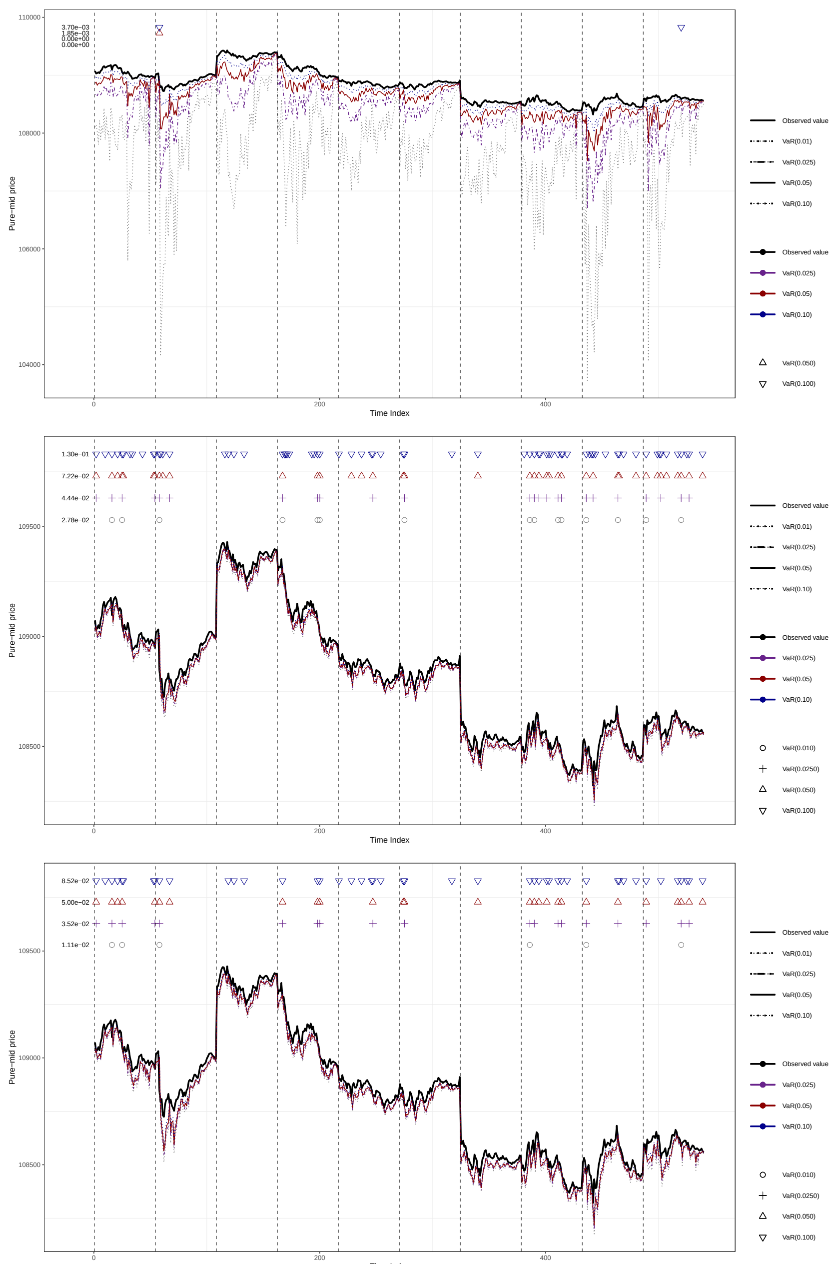

Figure 5.13: VaR$(10, \alpha)$ values during 10 days in 3rd and 4th weeks of July 2024, from up to bottom, for discrete stable, Poisson and Poisson-Tweedie used for jumps, bootstrap sampling for standard residuals in the GARCH model for interarrival times, and $\alpha \in \{0.01, 0.025, 0.05, 0.1\}$. The occurrence of exceptions are marked for any confidence level.

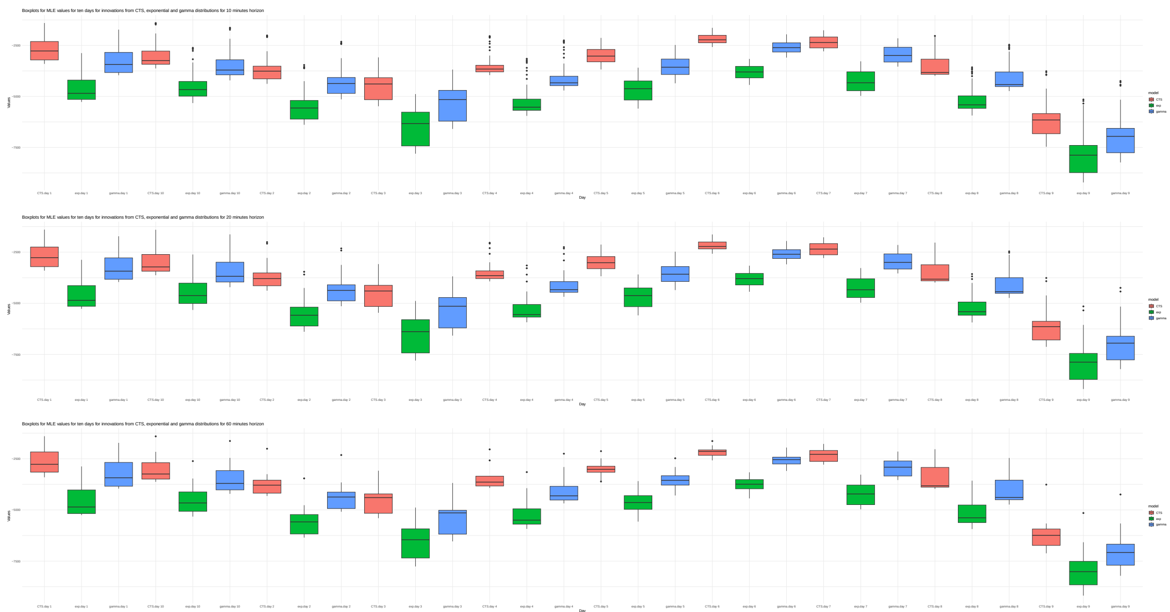

Figure 5.14: Box plots of maximum values of log-likelihood functions in fitting Tweedie, exponential and gamma distributions on standard residuals in the GARCH model for 10 days. Three figures are for different time horizons, 10, 20 and 60 minutes from top to bottom, respectively.

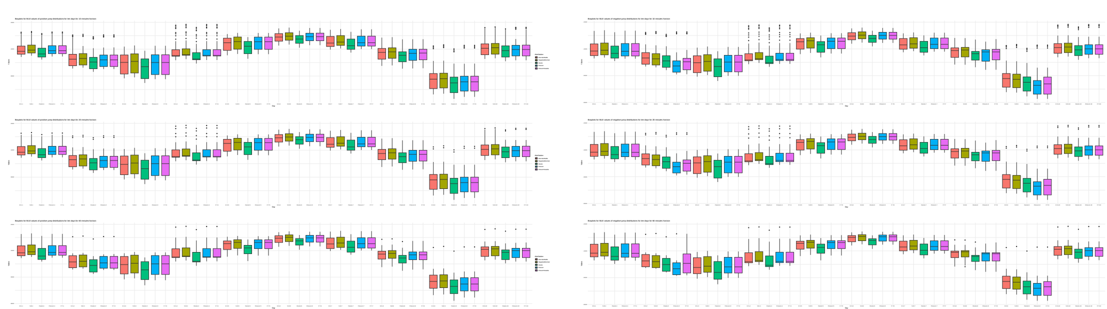

Figure 5.15: Box plots of maximum values of log-likelihood functions in fitting discrete stable, negative binomial, DTS (Pareto tempering), Poisson and DTS (Poisson Tweedie) on positive (left figures) and negative (right figures) for 10 days. Three figures on each side are for different time horizons, 10, 20 and 60 minutes from top to bottom, respectively.

To test how stable the results from Monte Carlo simulations are, these simulations were repeated 50 times for negative binomial and DTS (Poisson Tweedie) distributions for modeling jumps and Tweedie, exponential and gamma distributions for modeling standard residuals in the GARCH model for interarrival times. Table 5.5 shows the results from these simulations in which the average and standard deviations were reported in each case. Standard deviations are small which indicate stable results.

The results from exception counting and conditional coverage tests in Tables 5.1 to 5.4 show that the negative binomial and Poisson-Tweedie distributions followed by Poisson distribution have a good performance in terms of exception rates. In almost all of the cases we have some combination of these distribution with large p-values, thus these models are acceptable based on exception rates. However, in independent tests, especially when α is 0.01 or 0.025, we can observe that $\hat{\pi}_1$, is zero, which shows that it is very unlikely to see two consecutive exceptions. This is not very surprising since, especially when the sample size is small, we expect to see few exceptions. In these cases if we accept that $\hat{\pi}_1^{T_{11}} = 0^0 = 1$ then p-values are more meaningful. In other cases, when α is 0.05 or 0.10, we have much more satisfactory results. We have many cases where $\hat{\pi}_1$ is not zero and it is very close to $\hat{\pi}_0$ and the p-values from conditional coverage now are more reliable. Thus, we can say in these cases, we have a good exception rate, for which the exceptions are well spread.

Table 5.1: Results from exception counting and conditional coverage tests for $\alpha = 0.01$

Jump Distribution	Time Distribution	Horizon (in minutes)	α	Exception rates	p-value for LR_{uc}	$\hat{\pi}_0$	$\hat{\pi}_1$	$\hat{\pi}$	p-value for LR_{ind}	p-value for LR_{cc}
Discrete Stable	bootstrap	10	0.01	0.00e+00	9.86e-04 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03 (**)
Discrete Stable	Tweedie	10	0.01	0.00e+00	9.86e-04 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03 (**)
Discrete Stable	exponential	10	0.01	0.00e+00	9.86e-04 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03 (**)
Discrete Stable	gamma	10	0.01	0.00e+00	9.86e-04 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03 (**)
Discrete Stable	bootstrap	20	0.01	0.00e+00	1.98e-02 (*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02
Discrete Stable	Tweedie	20	0.01	0.00e+00	1.98e-02 (*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02
Discrete Stable	exponential	20	0.01	0.00e+00	1.98e-02 (*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02
Discrete Stable	gamma	20	0.01	0.00e+00	1.98e-02 (*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02
Discrete Stable	bootstrap	60	0.01	0.00e+00	1.79e-01	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01
Discrete Stable	Tweedie	60	0.01	0.00e+00	1.79e-01	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01
Discrete Stable	exponential	60	0.01	0.00e+00	1.79e-01	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01
Discrete Stable	gamma	60	0.01	0.00e+00	1.79e-01	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01
Negative Binomial	bootstrap	10	0.01	2.04e-02	3.37e-02 (*)	2.08e-02	0.00e+00	2.04e-02	4.98e-01	8.33e-02
Negative Binomial	Tweedie	10	0.01	2.78e-02	6.51e-04 (***)	2.86e-02	0.00e+00	2.78e-02	3.54e-01	1.95e-03 (**)
Negative Binomial	exponential	10	0.01	**1.11e-02**	7.99e-01	1.13e-02	0.00e+00	1.11e-02	7.13e-01	9.05e-01
Negative Binomial	gamma	10	0.01	1.48e-02	2.94e-01	1.51e-02	0.00e+00	1.48e-02	6.23e-01	5.11e-01
Negative Binomial	bootstrap	20	0.01	1.85e-02	2.09e-01	1.89e-02	0.00e+00	1.86e-02	6.63e-01	4.12e-01
Negative Binomial	Tweedie	20	0.01	3.33e-02	2.39e-03 (**)	3.46e-02	0.00e+00	3.35e-02	4.30e-01	7.28e-03 (**)
Negative Binomial	exponential	20	0.01	**1.11e-02**	8.57e-01	1.13e-02	0.00e+00	1.12e-02	7.95e-01	9.51e-01
Negative Binomial	gamma	20	0.01	1.48e-02	4.58e-01	1.51e-02	0.00e+00	1.49e-02	7.28e-01	7.15e-01
Negative Binomial	bootstrap	60	0.01	2.22e-02	3.15e-01	2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01
Negative Binomial	Tweedie	60	0.01	3.33e-02	7.96e-02	3.49e-02	0.00e+00	3.37e-02	6.47e-01	1.94e-01
Negative Binomial	exponential	60	0.01	**1.11e-02**	9.17e-01	1.14e-02	0.00e+00	1.12e-02	8.80e-01	9.83e-01
Negative Binomial	gamma	60	0.01	2.22e-02	3.15e-01	2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01
DTS (Pareto Tempering)	bootstrap	10	0.01	0.00e+00	9.86e-04 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03 (**)
DTS (Pareto Tempering)	Tweedie	10	0.01	0.00e+00	9.86e-04 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03 (**)
DTS (Pareto Tempering)	exponential	10	0.01	0.00e+00	9.86e-04 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03 (**)
DTS (Pareto Tempering)	gamma	10	0.01	0.00e+00	9.86e-04 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03 (**)
DTS (Pareto Tempering)	bootstrap	20	0.01	0.00e+00	1.98e-02 (*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02
DTS (Pareto Tempering)	Tweedie	20	0.01	0.00e+00	1.98e-02 (*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02
DTS (Pareto Tempering)	exponential	20	0.01	0.00e+00	1.98e-02 (*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02
DTS (Pareto Tempering)	gamma	20	0.01	0.00e+00	1.98e-02 (*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02
DTS (Pareto Tempering)	bootstrap	60	0.01	0.00e+00	1.79e-01	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01
DTS (Pareto Tempering)	Tweedie	60	0.01	0.00e+00	1.79e-01	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01
DTS (Pareto Tempering)	exponential	60	0.01	0.00e+00	1.79e-01	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01
DTS (Pareto Tempering)	gamma	60	0.01	0.00e+00	1.79e-01	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01
Poisson	bootstrap	10	0.01	2.78e-02	6.51e-04 (***)	2.86e-02	0.00e+00	2.78e-02	3.54e-01	1.95e-03 (**)
Poisson	Tweedie	10	0.01	2.96e-02	2.07e-04 (***)	3.06e-02	0.00e+00	2.97e-02	3.22e-01	6.27e-04 (***)
Poisson	exponential	10	0.01	**1.11e-02**	7.99e-01	1.13e-02	0.00e+00	1.11e-02	7.13e-01	9.05e-01
Poisson	gamma	10	0.01	2.22e-02	1.39e-02 (*)	2.28e-02	0.00e+00	2.23e-02	4.60e-01	3.70e-02 (*)
Poisson	bootstrap	20	0.01	2.22e-02	8.21e-02	2.28e-02	0.00e+00	2.23e-02	6.01e-01	1.92e-01
Poisson	Tweedie	20	0.01	2.96e-02	8.69e-03 (**)	3.07e-02	0.00e+00	2.97e-02	4.84e-01	2.50e-02 (*)
Poisson	exponential	20	0.01	**1.11e-02**	8.57e-01	1.13e-02	0.00e+00	1.12e-02	7.95e-01	9.51e-01
Poisson	gamma	20	0.01	1.48e-02	4.58e-01	1.51e-02	0.00e+00	1.49e-02	7.28e-01	7.15e-01
Poisson	bootstrap	60	0.01	2.22e-02	3.15e-01	2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01
Poisson	Tweedie	60	0.01	3.33e-02	7.96e-02	3.49e-02	0.00e+00	3.37e-02	6.47e-01	1.94e-01
Poisson	exponential	60	0.01	**1.11e-02**	9.17e-01	1.14e-02	0.00e+00	1.12e-02	8.80e-01	9.83e-01
Poisson	gamma	60	0.01	2.22e-02	3.15e-01	2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01
DTS (Poisson Tweedie)	bootstrap	10	0.01	2.78e-02	6.51e-04 (***)	2.86e-02	0.00e+00	2.78e-02	3.54e-01	1.95e-03 (**)
DTS (Poisson Tweedie)	Tweedie	10	0.01	2.96e-02	2.07e-04 (***)	3.06e-02	0.00e+00	2.97e-02	3.22e-01	6.27e-04 (***)
DTS (Poisson Tweedie)	exponential	10	0.01	**1.11e-02**	7.99e-01	1.13e-02	0.00e+00	1.11e-02	7.13e-01	9.05e-01
DTS (Poisson Tweedie)	gamma	10	0.01	2.41e-02	5.37e-03 (**)	2.47e-02	0.00e+00	2.41e-02	4.23e-01	1.50e-02 (*)
DTS (Poisson Tweedie)	bootstrap	20	0.01	1.85e-02	2.09e-01	1.89e-02	0.00e+00	1.86e-02	6.63e-01	4.12e-01
DTS (Poisson Tweedie)	Tweedie	20	0.01	3.33e-02	2.39e-03 (**)	3.46e-02	0.00e+00	3.35e-02	4.30e-01	7.28e-03 (**)
DTS (Poisson Tweedie)	exponential	20	0.01	**1.11e-02**	8.57e-01	1.13e-02	0.00e+00	1.12e-02	7.95e-01	9.51e-01
DTS (Poisson Tweedie)	gamma	20	0.01	1.48e-02	4.58e-01	1.51e-02	0.00e+00	1.49e-02	7.28e-01	7.15e-01
DTS (Poisson Tweedie)	bootstrap	60	0.01	2.22e-02	3.15e-01	2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01
DTS (Poisson Tweedie)	Tweedie	60	0.01	3.33e-02	7.96e-02	3.49e-02	0.00e+00	3.37e-02	6.47e-01	1.94e-01
DTS (Poisson Tweedie)	exponential	60	0.01	**1.11e-02**	9.17e-01	1.14e-02	0.00e+00	1.12e-02	8.80e-01	9.83e-01
DTS (Poisson Tweedie)	gamma	60	0.01	2.22e-02	3.15e-01	2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01

Table 5.2: Results from exception counting and conditional coverage tests for $\alpha = 0.025$

Jump Distribution	Time Distribution	Horizon (in minutes)	α	Exception rates	p-value for LR_{uc}		$\hat{\pi}_0$	$\hat{\pi}_1$	$\hat{\pi}$	p-value for LR_{ind}	p-value for for LR_{cc}	
Discrete Stable	bootstrap	10	0.025	0.00e+00	1.70e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.15e-06	(***)
Discrete Stable	Tweedie	10	0.025	0.00e+00	1.70e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.15e-06	(***)
Discrete Stable	exponential	10	0.025	0.00e+00	1.70e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.15e-06	(***)
Discrete Stable	gamma	10	0.025	0.00e+00	1.70e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.15e-06	(***)
Discrete Stable	bootstrap	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
Discrete Stable	Tweedie	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
Discrete Stable	exponential	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
Discrete Stable	gamma	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
Discrete Stable	bootstrap	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
Discrete Stable	Tweedie	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
Discrete Stable	exponential	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
Discrete Stable	gamma	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
Negative Binomial	bootstrap	10	0.025	4.26e-02	1.71e-02	(*)	4.46e-02	0.00e+00	4.27e-02	1.52e-01	2.09e-02	(*)
Negative Binomial	Tweedie	10	0.025	4.63e-02	4.52e-03	(**)	4.86e-02	0.00e+00	4.64e-02	1.19e-01	5.26e-03	(**)
Negative Binomial	exponential	10	0.025	**2.96e-02**	5.03e-01		3.06e-02	0.00e+00	2.97e-02	3.22e-01	4.90e-01	
Negative Binomial	gamma	10	0.025	3.70e-02	9.41e-02		3.85e-02	0.00e+00	3.71e-02	2.14e-01	1.14e-01	
Negative Binomial	bootstrap	20	0.025	4.07e-02	1.28e-01		4.26e-02	0.00e+00	4.09e-02	3.33e-01	1.97e-01	
Negative Binomial	Tweedie	20	0.025	5.93e-02	2.12e-03	(**)	5.93e-02	6.25e-02	5.95e-02	9.58e-01	8.87e-03	(**)
Negative Binomial	exponential	20	0.025	1.48e-02	2.47e-01		1.51e-02	0.00e+00	1.49e-02	7.28e-01	4.81e-01	
Negative Binomial	gamma	20	0.025	**2.59e-02**	9.23e-01		2.67e-02	0.00e+00	2.60e-02	5.41e-01	8.26e-01	
Negative Binomial	bootstrap	60	0.025	4.44e-02	2.86e-01		4.71e-02	0.00e+00	4.49e-02	5.39e-01	4.69e-01	
Negative Binomial	Tweedie	60	0.025	6.67e-02	3.53e-02	(*)	7.23e-02	0.00e+00	6.74e-02	3.51e-01	7.06e-02	
Negative Binomial	exponential	60	0.025	1.11e-02	3.44e-01		1.14e-02	0.00e+00	1.12e-02	8.80e-01	6.32e-01	
Negative Binomial	gamma	60	0.025	**2.22e-02**	8.63e-01		2.30e-02	0.00e+00	2.25e-02	7.62e-01	9.41e-01	
DTS (Pareto Tempering)	bootstrap	10	0.025	1.85e-03	7.39e-06	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	4.33e-05	(***)
DTS (Pareto Tempering)	Tweedie	10	0.025	1.85e-03	7.39e-06	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	4.33e-05	(***)
DTS (Pareto Tempering)	exponential	10	0.025	1.85e-03	7.39e-06	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	4.33e-05	(***)
DTS (Pareto Tempering)	gamma	10	0.025	1.85e-03	7.39e-06	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	4.33e-05	(***)
DTS (Pareto Tempering)	bootstrap	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
DTS (Pareto Tempering)	Tweedie	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
DTS (Pareto Tempering)	exponential	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
DTS (Pareto Tempering)	gamma	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
DTS (Pareto Tempering)	bootstrap	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
DTS (Pareto Tempering)	Tweedie	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
DTS (Pareto Tempering)	exponential	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
DTS (Pareto Tempering)	gamma	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
Poisson	bootstrap	10	0.025	4.44e-02	8.97e-03	(**)	4.66e-02	0.00e+00	4.45e-02	1.35e-01	1.07e-02	(*)
Poisson	Tweedie	10	0.025	5.19e-02	4.64e-04	(***)	5.48e-02	0.00e+00	5.19e-02	7.97e-02	4.70e-04	(***)
Poisson	exponential	10	0.025	3.15e-02	3.53e-01		3.26e-02	0.00e+00	3.15e-02	2.93e-01	3.74e-01	
Poisson	gamma	10	0.025	3.89e-02	5.56e-02		4.05e-02	0.00e+00	3.90e-02	1.92e-01	6.83e-02	
Poisson	bootstrap	20	0.025	4.44e-02	6.46e-02		4.67e-02	0.00e+00	4.46e-02	2.90e-01	1.04e-01	
Poisson	Tweedie	20	0.025	6.30e-02	7.71e-04	(***)	6.35e-02	5.88e-02	6.32e-02	9.38e-01	3.49e-03	(**)
Poisson	exponential	20	0.025	1.85e-02	4.75e-01		1.89e-02	0.00e+00	1.86e-02	6.63e-01	7.05e-01	
Poisson	gamma	20	0.025	2.96e-02	6.36e-01		3.07e-02	0.00e+00	2.97e-02	4.84e-01	6.99e-01	
Poisson	bootstrap	60	0.025	5.56e-02	1.09e-01		5.95e-02	0.00e+00	5.62e-02	4.40e-01	2.05e-01	
Poisson	Tweedie	60	0.025	6.67e-02	3.53e-02	(*)	7.23e-02	0.00e+00	6.74e-02	3.51e-01	7.06e-02	
Poisson	exponential	60	0.025	1.11e-02	3.44e-01		1.14e-02	0.00e+00	1.12e-02	8.80e-01	6.32e-01	
Poisson	gamma	60	0.025	**2.22e-02**	8.63e-01		2.30e-02	0.00e+00	2.25e-02	7.62e-01	9.41e-01	
DTS (Poisson Tweedie)	bootstrap	10	0.025	4.63e-02	4.52e-03	(**)	4.86e-02	0.00e+00	4.64e-02	1.19e-01	5.26e-03	(**)
DTS (Poisson Tweedie)	Tweedie	10	0.025	4.81e-02	2.19e-03	(**)	5.07e-02	0.00e+00	4.82e-02	1.04e-01	2.46e-03	(**)
DTS (Poisson Tweedie)	exponential	10	0.025	3.52e-02	1.53e-01		3.65e-02	0.00e+00	3.53e-02	2.39e-01	1.80e-01	
DTS (Poisson Tweedie)	gamma	10	0.025	3.89e-02	5.56e-02		4.05e-02	0.00e+00	3.90e-02	1.92e-01	6.83e-02	
DTS (Poisson Tweedie)	bootstrap	20	0.025	4.81e-02	3.03e-02	(*)	4.69e-02	7.69e-02	4.83e-02	6.47e-01	8.63e-02	
DTS (Poisson Tweedie)	Tweedie	20	0.025	7.04e-02	8.68e-05	(***)	6.80e-02	1.05e-01	7.06e-02	5.65e-01	3.83e-04	(***)
DTS (Poisson Tweedie)	exponential	20	0.025	1.85e-02	4.75e-01		1.89e-02	0.00e+00	1.86e-02	6.63e-01	7.05e-01	
DTS (Poisson Tweedie)	gamma	20	0.025	3.70e-02	2.37e-01		3.86e-02	0.00e+00	3.72e-02	3.79e-01	3.37e-01	
DTS (Poisson Tweedie)	bootstrap	60	0.025	4.44e-02	2.86e-01		4.71e-02	0.00e+00	4.49e-02	5.39e-01	4.69e-01	
DTS (Poisson Tweedie)	Tweedie	60	0.025	7.78e-02	9.91e-03	(**)	8.54e-02	0.00e+00	7.87e-02	2.74e-01	1.98e-02	(*)
DTS (Poisson Tweedie)	exponential	60	0.025	1.11e-02	3.44e-01		1.14e-02	0.00e+00	1.12e-02	8.80e-01	6.32e-01	
DTS (Poisson Tweedie)	gamma	60	0.025	**2.22e-02**	8.63e-01		2.30e-02	0.00e+00	2.25e-02	7.62e-01	9.41e-01	

Table 5.3: Results from exception counting and conditional coverage tests for $\alpha = 0.05$

Jump Distribution	Time Distribution	Horizon (in minutes)	α	Exception rates	p-value for LR_{uc}		$\hat{\pi}_0$	$\hat{\pi}_1$	$\hat{\pi}$	p-value for LR_{ind}	p-value for LR_{cc}	
Discrete Stable	bootstrap	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
Discrete Stable	Tweedie	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
Discrete Stable	exponential	10	0.05	0.00e+00	9.85e-14	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.35e-13	(***)
Discrete Stable	gamma	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
Discrete Stable	bootstrap	20	0.05	0.00e+00	1.42e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.67e-07	(***)
Discrete Stable	Tweedie	20	0.05	0.00e+00	1.42e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.67e-07	(***)
Discrete Stable	exponential	20	0.05	0.00e+00	1.42e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.67e-07	(***)
Discrete Stable	gamma	20	0.05	0.00e+00	1.42e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.67e-07	(***)
Discrete Stable	bootstrap	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
Discrete Stable	Tweedie	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
Discrete Stable	exponential	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
Discrete Stable	gamma	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
Negative Binomial	bootstrap	10	0.05	6.30e-02	1.83e-01		6.34e-02	5.88e-02	6.31e-02	9.15e-01	4.10e-01	
Negative Binomial	Tweedie	10	0.05	7.41e-02	1.62e-02	(*)	7.62e-02	5.00e-02	7.42e-02	5.22e-01	4.54e-02	(*)
Negative Binomial	exponential	10	0.05	4.44e-02	5.46e-01		4.66e-02	0.00e+00	4.45e-02	1.35e-01	2.72e-01	
Negative Binomial	gamma	10	0.05	6.11e-02	2.52e-01		6.13e-02	6.06e-02	6.12e-02	9.88e-01	5.18e-01	
Negative Binomial	bootstrap	20	0.05	7.04e-02	1.47e-01		7.17e-02	5.56e-02	7.06e-02	7.89e-01	3.37e-01	
Negative Binomial	Tweedie	20	0.05	1.07e-01	1.56e-04	(***)	1.04e-01	1.43e-01	1.08e-01	5.43e-01	6.51e-04	(***)
Negative Binomial	exponential	20	0.05	3.70e-02	3.07e-01		3.85e-02	0.00e+00	3.72e-02	4.05e-01	4.19e-01	
Negative Binomial	gamma	20	0.05	**5.93e-02**	4.97e-01		5.91e-02	6.67e-02	5.95e-02	9.05e-01	7.88e-01	
Negative Binomial	bootstrap	60	0.05	6.67e-02	4.89e-01		7.23e-02	0.00e+00	6.74e-02	3.51e-01	5.10e-01	
Negative Binomial	Tweedie	60	0.05	1.33e-01	2.40e-03	(**)	1.30e-01	1.67e-01	1.35e-01	7.35e-01	9.41e-03	(**)
Negative Binomial	exponential	60	0.05	2.22e-02	1.76e-01		2.30e-02	0.00e+00	2.25e-02	7.62e-01	3.83e-01	
Negative Binomial	gamma	60	0.05	3.33e-02	4.41e-01		3.49e-02	0.00e+00	3.37e-02	6.47e-01	6.69e-01	
DTS (Pareto Tempering)	bootstrap	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
DTS (Pareto Tempering)	Tweedie	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
DTS (Pareto Tempering)	exponential	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
DTS (Pareto Tempering)	gamma	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
DTS (Pareto Tempering)	bootstrap	20	0.05	0.00e+00	1.42e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.67e-07	(***)
DTS (Pareto Tempering)	Tweedie	20	0.05	0.00e+00	1.42e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.67e-07	(***)
DTS (Pareto Tempering)	exponential	20	0.05	0.00e+00	1.42e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.67e-07	(***)
DTS (Pareto Tempering)	gamma	20	0.05	0.00e+00	1.42e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.67e-07	(***)
DTS (Pareto Tempering)	bootstrap	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
DTS (Pareto Tempering)	Tweedie	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
DTS (Pareto Tempering)	exponential	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
DTS (Pareto Tempering)	gamma	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
Poisson	bootstrap	10	0.05	7.22e-02	2.59e-02	(*)	7.00e-02	1.03e-01	7.24e-02	4.73e-01	6.45e-02	
Poisson	Tweedie	10	0.05	7.59e-02	9.97e-03	(**)	8.03e-02	2.44e-02	7.61e-02	1.38e-01	1.20e-02	(*)
Poisson	exponential	10	0.05	4.63e-02	6.89e-01		4.86e-02	0.00e+00	4.64e-02	1.19e-01	2.74e-01	
Poisson	gamma	10	0.05	6.11e-02	2.52e-01		6.13e-02	6.06e-02	6.12e-02	9.88e-01	5.18e-01	
Poisson	bootstrap	20	0.05	8.52e-02	1.54e-02	(*)	7.69e-02	1.82e-01	8.55e-02	1.32e-01	1.71e-02	(*)
Poisson	Tweedie	20	0.05	1.15e-01	2.50e-05	(***)	1.13e-01	1.33e-01	1.15e-01	7.47e-01	1.32e-04	(***)
Poisson	exponential	20	0.05	3.70e-02	3.07e-01		3.85e-02	0.00e+00	3.72e-02	4.05e-01	4.19e-01	
Poisson	gamma	20	0.05	6.67e-02	2.31e-01		6.75e-02	5.88e-02	6.69e-02	8.88e-01	4.83e-01	
Poisson	bootstrap	60	0.05	8.89e-02	1.25e-01		8.64e-02	1.25e-01	8.99e-02	7.28e-01	2.91e-01	
Poisson	Tweedie	60	0.05	1.33e-01	2.40e-03	(**)	1.30e-01	1.67e-01	1.35e-01	7.35e-01	9.41e-03	(**)
Poisson	exponential	60	0.05	3.33e-02	4.41e-01		3.49e-02	0.00e+00	3.37e-02	6.47e-01	6.69e-01	
Poisson	gamma	60	0.05	**5.56e-02**	8.12e-01		5.95e-02	0.00e+00	5.62e-02	4.40e-01	7.22e-01	
DTS (Poisson Tweedie)	bootstrap	10	0.05	6.67e-02	9.01e-02		6.76e-02	5.56e-02	6.68e-02	7.74e-01	2.28e-01	
DTS (Poisson Tweedie)	Tweedie	10	0.05	7.78e-02	5.98e-03	(**)	7.85e-02	7.14e-02	7.79e-02	8.69e-01	2.25e-02	(*)
DTS (Poisson Tweedie)	exponential	10	0.05	**5.00e-02**	1.00e+00		5.08e-02	3.70e-02	5.01e-02	7.39e-01	9.46e-01	
DTS (Poisson Tweedie)	gamma	10	0.05	5.93e-02	3.37e-01		5.92e-02	6.25e-02	5.94e-02	9.39e-01	6.29e-01	
DTS (Poisson Tweedie)	bootstrap	20	0.05	9.26e-02	3.89e-03	(**)	8.57e-02	1.67e-01	9.29e-02	2.31e-01	7.56e-03	(**)
DTS (Poisson Tweedie)	Tweedie	20	0.05	1.15e-01	2.50e-05	(***)	1.09e-01	1.67e-01	1.15e-01	3.73e-01	9.34e-05	(***)
DTS (Poisson Tweedie)	exponential	20	0.05	4.07e-02	4.71e-01		4.25e-02	0.00e+00	4.09e-02	3.56e-01	5.04e-01	
DTS (Poisson Tweedie)	gamma	20	0.05	6.67e-02	2.31e-01		6.75e-02	5.88e-02	6.69e-02	8.88e-01	4.83e-01	
DTS (Poisson Tweedie)	bootstrap	60	0.05	8.89e-02	1.25e-01		8.64e-02	1.25e-01	8.99e-02	7.28e-01	2.91e-01	
DTS (Poisson Tweedie)	Tweedie	60	0.05	1.33e-01	2.40e-03	(**)	1.30e-01	1.67e-01	1.35e-01	7.35e-01	9.41e-03	(**)
DTS (Poisson Tweedie)	exponential	60	0.05	3.33e-02	4.41e-01		3.49e-02	0.00e+00	3.37e-02	6.47e-01	6.69e-01	
DTS (Poisson Tweedie)	gamma	60	0.05	3.33e-02	4.41e-01		3.49e-02	0.00e+00	3.37e-02	6.47e-01	6.69e-01	

Table 5.4: Results from exception counting and conditional coverage tests for $\alpha = 0.10$

Jump Distribution	Time Distribution	Horizon (in minutes)	α	Exception rates	p-value for LR_{uc}	$\hat{\pi}_0$	$\hat{\pi}_1$	$\hat{\pi}$	p-value for LR_{ind}	p-value for for LR_{cc}
Discrete Stable	gamma	10	0.1	3.70e-03	0.00e+00 (***)	3.72e-03	0.00e+00	3.71e-03	9.03e-01	0.00e+00 (***)
Discrete Stable	bootstrap	20	0.1	3.70e-03	4.06e-12 (***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.58e-11 (***)
Discrete Stable	Tweedie	20	0.1	7.41e-03	8.77e-11 (***)	7.49e-03	0.00e+00	7.43e-03	8.63e-01	7.19e-10 (***)
Discrete Stable	exponential	20	0.1	3.70e-03	4.06e-12 (***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.58e-11 (***)
Discrete Stable	gamma	20	0.1	3.70e-03	4.06e-12 (***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.58e-11 (***)
Discrete Stable	bootstrap	60	0.1	0.00e+00	1.33e-05 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05 (***)
Discrete Stable	Tweedie	60	0.1	0.00e+00	1.33e-05 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05 (***)
Discrete Stable	exponential	60	0.1	0.00e+00	1.33e-05 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05 (***)
Discrete Stable	gamma	60	0.1	0.00e+00	1.33e-05 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05 (***)
Negative Binomial	bootstrap	10	0.1	1.24e-01	7.12e-02	1.23e-01	1.34e-01	1.24e-01	7.92e-01	1.90e-01
Negative Binomial	Tweedie	10	0.1	1.37e-01	6.29e-03 (**)	1.33e-01	1.62e-01	1.37e-01	5.12e-01	1.93e-02 (*)
Negative Binomial	exponential	10	0.1	8.52e-02	2.40e-01	8.32e-02	1.09e-01	8.53e-02	5.67e-01	4.26e-01
Negative Binomial	gamma	10	0.1	**1.07e-01**	5.70e-01	1.06e-01	1.21e-01	1.08e-01	7.37e-01	8.05e-01
Negative Binomial	bootstrap	20	0.1	1.56e-01	4.53e-03 (**)	1.49e-01	1.95e-01	1.56e-01	4.66e-01	1.36e-02 (*)
Negative Binomial	Tweedie	20	0.1	1.70e-01	3.97e-04 (***)	1.56e-01	2.44e-01	1.71e-01	1.68e-01	7.29e-04 (***)
Negative Binomial	exponential	20	0.1	7.41e-02	1.38e-01	7.20e-02	1.05e-01	7.43e-02	6.13e-01	2.94e-01
Negative Binomial	gamma	20	0.1	1.26e-01	1.70e-01	1.14e-01	2.12e-01	1.26e-01	1.39e-01	1.30e-01
Negative Binomial	bootstrap	60	0.1	2.00e-01	4.70e-03 (**)	1.69e-01	3.33e-01	2.02e-01	1.38e-01	6.13e-03 (**)
Negative Binomial	Tweedie	60	0.1	2.22e-01	6.93e-04 (***)	1.88e-01	3.50e-01	2.25e-01	1.41e-01	1.07e-03 (**)
Negative Binomial	exponential	60	0.1	8.89e-02	7.21e-01	6.17e-02	3.75e-01	8.99e-02	1.71e-02	5.47e-02
Negative Binomial	gamma	60	0.1	**1.11e-01**	7.29e-01	8.86e-02	3.00e-01	1.12e-01	8.10e-02	2.05e-01
DTS (Pareto Tempering)	bootstrap	10	0.1	5.56e-03	0.00e+00 (***)	5.60e-03	0.00e+00	5.57e-03	8.55e-01	0.00e+00 (***)
DTS (Pareto Tempering)	Tweedie	10	0.1	5.56e-03	0.00e+00 (***)	5.60e-03	0.00e+00	5.57e-03	8.55e-01	0.00e+00 (***)
DTS (Pareto Tempering)	exponential	10	0.1	1.85e-03	0.00e+00 (***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	0.00e+00 (***)
DTS (Pareto Tempering)	gamma	10	0.1	5.56e-03	0.00e+00 (***)	5.60e-03	0.00e+00	5.57e-03	8.55e-01	0.00e+00 (***)
DTS (Pareto Tempering)	bootstrap	20	0.1	1.11e-02	1.11e-09 (***)	1.13e-02	0.00e+00	1.12e-02	7.95e-01	8.44e-09 (***)
DTS (Pareto Tempering)	Tweedie	20	0.1	1.11e-02	1.11e-09 (***)	1.13e-02	0.00e+00	1.12e-02	7.95e-01	8.44e-09 (***)
DTS (Pareto Tempering)	exponential	20	0.1	3.70e-03	4.06e-12 (***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.58e-11 (***)
DTS (Pareto Tempering)	gamma	20	0.1	7.41e-03	8.77e-11 (***)	7.49e-03	0.00e+00	7.43e-03	8.63e-01	7.19e-10 (***)
DTS (Pareto Tempering)	bootstrap	60	0.1	0.00e+00	1.33e-05 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05 (***)
DTS (Pareto Tempering)	Tweedie	60	0.1	0.00e+00	1.33e-05 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05 (***)
DTS (Pareto Tempering)	exponential	60	0.1	0.00e+00	1.33e-05 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05 (***)
DTS (Pareto Tempering)	gamma	60	0.1	0.00e+00	1.33e-05 (***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05 (***)
Poisson	bootstrap	10	0.1	1.30e-01	2.74e-02 (*)	1.24e-01	1.71e-01	1.30e-01	2.84e-01	4.95e-02 (*)
Poisson	Tweedie	10	0.1	1.39e-01	4.20e-03 (**)	1.36e-01	1.60e-01	1.39e-01	5.80e-01	1.42e-02 (*)
Poisson	exponential	10	0.1	8.89e-02	3.81e-01	8.76e-02	1.04e-01	8.91e-02	7.06e-01	6.35e-01
Poisson	gamma	10	0.1	1.09e-01	4.79e-01	1.08e-01	1.19e-01	1.09e-01	8.13e-01	7.57e-01
Poisson	bootstrap	20	0.1	1.56e-01	4.53e-03 (**)	1.45e-01	2.20e-01	1.56e-01	2.43e-01	8.98e-03 (**)
Poisson	Tweedie	20	0.1	1.81e-01	4.97e-05 (***)	1.58e-01	2.71e-01	1.78e-01	7.72e-02	5.59e-05 (***)
Poisson	exponential	20	0.1	7.41e-02	1.38e-01	7.20e-02	1.05e-01	7.43e-02	6.13e-01	2.94e-01
Poisson	gamma	20	0.1	1.30e-01	1.19e-01	1.19e-01	2.06e-01	1.30e-01	1.85e-01	1.23e-01
Poisson	bootstrap	60	0.1	2.00e-01	4.70e-03 (**)	1.69e-01	3.33e-01	2.02e-01	1.38e-01	6.13e-03 (**)
Poisson	Tweedie	60	0.1	2.22e-01	6.93e-04 (***)	1.88e-01	3.50e-01	2.25e-01	1.41e-01	1.07e-03 (**)
Poisson	exponential	60	0.1	8.89e-02	7.21e-01	6.17e-02	3.75e-01	8.99e-02	1.71e-02	5.47e-02
Poisson	gamma	60	0.1	1.22e-01	4.96e-01	1.03e-01	2.73e-01	1.24e-01	1.47e-01	2.77e-01
DTS (Poisson Tweedie)	bootstrap	10	0.1	1.28e-01	3.82e-02 (*)	1.21e-01	1.74e-01	1.28e-01	2.39e-01	5.85e-02
DTS (Poisson Tweedie)	Tweedie	10	0.1	1.35e-01	9.28e-03 (**)	1.31e-01	1.64e-01	1.35e-01	4.48e-01	2.54e-02 (*)
DTS (Poisson Tweedie)	exponential	10	0.1	8.52e-02	2.40e-01	8.32e-02	1.09e-01	8.53e-02	5.67e-01	4.26e-01
DTS (Poisson Tweedie)	gamma	10	0.1	1.11e-01	3.97e-01	1.11e-01	1.17e-01	1.11e-01	8.90e-01	6.92e-01
DTS (Poisson Tweedie)	bootstrap	20	0.1	1.52e-01	7.81e-03 (**)	1.44e-01	2.00e-01	1.52e-01	3.79e-01	1.97e-02 (*)
DTS (Poisson Tweedie)	Tweedie	20	0.1	1.78e-01	1.02e-04 (***)	1.58e-01	2.55e-01	1.75e-01	1.24e-01	1.60e-04 (***)
DTS (Poisson Tweedie)	exponential	20	0.1	**8.15e-02**	2.96e-01	8.06e-02	9.52e-02	8.18e-02	8.19e-01	5.64e-01
DTS (Poisson Tweedie)	gamma	20	0.1	1.30e-01	1.19e-01	1.19e-01	2.06e-01	1.30e-01	1.85e-01	1.23e-01
DTS (Poisson Tweedie)	bootstrap	60	0.1	2.00e-01	4.70e-03 (**)	1.69e-01	3.33e-01	2.02e-01	1.38e-01	6.13e-03 (**)
DTS (Poisson Tweedie)	Tweedie	60	0.1	2.22e-01	6.93e-04 (***)	1.88e-01	3.50e-01	2.25e-01	1.41e-01	1.07e-03 (**)
DTS (Poisson Tweedie)	exponential	60	0.1	8.89e-02	7.21e-01	6.17e-02	3.75e-01	8.99e-02	1.71e-02	5.47e-02
DTS (Poisson Tweedie)	gamma	60	0.1	1.22e-01	4.96e-01	1.03e-01	2.73e-01	1.24e-01	1.47e-01	2.77e-01

Table 5.5: Testing stability of the Monte Carlo simulations for different distributions for jumps and interarrival times. For each value of α, 5 days were selected (these days are: July 15th, 18th, 19th, 23rd and 25th) and for each day, the simulation were repeated 50 times. The mean values of exception rates along with the standard deviations are reported in each case.

Jump distribution	Time distribution	day	α 0.01	Standard Deviation	mean	α 0.025	Standard Deviation	mean	α 0.05	Standard Deviation	mean	α 0.10	Standard Deviation	mean
DTS (Poisson Tweedie)	Tweedie	1	0.01	0.0092	0.0481	0.025	0.0093	0.0844	0.05	0.0103	0.1233	0.1	0.0026	0.2041
DTS (Poisson Tweedie)	Tweedie	2	0.01	0	0.0556	0.025	0	0.0556	0.05	0.013	0.0667	0.1	0	0.1852
DTS (Poisson Tweedie)	Tweedie	3	0.01	0	0	0.025	0	0.0185	0.05	0.0037	0.0733	0.1	0.0108	0.1326
DTS (Poisson Tweedie)	Tweedie	4	0.01	0	0	0.025	0.0026	0.0181	0.05	0	0.0185	0.1	0.013	0.0319
DTS (Poisson Tweedie)	Tweedie	5	0.01	0	0.037	0.025	0.0077	0.0596	0.05	0.0081	0.1089	0.1	0.0069	0.1881
Negative Binomial	Tweedie	1	0.01	0.0082	0.0419	0.025	0.0083	0.0748	0.05	0.0082	0.1159	0.1	0	0.2037
Negative Binomial	Tweedie	2	0.01	0.0026	0.0552	0.025	0	0.0556	0.05	0.0084	0.0607	0.1	0.0065	0.1826
Negative Binomial	Tweedie	3	0.01	0	0	0.025	0	0.0185	0.05	0.0056	0.0722	0.1	0.0122	0.1263
Negative Binomial	Tweedie	4	0.01	0	0	0.025	0.0069	0.0156	0.05	0	0.0185	0.1	0.0051	0.02
Negative Binomial	Tweedie	5	0.01	0.0086	0.0352	0.025	0	0.0556	0.05	0.0065	0.0952	0.1	0	0.1852
DTS (Poisson Tweedie)	Expponential	1	0.01	0	0.037	0.025	0.0037	0.0733	0.05	0.009	0.0941	0.1	0	0.1481
DTS (Poisson Tweedie)	Expponential	2	0.01	0.0026	4e-4	0.025	0	0.0556	0.05	0	0.0556	0.1	0	0.0556
DTS (Poisson Tweedie)	Expponential	3	0.01	0	0	0.025	0.0072	0.0033	0.05	0.0037	0.0193	0.1	0.0026	0.1107
DTS (Poisson Tweedie)	Expponential	4	0.01	0	0	0.025	0	0	0.05	0.0093	0.0085	0.1	0	0.0185
DTS (Poisson Tweedie)	Expponential	5	0.01	0.0044	0.0174	0.025	0.0086	0.0315	0.05	0	0.037	0.1	0	0.0926
Negative Binomial	Expponential	1	0.01	0	0.037	0.025	0.0075	0.0704	0.05	0.0072	0.0893	0.1	0	0.1481
Negative Binomial	Expponential	2	0.01	0	0	0.025	0	0.0556	0.05	0	0.0556	0.1	0	0.0556
Negative Binomial	Expponential	3	0.01	0	0	0.025	0.0056	0.0019	0.05	0	0.0185	0.1	0.009	0.1063
Negative Binomial	Expponential	4	0.01	0	0	0.025	0	0	0.05	0.0065	0.0026	0.1	0	0.0185
Negative Binomial	Expponential	5	0.01	0.0084	0.0133	0.025	0	0.0185	0.05	0	0.037	0.1	0.0072	0.10780
DTS (Poisson Tweedie)	Gamma	1	0.01	0.0026	0.0374	0.025	0	0.0741	0.05	0.0061	0.0948	0.1	0.0026	0.1848
DTS (Poisson Tweedie)	Gamma	2	0.01	0.0092	0.0504	0.025	0	0.0556	0.05	0	0.0556	0.1	0.0103	0.1341
DTS (Poisson Tweedie)	Gamma	3	0.01	0	0	0.025	0.0056	0.0167	0.05	0	0.0556	0.1	0	0.1111
DTS (Poisson Tweedie)	Gamma	4	0.01	0	0	0.025	0	0	0.05	0	0.0185	0.1	0	0.0185
DTS (Poisson Tweedie)	Gamma	5	0.01	0.0061	0.0207	0.025	0	0.037	0.05	0.0111	0.0859	0.1	0.0106	0.1322
Negative Binomial	Gamma	1	0.01	0	0.037	0.025	0.0044	0.073	0.05	0.0044	0.0937	0.1	0.0037	0.1844
Negative Binomial	Gamma	2	0.01	0.011	0.0404	0.025	0	0.0556	0.05	0	0.0556	0.1	0.012	0.1267
Negative Binomial	Gamma	3	0.01	0	0	0.025	0.0086	0.013	0.05	0.0069	0.0526	0.1	0	0.1111
Negative Binomial	Gamma	4	0.01	0	0	0.025	0	0	0.05	0	0.0185	0.1	0	0.0185
Negative Binomial	Gamma	5	0.01	0	0.0185	0.025	0	0.037	0.05	0.0086	0.087	0.1	0.0093	0.1215

5.4 Simplifying Assumptions

In this section, we consider a model which is simpler than the one proposed in the previous section. To evaluate this model, we again rely on exception counting and VaR estimation.

To simplify the proposed model, one can assume that the interarrival times of price changes are i.i.d. This assumption is not realistic, since we have already observed a strong dependence in these interarrival times. However, by this assumption, the model will be simpler and can be used for other purposes, e.g. option pricing, in an easier way.

For this simplified model, we assume that the price changes or jumps are following one of the discrete distributions that we also consider previously. These distributions are Poisson, negative binomial, discrete stable, DTS (Pareto tempering) and DTS (Poisson-Tweedie). Simulations from jumps are similar to the previous model. The

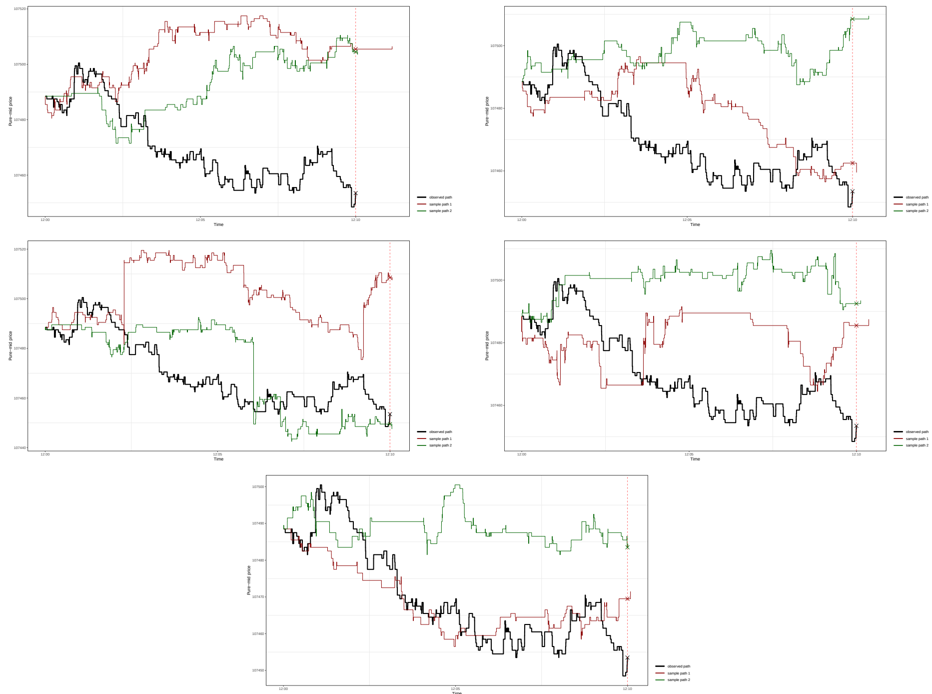

Figure 5.16: Solid black line is the observed path and two other colored and thinner ones are simulated paths using bootstrap sampling for interarrival times and different distributions for jumps as follow; top left: negative binomial, top right: DTS (Poisson-Tweedie) middle left: discrete stable, middle right: DTS (Pareto tempering), bottom: Poisson.

main difference here is in the modeling of interarrival times. In this new model, the interarrival times for the jumps are assumed to be i.i.d. and a continuous distribution e.g. gamma, exponential or Tweedie can be used for modeling.

To simulate a sample path for a time horizon H, n observations from the continuous distribution are simulated in a way that they add up to a number that is greater or equal to H and the sum of $n-1$ observations is less than H. Next, the price changes or jumps will be simulated in a similar way as in the previous sections.

Figure 5.16 shows some examples of simulated sample paths for a 10 minute time horizon from different distributions for jumps and bootstrap sampling for interarrival times. One can see how tempering the tail of a heavy tailed distribution, e.g. discrete stable, can result in a more similar sample path to the observed one.

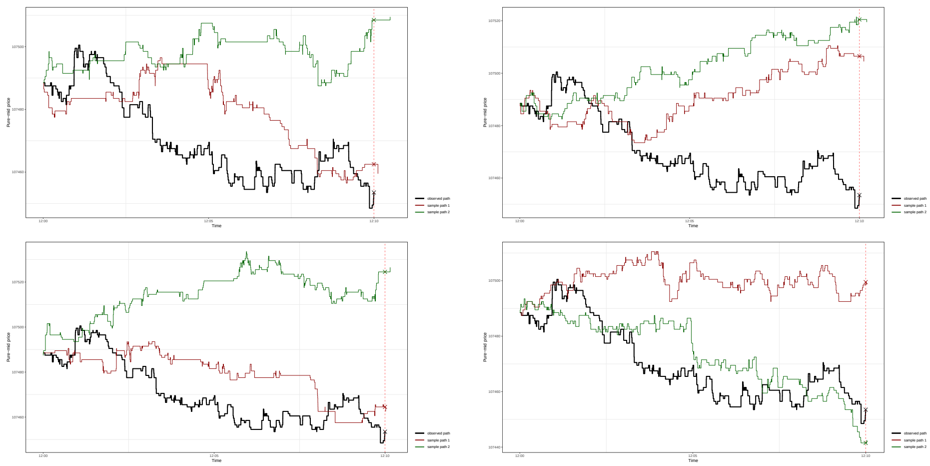

Figure 5.17: Solid black line is the observed path and two other colored and thinner ones are simulated paths using DTS (Poisson-Tweedie) distribution for jumps and different distributions for interarrival times as follow; top left: bootstrap sampling, top right: gamma, bottom left: Tweedie, Bottom right: exponential.

Figure 5.17 compares different distributions for simulating interarrival times. In these cases the Poisson-Tweedie distribution is used to simulate jumps. Figure 5.18 compares the box plots of maximum values of log-likelihood functions in fitting different distributions to interarrival times.

To evaluate this method, we consider the rate and independence of exceptions. Figure 5.19 shows VaR value estimations for different levels and different time horizons, for 10 days in third and fourth weeks of July 2024 using different distributions for modeling. The exceptions are marked when they happened.

Tables 5.6 to 5.9 compare the results from exception counting and conditional coverage tests. In general, the Poisson-Tweedie distribution is doing a better job then other distributions if one considers exception counting. The exception rates are generally higher than when the GARCH model was used in the proposed model in the previous sections. Thus, one can observe that removing the GARCH model and instead using an i.i.d. assumption for interarrival times will simplifies the modeling, but it increases the cost of performance in exception counting.

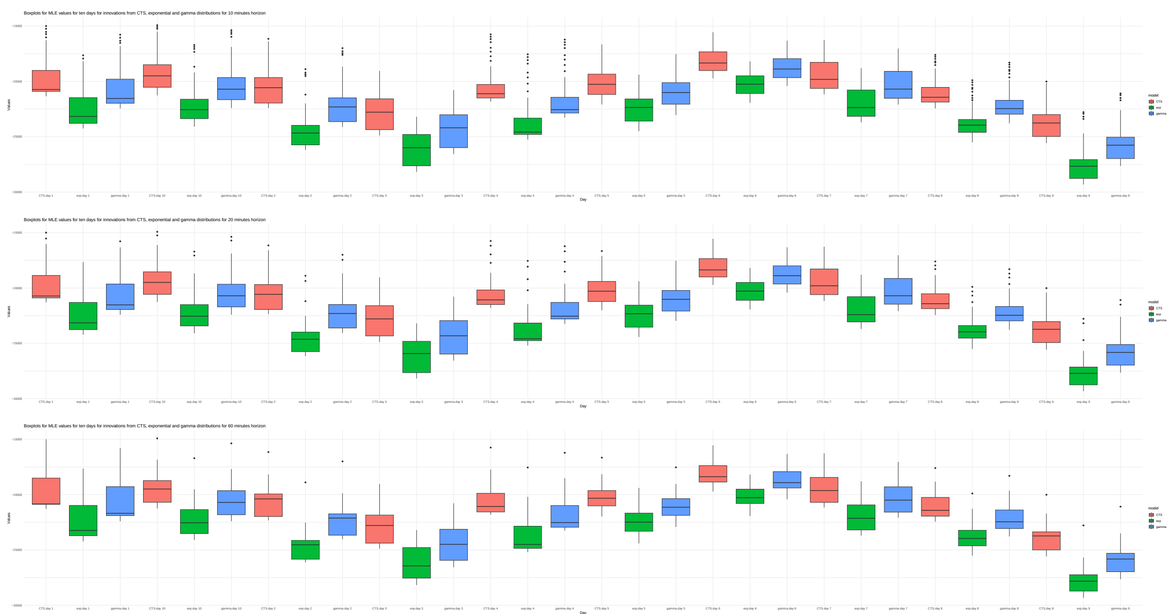

Figure 5.18: Box plots of maximum values of log-likelihood functions in fitting Tweedie, exponential and gamma distributions on interarrival times for 10 days. Three figures are for different time horizons, 10, 20 and 60 minutes from top to bottom, respectively.

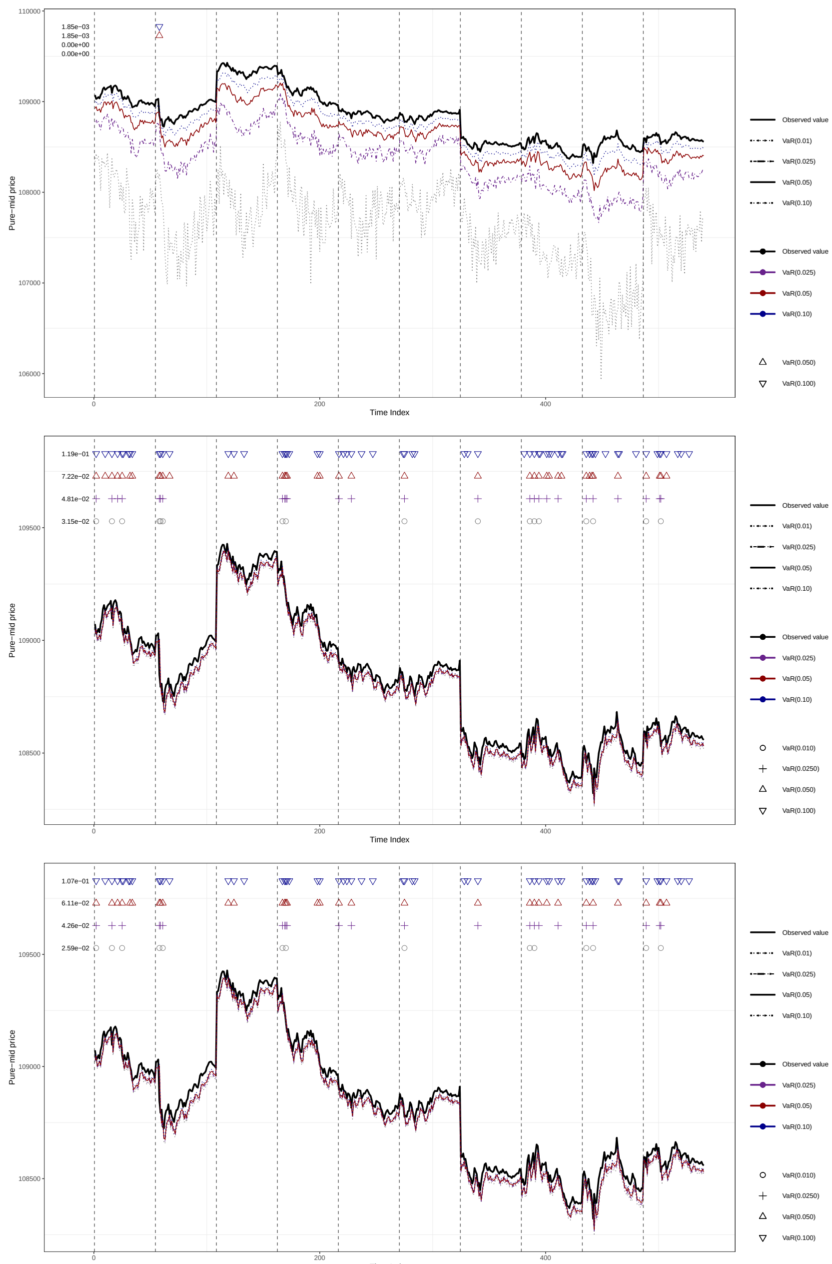

Figure 5.19: VaR$(10, \alpha)$ values during 10 days in 3rd and 4th weeks of July 2024, from up to bottom, for discrete stable, Poisson and Poisson-Tweedie used for jumps, bootstrap sampling for interarrival times, and $\alpha \in \{0.01, 0.025, 0.05, 0.1\}$. The occurrence of exceptions are marked for any confidence level.

Table 5.6: Results from exception counting and conditional coverage tests for $\alpha = 0.01$

Jump Distribution	Time Distribution	Horizon (in minutes)	α	Exception rates	p-value for LR_{uc}		$\hat{\pi}_0$	$\hat{\pi}_1$	$\hat{\pi}$	p-value for LR_{ind}	p-value for for LR_{cc}	
Discrete Stable	bootstrap	10	0.01	**0.00e+00**	9.86e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03	(**)
Discrete Stable	Tweedie	10	0.01	**0.00e+00**	9.86e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03	(**)
Discrete Stable	exponential	10	0.01	**0.00e+00**	9.86e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03	(**)
Discrete Stable	gamma	10	0.01	**0.00e+00**	9.86e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03	(**)
Discrete Stable	bootstrap	20	0.01	**0.00e+00**	1.98e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02	
Discrete Stable	Tweedie	20	0.01	**0.00e+00**	1.98e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02	
Discrete Stable	exponential	20	0.01	**0.00e+00**	1.98e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02	
Discrete Stable	gamma	20	0.01	**0.00e+00**	1.98e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02	
Discrete Stable	bootstrap	60	0.01	**0.00e+00**	1.79e-01		0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01	
Discrete Stable	Tweedie	60	0.01	**0.00e+00**	1.79e-01		0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01	
Discrete Stable	exponential	60	0.01	**0.00e+00**	1.79e-01		0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01	
Discrete Stable	gamma	60	0.01	**0.00e+00**	1.79e-01		0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01	
Negative Binomial	bootstrap	10	0.01	3.15e-02	6.18e-05	(***)	3.07e-02	5.88e-02	3.15e-02	5.57e-01	2.76e-04	(***)
Negative Binomial	Tweedie	10	0.01	2.96e-02	2.07e-04	(***)	2.87e-02	6.25e-02	2.97e-02	4.90e-01	8.07e-04	(***)
Negative Binomial	exponential	10	0.01	3.15e-02	6.18e-05	(***)	3.07e-02	5.88e-02	3.15e-02	5.57e-01	2.76e-04	(***)
Negative Binomial	gamma	10	0.01	3.15e-02	6.18e-05	(***)	3.07e-02	5.88e-02	3.15e-02	5.57e-01	2.76e-04	(***)
Negative Binomial	bootstrap	20	0.01	3.33e-02	2.39e-03	(**)	3.08e-02	1.11e-01	3.35e-02	2.90e-01	5.68e-03	(**)
Negative Binomial	Tweedie	20	0.01	3.33e-02	2.39e-03	(**)	3.08e-02	1.11e-01	3.35e-02	2.90e-01	5.68e-03	(**)
Negative Binomial	exponential	20	0.01	3.33e-02	2.39e-03	(**)	3.08e-02	1.11e-01	3.35e-02	2.90e-01	5.68e-03	(**)
Negative Binomial	gamma	20	0.01	3.33e-02	2.39e-03	(**)	3.08e-02	1.11e-01	3.35e-02	2.90e-01	5.68e-03	(**)
Negative Binomial	bootstrap	60	0.01	2.22e-02	3.15e-01		2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01	
Negative Binomial	Tweedie	60	0.01	2.22e-02	3.15e-01		2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01	
Negative Binomial	exponential	60	0.01	2.22e-02	3.15e-01		2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01	
Negative Binomial	gamma	60	0.01	3.33e-02	7.96e-02		3.49e-02	0.00e+00	3.37e-02	6.47e-01	1.94e-01	
DTS (Pareto Tempering)	bootstrap	10	0.01	**0.00e+00**	9.86e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03	(**)
DTS (Pareto Tempering)	Tweedie	10	0.01	**0.00e+00**	9.86e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03	(**)
DTS (Pareto Tempering)	exponential	10	0.01	**0.00e+00**	9.86e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03	(**)
DTS (Pareto Tempering)	gamma	10	0.01	**0.00e+00**	9.86e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.40e-03	(**)
DTS (Pareto Tempering)	bootstrap	20	0.01	**0.00e+00**	1.98e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02	
DTS (Pareto Tempering)	Tweedie	20	0.01	**0.00e+00**	1.98e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02	
DTS (Pareto Tempering)	exponential	20	0.01	**0.00e+00**	1.98e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02	
DTS (Pareto Tempering)	gamma	20	0.01	**0.00e+00**	1.98e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	6.63e-02	
DTS (Pareto Tempering)	bootstrap	60	0.01	**0.00e+00**	1.79e-01		0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01	
DTS (Pareto Tempering)	Tweedie	60	0.01	**0.00e+00**	1.79e-01		0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01	
DTS (Pareto Tempering)	exponential	60	0.01	**0.00e+00**	1.79e-01		0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01	
DTS (Pareto Tempering)	gamma	60	0.01	**0.00e+00**	1.79e-01		0.00e+00	0.00e+00	0.00e+00	1.00e+00	4.05e-01	
Poisson	bootstrap	10	0.01	3.15e-02	6.18e-05	(***)	3.07e-02	5.88e-02	3.15e-02	5.57e-01	2.76e-04	(***)
Poisson	Tweedie	10	0.01	3.15e-02	6.18e-05	(***)	3.07e-02	5.88e-02	3.15e-02	5.57e-01	2.76e-04	(***)
Poisson	exponential	10	0.01	3.15e-02	6.18e-05	(***)	3.07e-02	5.88e-02	3.15e-02	5.57e-01	2.76e-04	(***)
Poisson	gamma	10	0.01	3.15e-02	6.18e-05	(***)	3.07e-02	5.88e-02	3.15e-02	5.57e-01	2.76e-04	(***)
Poisson	bootstrap	20	0.01	3.33e-02	2.39e-03	(**)	3.08e-02	1.11e-01	3.35e-02	2.90e-01	5.68e-03	(**)
Poisson	Tweedie	20	0.01	3.33e-02	2.39e-03	(**)	3.08e-02	1.11e-01	3.35e-02	2.90e-01	5.68e-03	(**)
Poisson	exponential	20	0.01	3.33e-02	2.39e-03	(**)	3.08e-02	1.11e-01	3.35e-02	2.90e-01	5.68e-03	(**)
Poisson	gamma	20	0.01	3.33e-02	2.39e-03	(**)	3.08e-02	1.11e-01	3.35e-02	2.90e-01	5.68e-03	(**)
Poisson	bootstrap	60	0.01	3.33e-02	7.96e-02		3.49e-02	0.00e+00	3.37e-02	6.47e-01	1.94e-01	
Poisson	Tweedie	60	0.01	2.22e-02	3.15e-01		2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01	
Poisson	exponential	60	0.01	3.33e-02	7.96e-02		3.49e-02	0.00e+00	3.37e-02	6.47e-01	1.94e-01	
Poisson	gamma	60	0.01	2.22e-02	3.15e-01		2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01	
DTS (Poisson Tweedie)	bootstrap	10	0.01	2.59e-02	1.93e-03	(**)	2.67e-02	0.00e+00	2.60e-02	3.88e-01	5.63e-03	(**)
DTS (Poisson Tweedie)	Tweedie	10	0.01	2.41e-02	5.37e-03	(**)	2.47e-02	0.00e+00	2.41e-02	4.23e-01	1.50e-02	(*)
DTS (Poisson Tweedie)	exponential	10	0.01	2.59e-02	1.93e-03	(**)	2.67e-02	0.00e+00	2.60e-02	3.88e-01	5.63e-03	(**)
DTS (Poisson Tweedie)	gamma	10	0.01	2.41e-02	5.37e-03	(**)	2.47e-02	0.00e+00	2.41e-02	4.23e-01	1.50e-02	(*)
DTS (Poisson Tweedie)	bootstrap	20	0.01	2.96e-02	8.69e-03	(**)	2.68e-02	1.25e-01	2.97e-02	2.20e-01	1.51e-02	(*)
DTS (Poisson Tweedie)	Tweedie	20	0.01	2.96e-02	8.69e-03	(**)	2.68e-02	1.25e-01	2.97e-02	2.20e-01	1.51e-02	(*)
DTS (Poisson Tweedie)	exponential	20	0.01	2.96e-02	8.69e-03	(**)	2.68e-02	1.25e-01	2.97e-02	2.20e-01	1.51e-02	(*)
DTS (Poisson Tweedie)	gamma	20	0.01	2.96e-02	8.69e-03	(**)	2.68e-02	1.25e-01	2.97e-02	2.20e-01	1.51e-02	(*)
DTS (Poisson Tweedie)	bootstrap	60	0.01	2.22e-02	3.15e-01		2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01	
DTS (Poisson Tweedie)	Tweedie	60	0.01	2.22e-02	3.15e-01		2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01	
DTS (Poisson Tweedie)	exponential	60	0.01	2.22e-02	3.15e-01		2.30e-02	0.00e+00	2.25e-02	7.62e-01	5.77e-01	
DTS (Poisson Tweedie)	gamma	60	0.01	3.33e-02	7.96e-02		3.49e-02	0.00e+00	3.37e-02	6.47e-01	1.94e-01	

Table 5.7: Results from exception counting and conditional coverage tests for $\alpha = 0.025$

Jump Distribution	Time Distribution	Horizon (in minutes)	α	Exception rates	p-value for LR_{uc}		$\hat{\pi}_0$	$\hat{\pi}_1$	$\hat{\pi}$	p-value for LR_{ind}	p-value for for LR_{cc}	
Discrete Stable	bootstrap	10	0.025	0.00e+00	1.70e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.15e-06	(***)
Discrete Stable	Tweedie	10	0.025	0.00e+00	1.70e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.15e-06	(***)
Discrete Stable	exponential	10	0.025	0.00e+00	1.70e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.15e-06	(***)
Discrete Stable	gamma	10	0.025	0.00e+00	1.70e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.15e-06	(***)
Discrete Stable	bootstrap	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
Discrete Stable	Tweedie	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
Discrete Stable	exponential	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
Discrete Stable	gamma	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
Discrete Stable	bootstrap	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
Discrete Stable	Tweedie	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
Discrete Stable	exponential	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
Discrete Stable	gamma	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
Negative Binomial	bootstrap	10	0.025	4.63e-02	4.52e-03	(**)	4.09e-02	1.60e-01	4.64e-02	2.61e-02	1.50e-03	(**)
Negative Binomial	Tweedie	10	0.025	4.63e-02	4.52e-03	(**)	4.09e-02	1.60e-01	4.64e-02	2.61e-02	1.50e-03	(**)
Negative Binomial	exponential	10	0.025	4.63e-02	4.52e-03	(**)	4.09e-02	1.60e-01	4.64e-02	2.61e-02	1.50e-03	(**)
Negative Binomial	gamma	10	0.025	4.63e-02	4.52e-03	(**)	4.09e-02	1.60e-01	4.64e-02	2.61e-02	1.50e-03	(**)
Negative Binomial	bootstrap	20	0.025	4.81e-02	3.03e-02	(*)	4.69e-02	7.69e-02	4.83e-02	6.47e-01	8.63e-02	
Negative Binomial	Tweedie	20	0.025	4.81e-02	3.03e-02	(*)	4.69e-02	7.69e-02	4.83e-02	6.47e-01	8.63e-02	
Negative Binomial	exponential	20	0.025	4.81e-02	3.03e-02	(*)	4.69e-02	7.69e-02	4.83e-02	6.47e-01	8.63e-02	
Negative Binomial	gamma	20	0.025	4.81e-02	3.03e-02	(*)	4.69e-02	7.69e-02	4.83e-02	6.47e-01	8.63e-02	
Negative Binomial	bootstrap	60	0.025	**4.44e-02**	2.86e-01		4.71e-02	0.00e+00	4.49e-02	5.39e-01	4.69e-01	
Negative Binomial	Tweedie	60	0.025	**4.44e-02**	2.86e-01		4.71e-02	0.00e+00	4.49e-02	5.39e-01	4.69e-01	
Negative Binomial	exponential	60	0.025	**4.44e-02**	2.86e-01		4.71e-02	0.00e+00	4.49e-02	5.39e-01	4.69e-01	
Negative Binomial	gamma	60	0.025	**4.44e-02**	2.86e-01		4.71e-02	0.00e+00	4.49e-02	5.39e-01	4.69e-01	
DTS (Pareto Tempering)	bootstrap	10	0.025	1.85e-03	7.39e-06	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	4.33e-05	(***)
DTS (Pareto Tempering)	Tweedie	10	0.025	1.85e-03	7.39e-06	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	4.33e-05	(***)
DTS (Pareto Tempering)	exponential	10	0.025	1.85e-03	7.39e-06	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	4.33e-05	(***)
DTS (Pareto Tempering)	gamma	10	0.025	1.85e-03	7.39e-06	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	4.33e-05	(***)
DTS (Pareto Tempering)	bootstrap	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
DTS (Pareto Tempering)	Tweedie	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
DTS (Pareto Tempering)	exponential	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
DTS (Pareto Tempering)	gamma	20	0.025	0.00e+00	2.18e-04	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.07e-03	(**)
DTS (Pareto Tempering)	bootstrap	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
DTS (Pareto Tempering)	Tweedie	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
DTS (Pareto Tempering)	exponential	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
DTS (Pareto Tempering)	gamma	60	0.025	0.00e+00	3.28e-02	(*)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	1.02e-01	
Poisson	bootstrap	10	0.025	4.81e-02	2.19e-03	(**)	4.29e-02	1.54e-01	4.82e-02	3.52e-02	1.00e-03	(**)
Poisson	Tweedie	10	0.025	4.63e-02	4.52e-03	(**)	4.09e-02	1.60e-01	4.64e-02	2.61e-02	1.50e-03	(**)
Poisson	exponential	10	0.025	4.63e-02	4.52e-03	(**)	4.09e-02	1.60e-01	4.64e-02	2.61e-02	1.50e-03	(**)
Poisson	gamma	10	0.025	4.81e-02	2.19e-03	(**)	4.29e-02	1.54e-01	4.82e-02	3.52e-02	1.00e-03	(**)
Poisson	bootstrap	20	0.025	4.81e-02	3.03e-02	(*)	4.69e-02	7.69e-02	4.83e-02	6.47e-01	8.63e-02	
Poisson	Tweedie	20	0.025	4.81e-02	3.03e-02	(*)	4.69e-02	7.69e-02	4.83e-02	6.47e-01	8.63e-02	
Poisson	exponential	20	0.025	4.81e-02	3.03e-02	(*)	4.69e-02	7.69e-02	4.83e-02	6.47e-01	8.63e-02	
Poisson	gamma	20	0.025	4.81e-02	3.03e-02	(*)	4.69e-02	7.69e-02	4.83e-02	6.47e-01	8.63e-02	
Poisson	bootstrap	60	0.025	5.56e-02	1.09e-01		5.95e-02	0.00e+00	5.62e-02	4.40e-01	2.05e-01	
Poisson	Tweedie	60	0.025	**4.44e-02**	2.86e-01		4.71e-02	0.00e+00	4.49e-02	5.39e-01	4.69e-01	
Poisson	exponential	60	0.025	5.56e-02	1.09e-01		5.95e-02	0.00e+00	5.62e-02	4.40e-01	2.05e-01	
Poisson	gamma	60	0.025	**4.44e-02**	2.86e-01		4.71e-02	0.00e+00	4.49e-02	5.39e-01	4.69e-01	
DTS (Poisson Tweedie)	bootstrap	10	0.025	**4.07e-02**	3.15e-02	(*)	3.68e-02	1.36e-01	4.08e-02	6.23e-02	1.74e-02	(*)
DTS (Poisson Tweedie)	Tweedie	10	0.025	**4.07e-02**	3.15e-02	(*)	3.68e-02	1.36e-01	4.08e-02	6.23e-02	1.74e-02	(*)
DTS (Poisson Tweedie)	exponential	10	0.025	4.26e-02	1.71e-02	(*)	3.68e-02	1.74e-01	4.27e-02	1.36e-02	2.78e-03	(**)
DTS (Poisson Tweedie)	gamma	10	0.025	**4.07e-02**	3.15e-02	(*)	3.68e-02	1.36e-01	4.08e-02	6.23e-02	1.74e-02	(*)
DTS (Poisson Tweedie)	bootstrap	20	0.025	**4.07e-02**	1.28e-01		3.88e-02	9.09e-02	4.09e-02	4.55e-01	2.38e-01	
DTS (Poisson Tweedie)	Tweedie	20	0.025	4.44e-02	6.46e-02		4.28e-02	8.33e-02	4.46e-02	5.49e-01	1.52e-01	
DTS (Poisson Tweedie)	exponential	20	0.025	4.44e-02	6.46e-02		4.28e-02	8.33e-02	4.46e-02	5.49e-01	1.52e-01	
DTS (Poisson Tweedie)	gamma	20	0.025	4.44e-02	6.46e-02		4.28e-02	8.33e-02	4.46e-02	5.49e-01	1.52e-01	
DTS (Poisson Tweedie)	bootstrap	60	0.025	**4.44e-02**	2.86e-01		4.71e-02	0.00e+00	4.49e-02	5.39e-01	4.69e-01	
DTS (Poisson Tweedie)	Tweedie	60	0.025	**4.44e-02**	2.86e-01		4.71e-02	0.00e+00	4.49e-02	5.39e-01	4.69e-01	
DTS (Poisson Tweedie)	exponential	60	0.025	**4.44e-02**	2.86e-01		4.71e-02	0.00e+00	4.49e-02	5.39e-01	4.69e-01	
DTS (Poisson Tweedie)	gamma	60	0.025	**4.44e-02**	2.86e-01		4.71e-02	0.00e+00	4.49e-02	5.39e-01	4.69e-01	

Table 5.8: Results from exception counting and conditional coverage tests for $\alpha = 0.05$

Jump Distribution	Time Distribution	Horizon (in minutes)	α	Exception rates	p-value for LR_{uc}		$\hat{\pi}_0$	$\hat{\pi}_1$	$\hat{\pi}$	p-value for LR_{ind}	p-value for for LR_{cc}	
Discrete Stable	bootstrap	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
Discrete Stable	Tweedie	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
Discrete Stable	exponential	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
Discrete Stable	gamma	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
Discrete Stable	bootstrap	20	0.05	0.00e+00	1.42e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.67e-07	(***)
Discrete Stable	Tweedie	20	0.05	0.00e+00	1.42e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.67e-07	(***)
Discrete Stable	exponential	20	0.05	0.00e+00	1.42e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.67e-07	(***)
Discrete Stable	gamma	20	0.05	0.00e+00	1.42e-07	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.67e-07	(***)
Discrete Stable	bootstrap	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
Discrete Stable	Tweedie	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
Discrete Stable	exponential	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
Discrete Stable	gamma	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
Negative Binomial	bootstrap	10	0.05	6.85e-02	6.09e-02		6.37e-02	1.35e-01	6.86e-02	1.36e-01	5.68e-02	
Negative Binomial	Tweedie	10	0.05	7.04e-02	4.02e-02	(*)	6.59e-02	1.32e-01	7.05e-02	1.66e-01	4.66e-02	(*)
Negative Binomial	exponential	10	0.05	6.85e-02	6.09e-02		6.37e-02	1.35e-01	6.86e-02	1.36e-01	5.68e-02	
Negative Binomial	gamma	10	0.05	7.22e-02	2.59e-02	(*)	6.80e-02	1.28e-01	7.24e-02	2.00e-01	3.67e-02	(*)
Negative Binomial	bootstrap	20	0.05	9.26e-02	3.89e-03	(**)	9.02e-02	1.20e-01	9.29e-02	6.37e-01	1.39e-02	(*)
Negative Binomial	Tweedie	20	0.05	9.26e-02	3.89e-03	(**)	8.61e-02	1.60e-01	9.29e-02	2.62e-01	8.26e-03	(**)
Negative Binomial	exponential	20	0.05	9.63e-02	1.84e-03	(**)	9.05e-02	1.54e-01	9.67e-02	3.30e-01	4.87e-03	(**)
Negative Binomial	gamma	20	0.05	9.26e-02	3.89e-03	(**)	8.61e-02	1.60e-01	9.29e-02	2.62e-01	8.26e-03	(**)
Negative Binomial	bootstrap	60	0.05	8.89e-02	1.25e-01		9.88e-02	0.00e+00	8.99e-02	2.08e-01	1.40e-01	
Negative Binomial	Tweedie	60	0.05	7.78e-02	2.62e-01		8.54e-02	0.00e+00	7.87e-02	2.74e-01	2.93e-01	
Negative Binomial	exponential	60	0.05	8.89e-02	1.25e-01		9.88e-02	0.00e+00	8.99e-02	2.08e-01	1.40e-01	
Negative Binomial	gamma	60	0.05	8.89e-02	1.25e-01		9.88e-02	0.00e+00	8.99e-02	2.08e-01	1.40e-01	
DTS (Pareto Tempering)	bootstrap	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
DTS (Pareto Tempering)	Tweedie	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
DTS (Pareto Tempering)	exponential	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
DTS (Pareto Tempering)	gamma	10	0.05	1.85e-03	8.25e-12	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	7.20e-11	(***)
DTS (Pareto Tempering)	bootstrap	20	0.05	3.70e-03	6.30e-06	(***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.71e-05	(***)
DTS (Pareto Tempering)	Tweedie	20	0.05	3.70e-03	6.30e-06	(***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.71e-05	(***)
DTS (Pareto Tempering)	exponential	20	0.05	3.70e-03	6.30e-06	(***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.71e-05	(***)
DTS (Pareto Tempering)	gamma	20	0.05	3.70e-03	6.30e-06	(***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.71e-05	(***)
DTS (Pareto Tempering)	bootstrap	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
DTS (Pareto Tempering)	Tweedie	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
DTS (Pareto Tempering)	exponential	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
DTS (Pareto Tempering)	gamma	60	0.05	0.00e+00	2.38e-03	(**)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	9.89e-03	(**)
Poisson	bootstrap	10	0.05	7.22e-02	2.59e-02	(*)	6.80e-02	1.28e-01	7.24e-02	2.00e-01	3.67e-02	(*)
Poisson	Tweedie	10	0.05	7.59e-02	9.97e-03	(**)	7.23e-02	1.22e-01	7.61e-02	2.82e-01	2.03e-02	(*)
Poisson	exponential	10	0.05	7.78e-02	5.98e-03	(**)	7.44e-02	1.19e-01	7.79e-02	3.31e-01	1.42e-02	(*)
Poisson	gamma	10	0.05	7.41e-02	1.62e-02	(*)	7.01e-02	1.25e-01	7.42e-02	2.39e-01	2.78e-02	(*)
Poisson	bootstrap	20	0.05	9.26e-02	3.89e-03	(**)	8.61e-02	1.60e-01	9.29e-02	2.62e-01	8.26e-03	(**)
Poisson	Tweedie	20	0.05	9.26e-02	3.89e-03	(**)	8.61e-02	1.60e-01	9.29e-02	2.62e-01	8.26e-03	(**)
Poisson	exponential	20	0.05	9.63e-02	1.84e-03	(**)	9.05e-02	1.54e-01	9.67e-02	3.30e-01	4.87e-03	(**)
Poisson	gamma	20	0.05	9.26e-02	3.89e-03	(**)	8.61e-02	1.60e-01	9.29e-02	2.62e-01	8.26e-03	(**)
Poisson	bootstrap	60	0.05	8.89e-02	1.25e-01		9.88e-02	0.00e+00	8.99e-02	2.08e-01	1.40e-01	
Poisson	Tweedie	60	0.05	8.89e-02	1.25e-01		9.88e-02	0.00e+00	8.99e-02	2.08e-01	1.40e-01	
Poisson	exponential	60	0.05	8.89e-02	1.25e-01		9.88e-02	0.00e+00	8.99e-02	2.08e-01	1.40e-01	
Poisson	gamma	60	0.05	1.00e-01	5.38e-02		1.13e-01	0.00e+00	1.01e-01	1.54e-01	5.65e-02	
DTS (Poisson Tweedie)	bootstrap	10	0.05	**6.11e-02**	2.52e-01		5.73e-02	1.21e-01	6.12e-02	1.83e-01	2.14e-01	
DTS (Poisson Tweedie)	Tweedie	10	0.05	6.30e-02	1.83e-01		5.94e-02	1.18e-01	6.31e-02	2.20e-01	1.94e-01	
DTS (Poisson Tweedie)	exponential	10	0.05	**6.11e-02**	2.52e-01		5.73e-02	1.21e-01	6.12e-02	1.83e-01	2.14e-01	
DTS (Poisson Tweedie)	gamma	10	0.05	**6.11e-02**	2.52e-01		5.73e-02	1.21e-01	6.12e-02	1.83e-01	2.14e-01	
DTS (Poisson Tweedie)	bootstrap	20	0.05	**8.52e-02**	1.54e-02	(*)	8.13e-02	1.30e-01	8.55e-02	4.48e-01	3.99e-02	(*)
DTS (Poisson Tweedie)	Tweedie	20	0.05	**8.52e-02**	1.54e-02	(*)	8.13e-02	1.30e-01	8.55e-02	4.48e-01	3.99e-02	(*)
DTS (Poisson Tweedie)	exponential	20	0.05	**8.52e-02**	1.54e-02	(*)	8.13e-02	1.30e-01	8.55e-02	4.48e-01	3.99e-02	(*)
DTS (Poisson Tweedie)	gamma	20	0.05	**8.52e-02**	1.54e-02	(*)	8.13e-02	1.30e-01	8.55e-02	4.48e-01	3.99e-02	(*)
DTS (Poisson Tweedie)	bootstrap	60	0.05	7.78e-02	2.62e-01		8.54e-02	0.00e+00	7.87e-02	2.74e-01	2.93e-01	
DTS (Poisson Tweedie)	Tweedie	60	0.05	7.78e-02	2.62e-01		8.54e-02	0.00e+00	7.87e-02	2.74e-01	2.93e-01	
DTS (Poisson Tweedie)	exponential	60	0.05	**6.67e-02**	4.89e-01		7.23e-02	0.00e+00	6.74e-02	3.51e-01	5.10e-01	
DTS (Poisson Tweedie)	gamma	60	0.05	**6.67e-02**	4.89e-01		7.23e-02	0.00e+00	6.74e-02	3.51e-01	5.10e-01	

Table 5.9: Results from exception counting and conditional coverage tests for $\alpha = 0.10$

Jump Distribution	Time Distribution	Horizon (in minutes)	α	Exception rates	p-value for LR_{uc}		$\hat{\pi}_0$	$\hat{\pi}_1$	$\hat{\pi}$	p-value for LR_{ind}	p-value for for LR_{cc}	
Discrete Stable	bootstrap	10	0.1	1.85e-03	0.00e+00	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	0.00e+00	(***)
Discrete Stable	Tweedie	10	0.1	1.85e-03	0.00e+00	(***)	1.86e-03	0.00e+00	1.86e-03	9.51e-01	0.00e+00	(***)
Discrete Stable	exponential	10	0.1	3.70e-03	0.00e+00	(***)	3.72e-03	0.00e+00	3.71e-03	9.03e-01	0.00e+00	(***)
Discrete Stable	gamma	10	0.1	3.70e-03	0.00e+00	(***)	3.72e-03	0.00e+00	3.71e-03	9.03e-01	0.00e+00	(***)
Discrete Stable	bootstrap	20	0.1	3.70e-03	4.06e-12	(***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.58e-11	(***)
Discrete Stable	Tweedie	20	0.1	3.70e-03	4.06e-12	(***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.58e-11	(***)
Discrete Stable	exponential	20	0.1	3.70e-03	4.06e-12	(***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.58e-11	(***)
Discrete Stable	gamma	20	0.1	3.70e-03	4.06e-12	(***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.58e-11	(***)
Discrete Stable	bootstrap	60	0.1	0.00e+00	1.33e-05	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05	(***)
Discrete Stable	Tweedie	60	0.1	0.00e+00	1.33e-05	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05	(***)
Discrete Stable	exponential	60	0.1	0.00e+00	1.33e-05	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05	(***)
Discrete Stable	gamma	60	0.1	0.00e+00	1.33e-05	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05	(***)
Negative Binomial	bootstrap	10	0.1	1.19e-01	1.62e-01		1.12e-01	1.72e-01	1.19e-01	1.82e-01	1.54e-01	
Negative Binomial	Tweedie	10	0.1	1.17e-01	2.07e-01		1.11e-01	1.59e-01	1.17e-01	2.90e-01	2.58e-01	
Negative Binomial	exponential	10	0.1	1.15e-01	2.61e-01		1.09e-01	1.61e-01	1.15e-01	2.45e-01	2.71e-01	
Negative Binomial	gamma	10	0.1	1.19e-01	1.62e-01		1.12e-01	1.72e-01	1.19e-01	1.82e-01	1.54e-01	
Negative Binomial	bootstrap	20	0.1	1.33e-01	8.07e-02		1.20e-01	1.94e-01	1.30e-01	2.41e-01	1.09e-01	
Negative Binomial	Tweedie	20	0.1	1.33e-01	8.07e-02		1.24e-01	1.67e-01	1.30e-01	4.97e-01	1.73e-01	
Negative Binomial	exponential	20	0.1	1.33e-01	8.07e-02		1.20e-01	1.94e-01	1.30e-01	2.41e-01	1.09e-01	
Negative Binomial	gamma	20	0.1	1.33e-01	8.07e-02		1.20e-01	1.94e-01	1.30e-01	2.41e-01	1.09e-01	
Negative Binomial	bootstrap	60	0.1	1.56e-01	1.01e-01		1.47e-01	2.14e-01	1.57e-01	5.38e-01	2.16e-01	
Negative Binomial	Tweedie	60	0.1	1.67e-01	5.18e-02		1.49e-01	2.67e-01	1.69e-01	2.89e-01	8.61e-02	
Negative Binomial	exponential	60	0.1	1.67e-01	5.18e-02		1.49e-01	2.67e-01	1.69e-01	2.89e-01	8.61e-02	
Negative Binomial	gamma	60	0.1	1.67e-01	5.18e-02		1.49e-01	2.67e-01	1.69e-01	2.89e-01	8.61e-02	
DTS (Pareto Tempering)	bootstrap	10	0.1	7.41e-03	0.00e+00	(***)	7.48e-03	0.00e+00	7.42e-03	8.07e-01	0.00e+00	(***)
DTS (Pareto Tempering)	Tweedie	10	0.1	7.41e-03	0.00e+00	(***)	7.48e-03	0.00e+00	7.42e-03	8.07e-01	0.00e+00	(***)
DTS (Pareto Tempering)	exponential	10	0.1	9.26e-03	0.00e+00	(***)	9.36e-03	0.00e+00	9.28e-03	7.60e-01	0.00e+00	(***)
DTS (Pareto Tempering)	gamma	10	0.1	7.41e-03	0.00e+00	(***)	7.48e-03	0.00e+00	7.42e-03	8.07e-01	0.00e+00	(***)
DTS (Pareto Tempering)	bootstrap	20	0.1	3.70e-03	4.06e-12	(***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.58e-11	(***)
DTS (Pareto Tempering)	Tweedie	20	0.1	3.70e-03	4.06e-12	(***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.58e-11	(***)
DTS (Pareto Tempering)	exponential	20	0.1	3.70e-03	4.06e-12	(***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.58e-11	(***)
DTS (Pareto Tempering)	gamma	20	0.1	3.70e-03	4.06e-12	(***)	3.73e-03	0.00e+00	3.72e-03	9.31e-01	3.58e-11	(***)
DTS (Pareto Tempering)	bootstrap	60	0.1	1.11e-02	4.36e-04	(***)	1.14e-02	0.00e+00	1.12e-02	8.80e-01	2.04e-03	(**)
DTS (Pareto Tempering)	Tweedie	60	0.1	0.00e+00	1.33e-05	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05	(***)
DTS (Pareto Tempering)	exponential	60	0.1	1.11e-02	4.36e-04	(***)	1.14e-02	0.00e+00	1.12e-02	8.80e-01	2.04e-03	(**)
DTS (Pareto Tempering)	gamma	60	0.1	0.00e+00	1.33e-05	(***)	0.00e+00	0.00e+00	0.00e+00	1.00e+00	7.62e-05	(***)
Poisson	bootstrap	10	0.1	1.19e-01	1.62e-01		1.12e-01	1.72e-01	1.19e-01	1.82e-01	1.54e-01	
Poisson	Tweedie	10	0.1	1.20e-01	1.25e-01		1.14e-01	1.69e-01	1.21e-01	2.19e-01	1.45e-01	
Poisson	exponential	10	0.1	1.20e-01	1.25e-01		1.14e-01	1.69e-01	1.21e-01	2.19e-01	1.45e-01	
Poisson	gamma	10	0.1	1.15e-01	2.61e-01		1.09e-01	1.61e-01	1.15e-01	2.45e-01	2.71e-01	
Poisson	bootstrap	20	0.1	1.33e-01	8.07e-02		1.20e-01	1.94e-01	1.30e-01	2.41e-01	1.09e-01	
Poisson	Tweedie	20	0.1	1.37e-01	5.33e-02		1.25e-01	1.89e-01	1.34e-01	3.07e-01	9.18e-02	
Poisson	exponential	20	0.1	1.37e-01	5.33e-02		1.16e-01	2.43e-01	1.34e-01	5.05e-02	2.28e-02	(*)
Poisson	gamma	20	0.1	1.37e-01	5.33e-02		1.25e-01	1.89e-01	1.34e-01	3.07e-01	9.18e-02	
Poisson	bootstrap	60	0.1	1.67e-01	5.18e-02		1.49e-01	2.67e-01	1.69e-01	2.89e-01	8.61e-02	
Poisson	Tweedie	60	0.1	1.67e-01	5.18e-02		1.49e-01	2.67e-01	1.69e-01	2.89e-01	8.61e-02	
Poisson	exponential	60	0.1	1.67e-01	5.18e-02		1.49e-01	2.67e-01	1.69e-01	2.89e-01	8.61e-02	
Poisson	gamma	60	0.1	1.67e-01	5.18e-02		1.49e-01	2.67e-01	1.69e-01	2.89e-01	8.61e-02	
DTS (Poisson Tweedie)	bootstrap	10	0.1	**1.07e-01**	5.70e-01		9.98e-02	1.72e-01	1.08e-01	1.14e-01	2.43e-01	
DTS (Poisson Tweedie)	Tweedie	10	0.1	**1.07e-01**	5.70e-01		9.98e-02	1.72e-01	1.08e-01	1.14e-01	2.43e-01	
DTS (Poisson Tweedie)	exponential	10	0.1	**1.07e-01**	5.70e-01		1.02e-01	1.55e-01	1.08e-01	2.39e-01	4.25e-01	
DTS (Poisson Tweedie)	gamma	10	0.1	**1.07e-01**	5.70e-01		9.98e-02	1.72e-01	1.08e-01	1.14e-01	2.43e-01	
DTS (Poisson Tweedie)	bootstrap	20	0.1	1.19e-01	3.23e-01		1.10e-01	1.56e-01	1.15e-01	4.57e-01	4.65e-01	
DTS (Poisson Tweedie)	Tweedie	20	0.1	**1.15e-01**	4.27e-01		1.05e-01	1.61e-01	1.12e-01	3.73e-01	4.90e-01	
DTS (Poisson Tweedie)	exponential	20	0.1	1.19e-01	3.23e-01		1.10e-01	1.56e-01	1.15e-01	4.57e-01	4.65e-01	
DTS (Poisson Tweedie)	gamma	20	0.1	**1.15e-01**	4.27e-01		1.05e-01	1.61e-01	1.12e-01	3.73e-01	4.90e-01	
DTS (Poisson Tweedie)	bootstrap	60	0.1	**1.44e-01**	1.84e-01		1.32e-01	2.31e-01	1.46e-01	3.75e-01	2.80e-01	
DTS (Poisson Tweedie)	Tweedie	60	0.1	**1.44e-01**	1.84e-01		1.32e-01	2.31e-01	1.46e-01	3.75e-01	2.80e-01	
DTS (Poisson Tweedie)	exponential	60	0.1	**1.44e-01**	1.84e-01		1.32e-01	2.31e-01	1.46e-01	3.75e-01	2.80e-01	
DTS (Poisson Tweedie)	gamma	60	0.1	**1.44e-01**	1.84e-01		1.32e-01	2.31e-01	1.46e-01	3.75e-01	2.80e-01	

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APPENDIX A: Figures from High-Frequency Financial Data Modeling

In this section, we include all of the VaR estimations for different models. In any of these cases, exceptions are labeled for different values of α. The first 5 figures are for models for which a GARCH model was used for modeling interarrival times, and in the last 5 figures the interarrival times are assumed to be i.i.d.

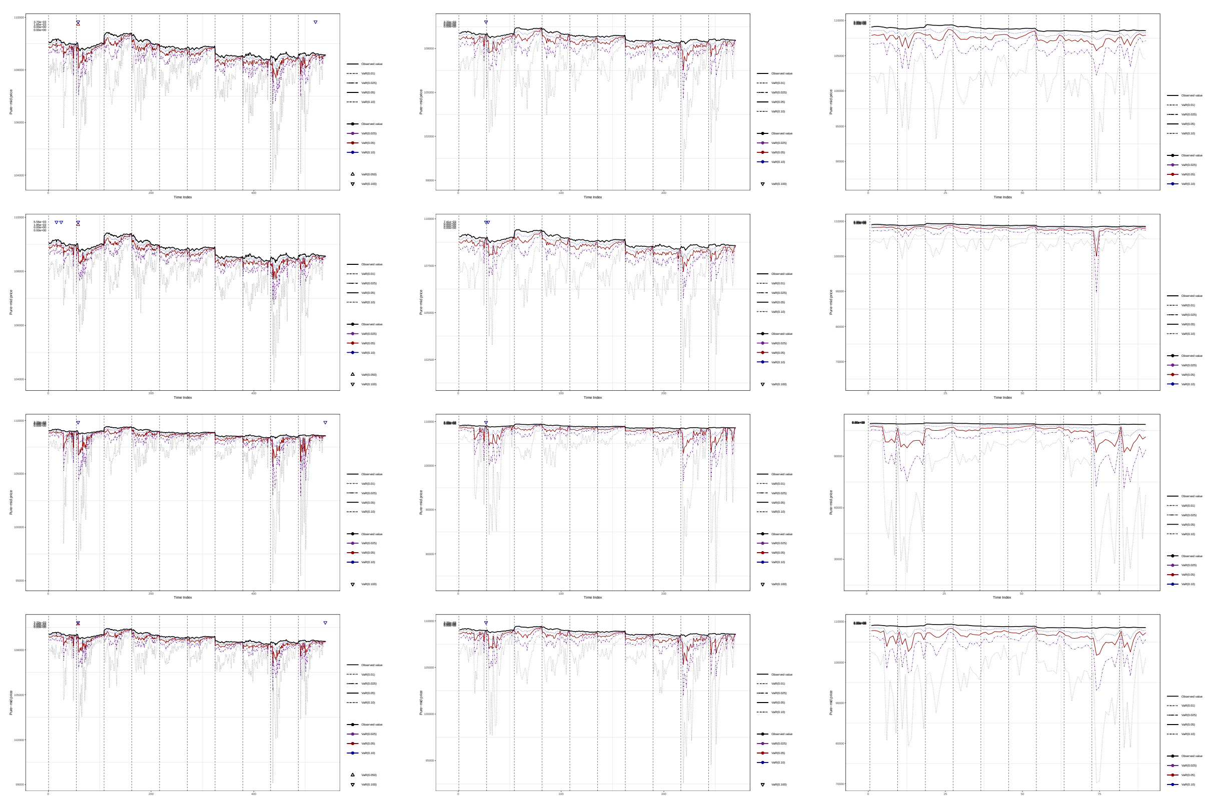

Figure A.1: VaR(T, α) for discrete stable distribution. Columns show different time horizons 10, 20, 60 minutes from left to right, respectively. Rows show different distributions for modeling standard residuals in the GARCH model for interarrival times which are from top to bottom, bootstrap sampling, Tweedie, exponential and gamma, respectively.

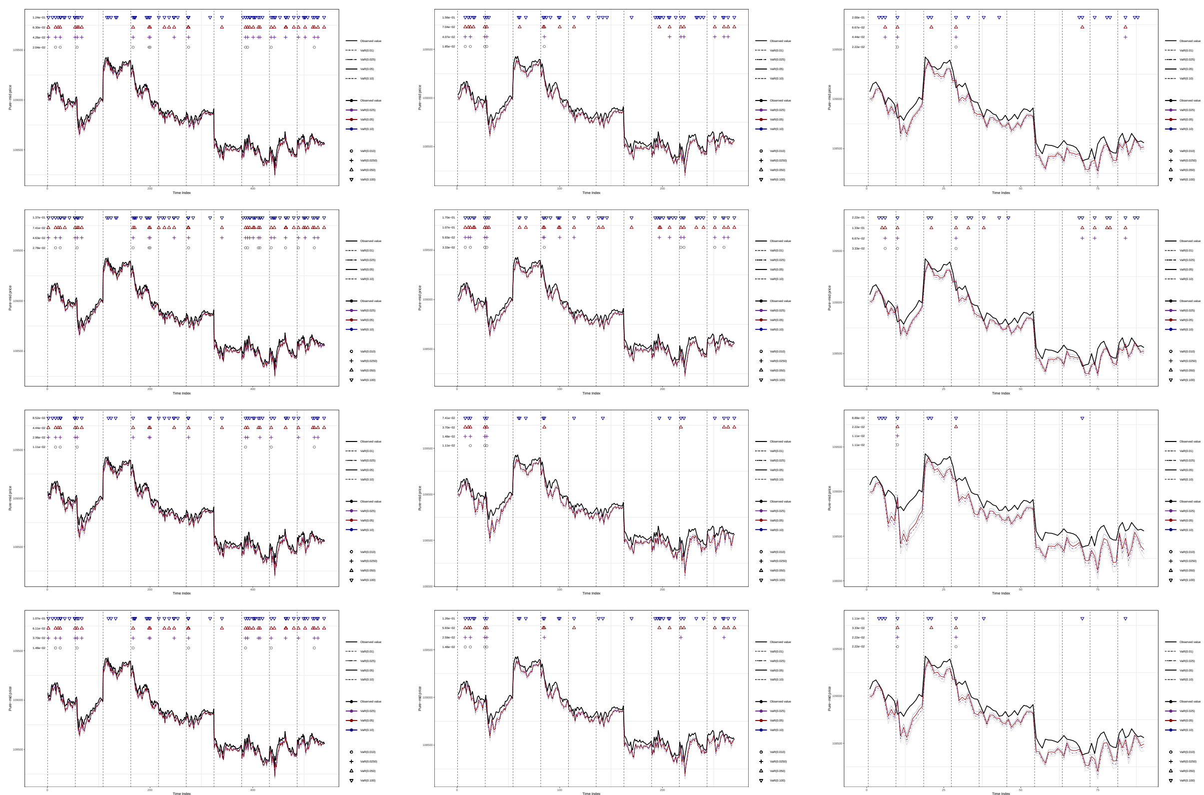

Figure A.2: VaR(T, α) for negative binomial distribution. Columns show different time horizons 10, 20, 60 minutes from left to right, respectively. Rows show different distributions for modeling standard residuals in the GARCH model for interarrival times which are from top to bottom, bootstrap sampling, Tweedie, exponential and gamma, respectively.

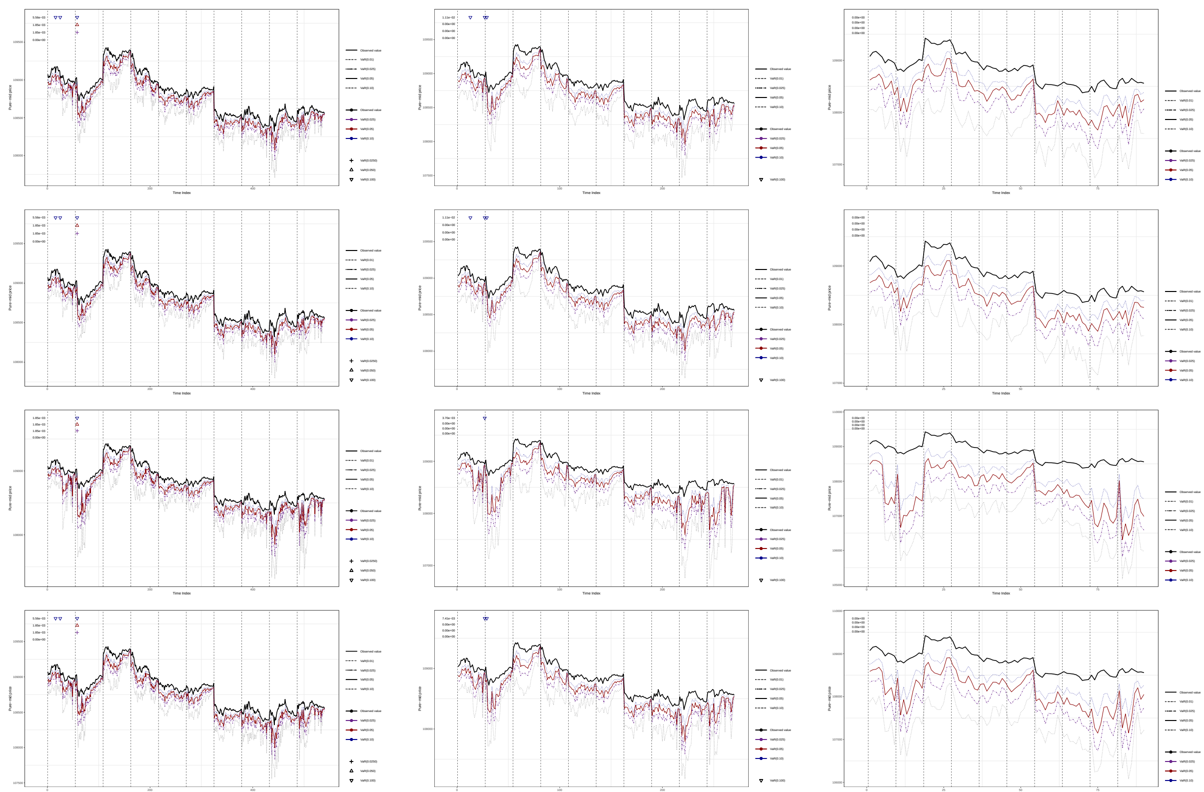

Figure A.3: VaR(T, α) for DTS (Pareto tempering) distribution. Columns show different time horizons 10, 20, 60 minutes from left to right, respectively. Rows show different distributions for modeling standard residuals in the GARCH model for interarrival times which are from top to bottom, bootstrap sampling, Tweedie, exponential and gamma, respectively.

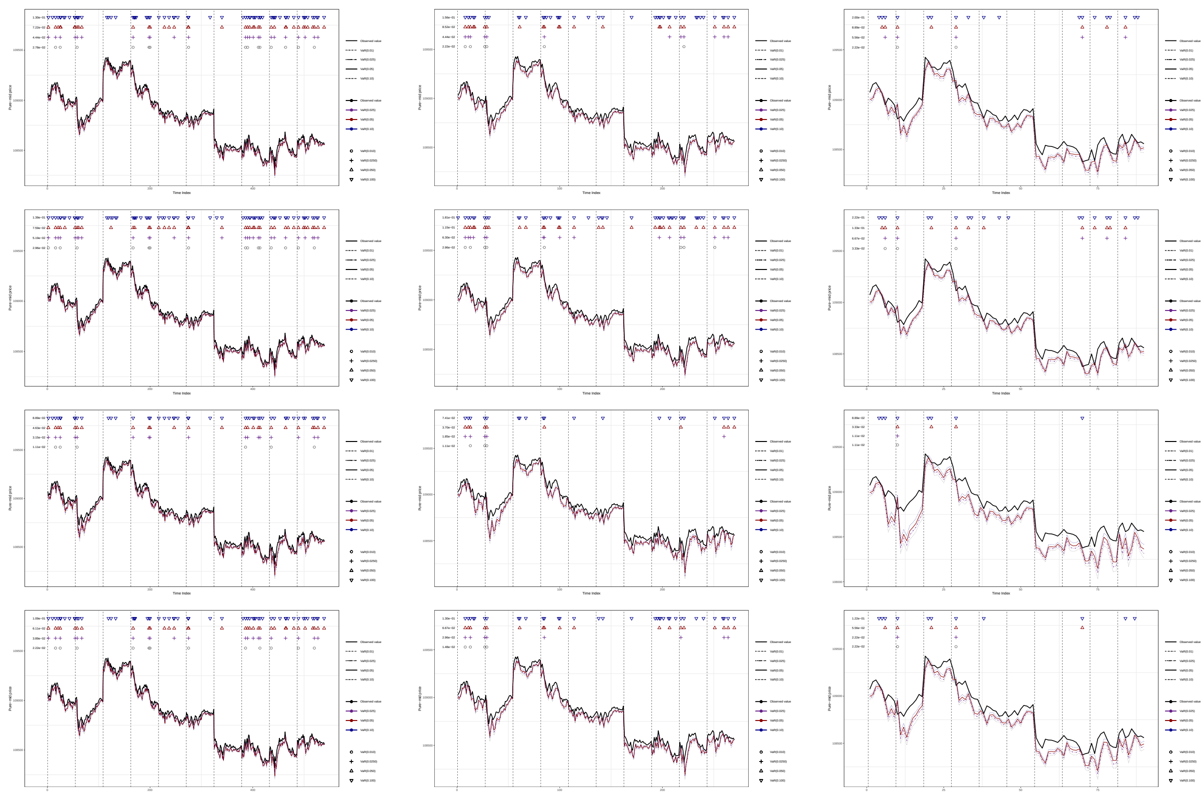

Figure A.4: VaR(T, α) for Poisson distribution. Columns show different time horizons 10, 20, 60 minutes from left to right, respectively. Rows show different distributions for modeling standard residuals in the GARCH model for interarrival times which are from top to bottom, bootstrap sampling, Tweedie, exponential and gamma, respectively.

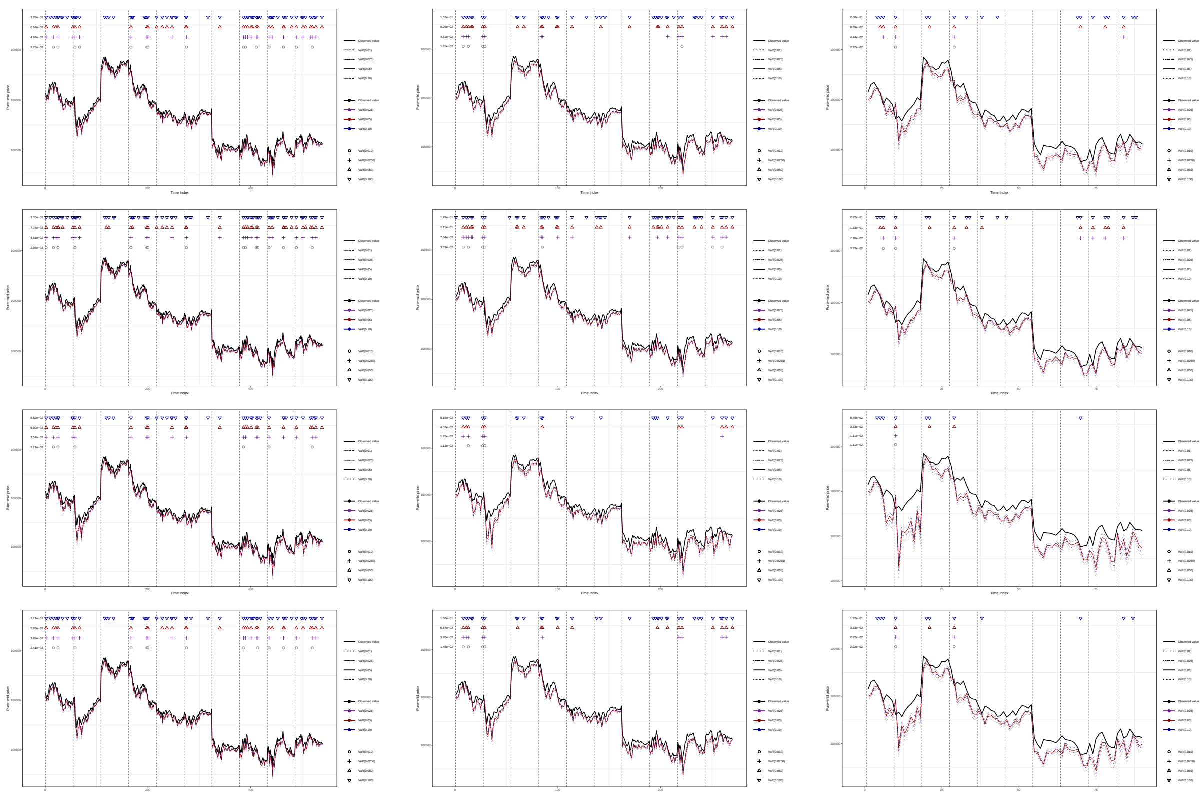

Figure A.5: VaR(T, α) for DTS (Poisson Tweedie) distribution. Columns show different time horizons 10, 20, 60 minutes from left to right, respectively. Rows show different distributions for modeling standard residuals in the GARCH model for inter-arrival times which are from top to bottom, bootstrap sampling, Tweedie, exponential and gamma, respectively.

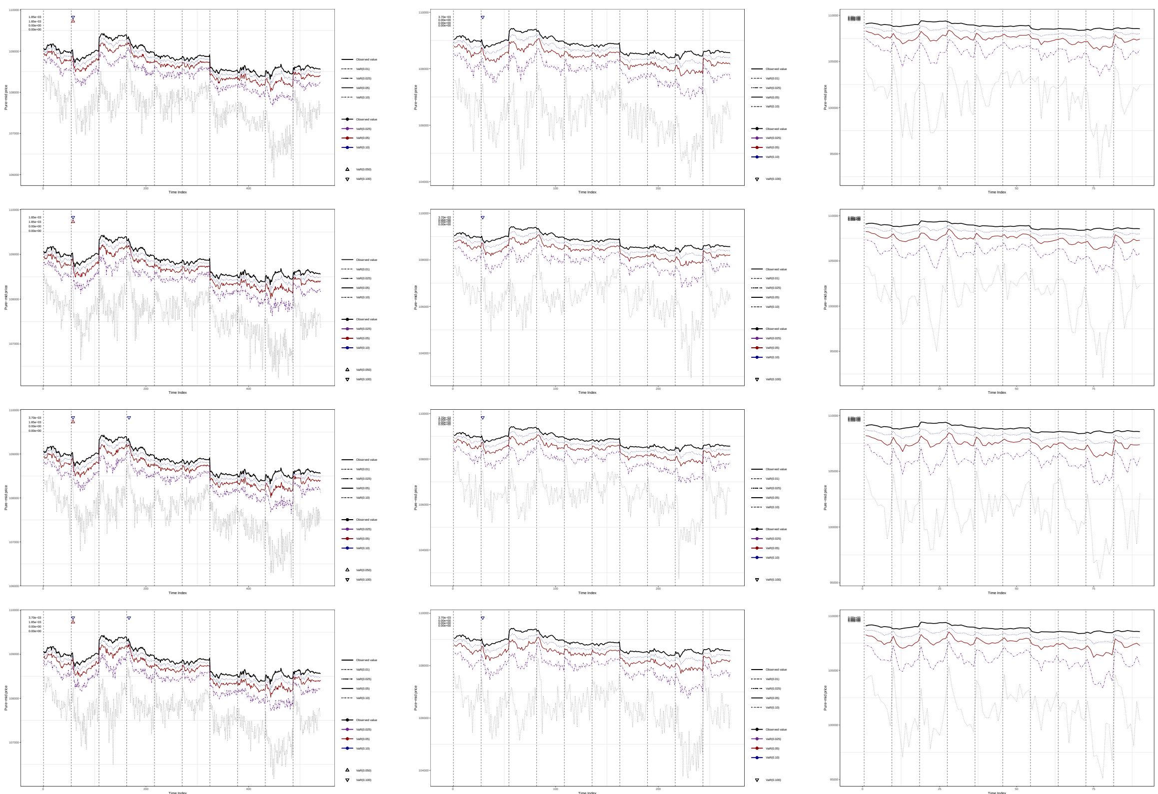

Figure A.6: VaR(T, α) for discrete stable distribution. Columns show different time horizons 10, 20, 60 minutes from left to right, respectively. Rows show different distributions for modeling interarrival times which are from top to bottom, bootstrap sampling, Tweedie, exponential and gamma, respectively.

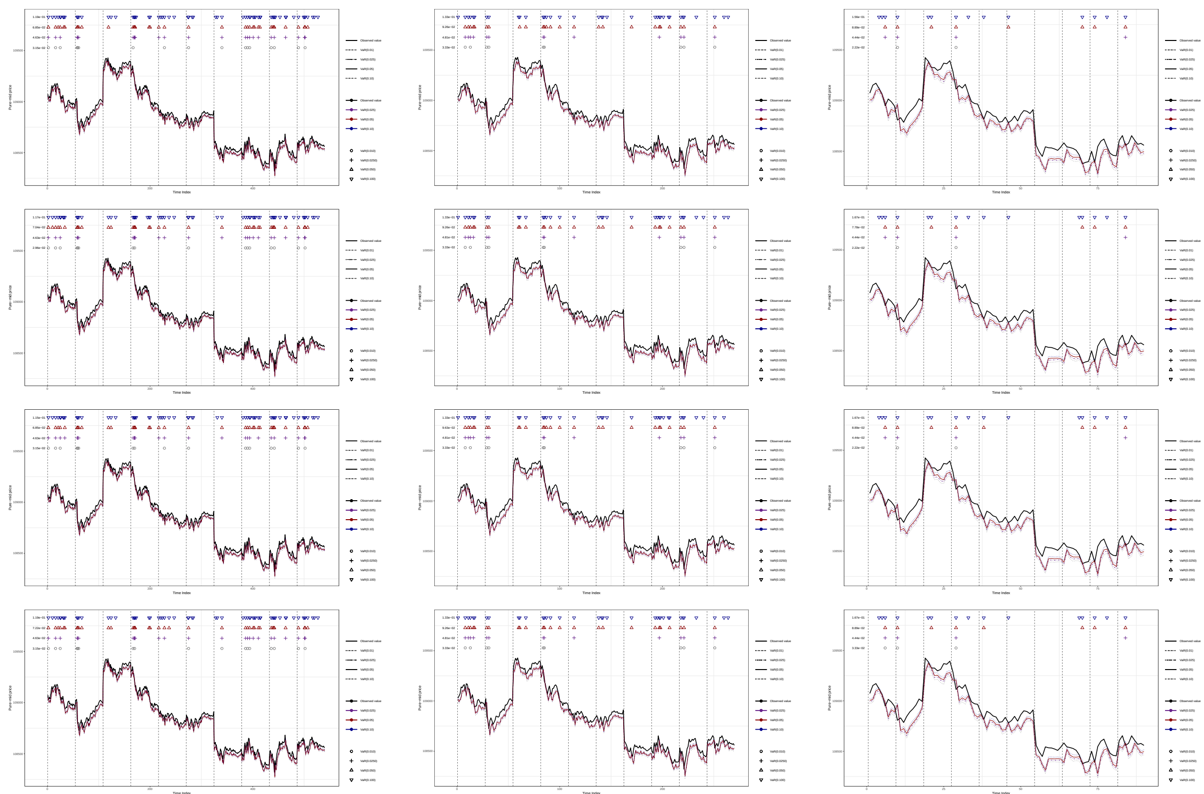

Figure A.7: VaR(T, α) for negative binomial distribution. Columns show different time horizons 10, 20, 60 minutes from left to right, respectively. Rows show different distributions for interarrival times which are from top to bottom, bootstrap sampling, Tweedie, exponential and gamma, respectively.

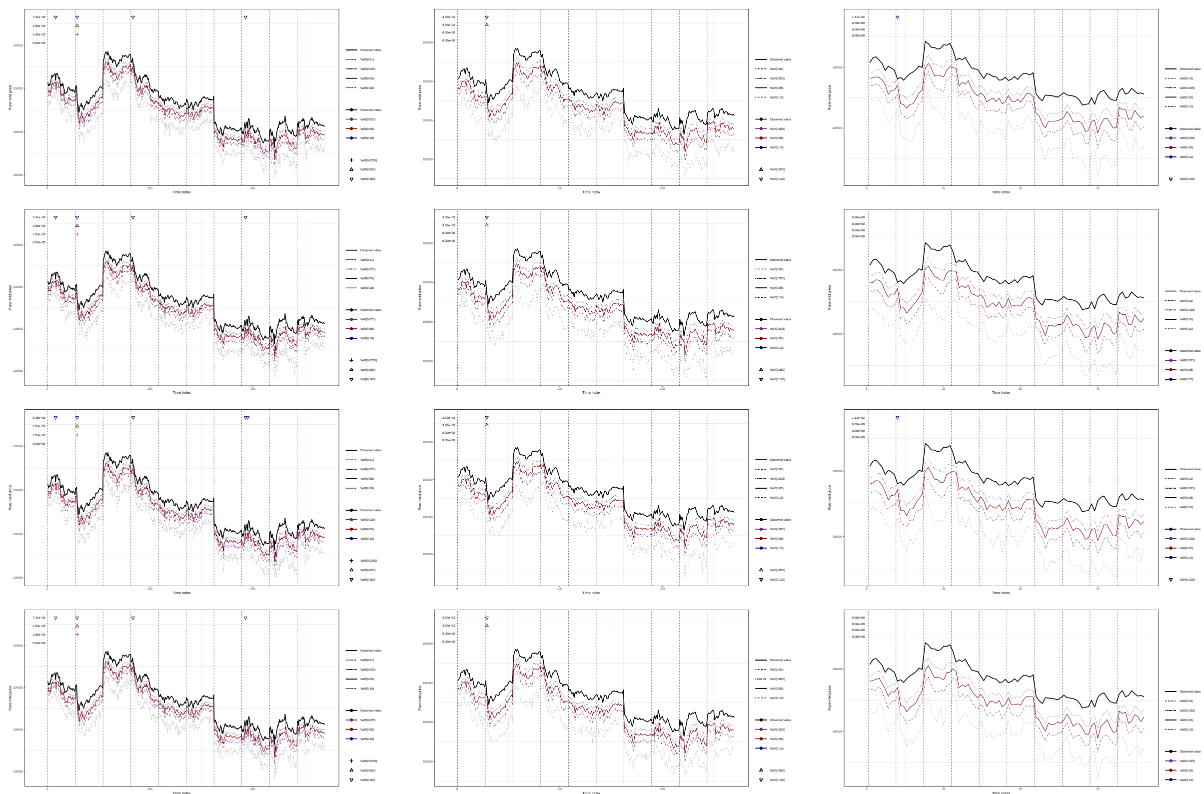

Figure A.8: VaR(T, α) for DTS (Pareto tempering) distribution. Columns show different time horizons 10, 20, 60 minutes from left to right, respectively. Rows show different distributions for interarrival times which are from top to bottom, bootstrap sampling, Tweedie, exponential and gamma, respectively.

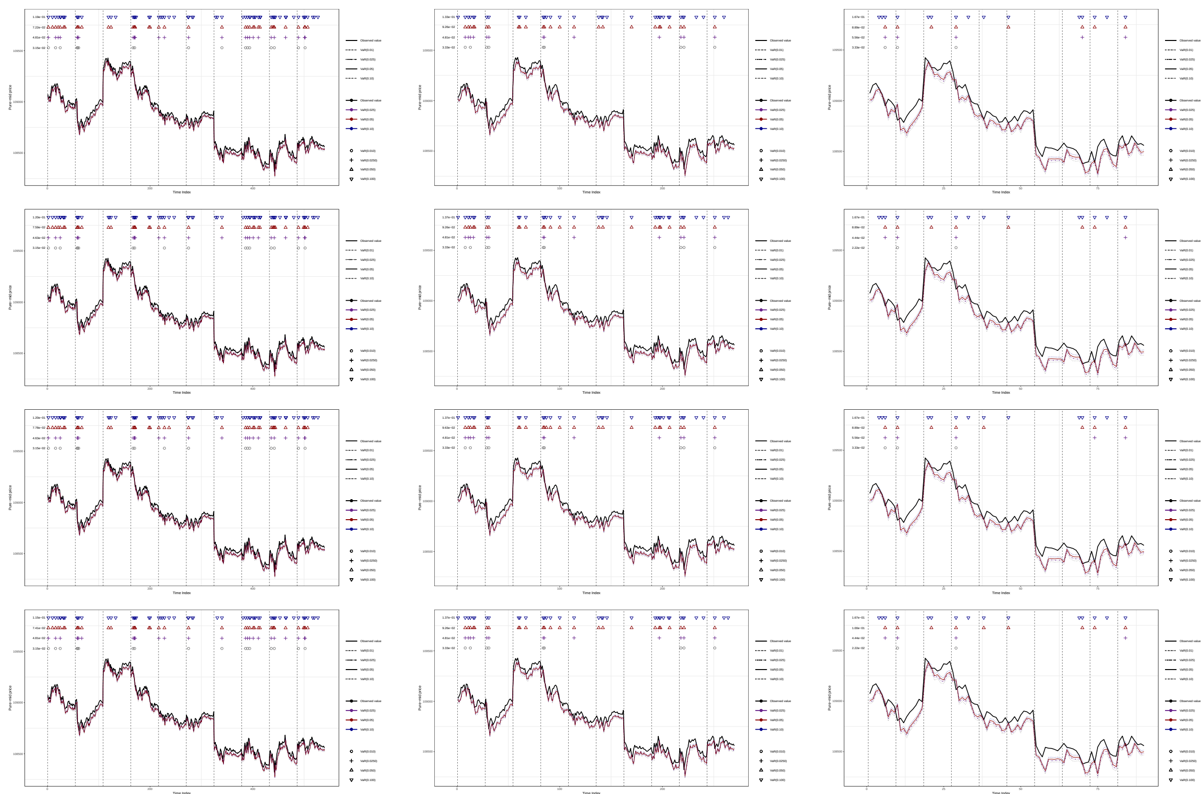

Figure A.9: VaR(T, α) for Poisson distribution. Columns show different time horizons 10, 20, 60 minutes from left to right, respectively. Rows show different distributions for modeling interarrival times which are from top to bottom, bootstrap sampling, Tweedie, exponential and gamma, respectively.

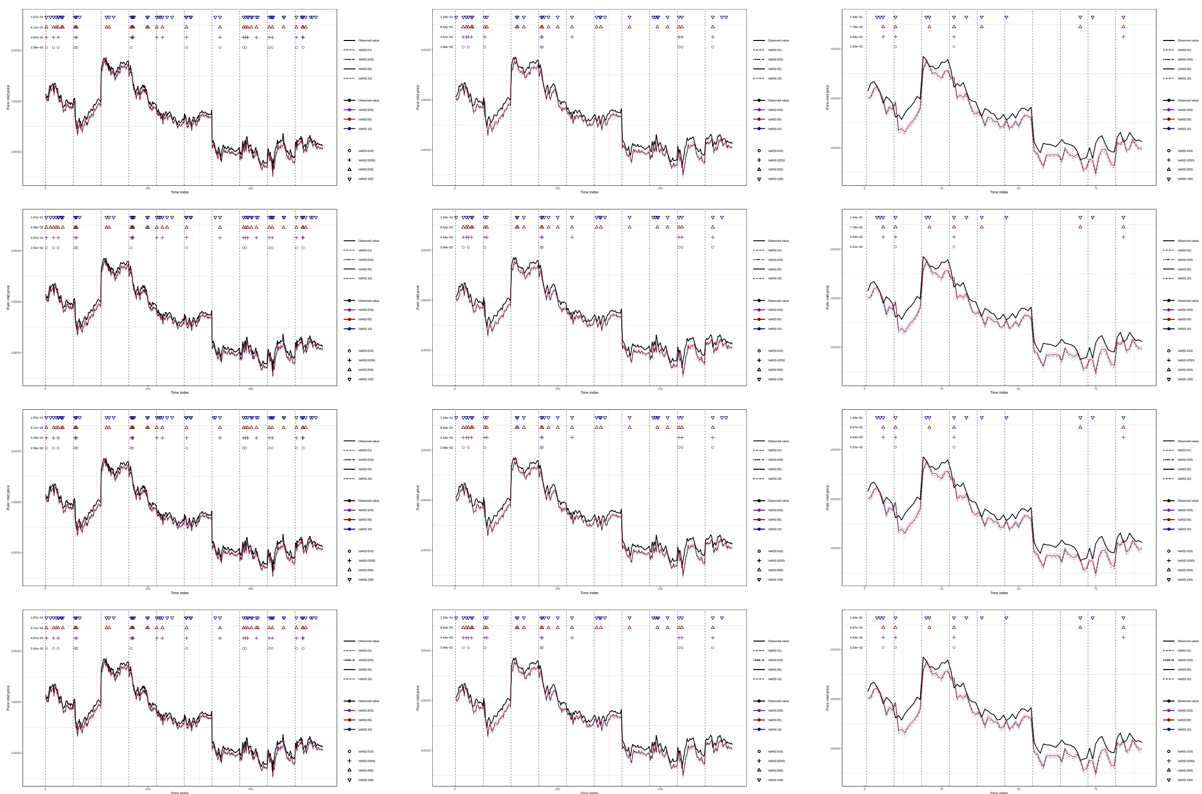

Figure A.10: VaR(T, α) for DTS (Poisson Tweedie) distribution. Columns show different time horizons 10, 20, 60 minutes from left to right, respectively. Rows show different distributions for interarrival times which are from top to bottom, bootstrap sampling, Tweedie, exponential and gamma, respectively.